Lie Groups

Michaelmas 2007

Question Sheet 3

Connectedness, group topology, Lie correspondence.

1. Show that an open subgroup H of a linear group G is also closed.

2. Show that the set G_I of all elements in G that can be connected to the idenity by a path is a normal subgroup of G.

3. Show that a discrete normal subgroup H in a connected group G is in the center of G.

4. Show that $\exp : L(SO_n) \to SO_n$ is surjective. [*Hint: Every $x \in SO_n$ is conjugate to a block-diagonal matrix with 2×2 blocks given by elements in SO_2 together with a single 1×1 block with entry 1 when n is odd.*]

   Deduce that SO_n is connected.

5. Let G be a connected linear group. Show that G is abelian if and only if $L(G)$ is abelian (i.e. $[X, Y] = 0$ for all $X, Y \in L(G)$).

   Hence describe all connected abelian linear groups (upto homeomorphisms).

6. Show that $S^1 \times SU_n \to U_n$ given by $(\lambda, A) \to \lambda A$ is a homomorphism of linear groups. Find its image and kernel. Describe the map induced on their Lie algebras. Do the same with $\det : U_n \to S^1$.