Lie Groups

Michaelmas 2007

Question Sheet 5

Classical Lie groups, Cartan subgroups, Weyl groups, root system.

1. Let $T \subset G$ be a maximal torus in a compact matrix group G, and let $Q \subset N_G(T)$ be a connected group in the normaliser of T in G.

   i. Prove that an automorphism of T induces and automorphism of $L(T)$ and of $\ker(\exp : L(T) \to T)$.

   ii. By considering the induced action of Q on $\ker(\exp)$, prove that the action of Q on T is trivial.

   iii. Deduce that the Weyl group is a finite discrete group.

2. Show that the Weyl group of SU_n is the symmetric group Σ_n.

   [*Hint: Conjugation takes a diagonal matrix to a diagonal matrix with the same eigenvalues.*]

3. Find the roots of $Sp_n\mathbb{C}$.

4. Let α, β be two roots and let E_α, E_β be eigenvectors for α and β. Show that

   (i) if $\alpha + \beta$ is a root, $[E_\alpha, E_\beta]$ is a non-zero multiple of $E_{\alpha+\beta}$, or

   (ii) if $\beta = -\alpha$, $[E_\alpha, E_\beta]$ is an element of $L(H)$, or

   (iii) $[E_\alpha, E_\beta]$ is zero.

5. Let L be the real subspace of $L(H)^*$ spanned by the coordinate function λ_i where $H = T_\mathbb{C} \subset G_\mathbb{C}$. For a root α, define $s_\alpha : L \to L$ by $s_\alpha(\lambda) = \lambda - 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$.

   i. Show that s_α is a reflection along α, i.e. a reflection in the hyperplane $< \alpha >^\perp$.

   ii. In the case of $G_\mathbb{C} = SL_n\mathbb{C}$, show that s_α is an element in the Weyl group, i.e. there is an element $s \in W$ with $s.\lambda = s_\alpha(\lambda)$. Furthermore, show that the s_α generate W.