Lie Groups

Michaelmas 2009

Question Sheet 1

Exponential map; one parameter subgroups.

1. Let $A \in M_n\mathbb{R}$ and λ be the largest eigenvalue of $A^T A$ where A^T is the transpose of A. Prove that $||A|| = \sqrt{\lambda}$. [*Hint: $A^T A$ is diagonisable*].

2. Show that the group of orthogonal matrices $O(n) \subset M_n(\mathbb{R})$ is bounded and closed. Deduce that $O(n)$ is a compact group of matrices. Do the same for the unitary group $U_n \subset M_n(\mathbb{C})$.

3. Show that $\exp \tau X$ is orthogonal (unitary) for all $\tau \in \mathbb{R}$ if and only if X is skew-symmetric (skew-Hermitian). Recall, X is skew-symmetric if $X^t = -X$ and skew-Hermitian if $\bar{X}^t = -X$.

4. Show that for any matrix X,
$$\det(\exp X) = e^{\operatorname{tr} X}.$$
Show that $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not the exponential of any real 2×2 matrix.

5. A matrix is called semisimple if it is diagonisable over $\mathbb{C}$. Show:
   (i) $\exp X$ is semisimple if X is;
   (ii) if a is an invertible semisimple matrix then there exists a semisimple matrix X so that $a = \exp X$, and no two distinct eigenvalues of X differ by a multiple of $2\pi i$.

6. Let $A, B \in M_n(\mathbb{R})$. Find a solution $\alpha : \mathbb{R} \to M_n(\mathbb{R})$ for the differential equation
$$\alpha'(t) = A\alpha(t) \quad \text{and} \quad \alpha(0) = B.$$
Prove that your solution is unique.