

LIE GROUPS
MICHAELMAS 2009
QUESTION SHEET 5

Classical Lie groups, Cartan subgroups, Weyl groups, root system.

1. Let $T \subset G$ be a maximal torus in a compact matrix group G , and let $Q \subset N_G(T)$ be a connected group in the normaliser of T in G .
 - i. Prove that an automorphism of T induces an automorphism of $L(T)$ and of $\ker(\exp : L(T) \rightarrow T)$.
 - ii. By considering the induced action of Q on $\ker(\exp)$, prove that the action of Q on T is trivial.
 - iii. Deduce that the Weyl group is a finite discrete group.

2. Show that the Weyl group of SU_n is the symmetric group Σ_n and that of Sp_n is $\Sigma_n \times (\mathbb{Z}/2\mathbb{Z})^n \subset \Sigma_{2n}$.

[Hint: Conjugation takes a diagonal matrix to a diagonal matrix with the same eigenvalues.]

3. Find the roots of $Sp_n\mathbb{C}$. What is the action of the Weyl group on the roots?

4. Let α, β be two roots and let E_α, E_β be eigenvectors for α and β . Show that
 - (i) if $\alpha + \beta$ is a root, $[E_\alpha, E_\beta]$ is a non-zero multiple of $E_{\alpha+\beta}$, or
 - (ii) if $\beta = -\alpha$, $[E_\alpha, E_\beta]$ is an element of $L(H)$, or
 - (iii) $[E_\alpha, E_\beta]$ is zero.

5. Let L be the real subspace of $L(H)^*$ spanned by the coordinate functions $\lambda_1, \dots, \lambda_n$ where $H = T_{\mathbb{C}} \subset G_{\mathbb{C}}$. Identifying this naturally as (a subspace of) \mathbb{R}^n , L is an innerproduct space. For a root α , define $s_\alpha : L \rightarrow L$ by $s_\alpha(\lambda) = \lambda - 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)}\alpha$.
 - i. Show that s_α is a reflection along α , i.e. a reflection in the hyperplane $\langle \alpha \rangle^\perp$.
 - ii. In the case of $G_{\mathbb{C}} = SL_n\mathbb{C}$, show that s_α is an element in the Weyl group, i.e. there is an element $s \in W$ with $s \cdot \lambda = s_\alpha(\lambda)$. Furthermore, show that

the s_α generate W .

- iii. For the type A_2 , $D_1 = A_1 \times A_1$ and $B_2 = C_2$, that is for SU_3 , SO_4 , SO_5 and Sp_2 , make a sketch of the real plane L with the lines of reflections s_α .