Lie Groups
Michaelmas 2007
Question Sheet 7

More characters, the representation ring, and elementary symmetric and alternating sums.

Note: All groups are assumed to be linear and all representations complex!

1. Let G be a compact group, (V, π) a G-representation. Show that $|\chi_\pi(g)| \leq \chi_\pi(I)$ with equality if and only if $\pi(g)$ is a scalar.

2. Let G be a compact group. Let $\{\pi^\lambda\}_\lambda$ be a complete set of non-equivalent irreducible G-representations, and define $K(G)$ to be the free abelian group generated by it, i.e. the set of finite sums

$$\sum_\lambda n_\lambda \pi^\lambda \qquad \text{with } n_\lambda \in \mathbb{Z}.$$

$K(G)$ has a multiplication defined by tensor product. Its elements are refered to as *virtual representations.*

i Prove that $\chi : K(G) \to \mathcal{C}(G)$ defines an injective ring homomorphisms into the ring of continuous, complex valued functions on G.

The image of χ, also written as $K(G)$, is called the *character ring.*

ii Determine the character ring of a torus of dimension n.

Let T be a maximal torus of G. Then the Weyl group W acts on the characters of T: for $s \in W, \psi$ a character of T, define $s.\psi(t) := \psi(sts^{-1})$.

iii Show that the restriction map $\chi \mapsto \psi = \chi|_T$ identifies $K(G)$ with a subring of $K(T)^W = \{\psi \in K(T) \,|\, s.\psi = \psi \text{ for all } s \in W\}$.

iv Determine $K(T)^W$ when $G = U_n$.

[It turns out that $K(G) \simeq K(T)^W$.]

3. Let G and H be two compact groups.

i Prove that if π and ρ are irreducible representations of G and H then $\pi \otimes \rho$ is an irreducible representation of $G \times H$. Here $(\pi \otimes \rho)(g, h) = \pi(g) \otimes \rho(h)$. [Hint: You may assume Fubini's theorem: $\int_{G\times H} = \int_G \int_H$.]

ii Vice versa, every irreducible representation of $G \times H$ is of this form. [Hint: Every representation V of a group K is isomorphic to

$$\oplus_\lambda \mathrm{Hom}_K(V_\lambda, V) \otimes V_\lambda$$

where Hom_K denotes the linear space of K-maps and the sum is taken over non-isomorphic irreducible representations of K. If you want to use this fact prove it by considering the map to V induced by evaluation.]

iii Deduce $K(G \times H) \simeq K(G) \otimes K(H)$ (where the tensor product is a tensor product over $\mathbb{Z}$ of abelian groups).

4. Describe the maximal torus of $G = U_2$, its roots, the Weyl group and its action on the roots. Hence show that for $G = U_2$ the formula $\triangle = A(\rho)$ holds.

5. Identify the adjoint representation of SU_2 as a direct sum its weight spaces. Hence write down its character. What is the highest weights? Deduce that the adjoint representation is irreducible. Using the Weyl Character Formula write down a formula for its character, and show that the two formulas for the character agree.