Tubular configurations: equivariant scanning and splitting

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Abstract

Replacing configurations of points by configurations of tubular neighbourhoods (or discs) in a manifold M we are able to define a natural scanning map that is equivariant under the action of the diffeomorphism group of the manifold. We also construct the so-called power set map of configuration spaces diffeomorphism equivariantly. Combining these two constructions yields stable splittings in the sense of Snaith and generalisations thereof that are equivariant. In particular one deduces stable splittings of homotopy orbit spaces. As an application the homology injectivity is proved for diffeomorphism of M that fix an increasing number of points. Throughout we work with configurations spaces with labels in a fibre bundle over M.

1 Introduction

There has been much recent interest in configuration spaces of manifolds. In one direction, the work on factorisation algebras and non-commutative Poincare duality by Lurie [Lur], see also Francis [Fra], is based on the classical work on configuration spaces of May [May72], Segal [Seg73], McDuff [McD75] and Salvatore [Sal01]. This has also ignited increased interest in the Goodwillie-Weiss embedding calculus [GW99]. From this point of view, one is more interested in configurations of embedded discs than points, and needs to understand the interaction with the diffeomorphism group of the background space M. Our approach to configurations spaces will address both these points.

In another direction, moduli spaces of manifolds and the scanning map have been central in the work on the Mumford conjecture and analogues; see [MW07], [Gal11] and also [Til12] for a survey. In this context, configuration spaces are moduli spaces of zero dimensional manifolds and have provided much intuition. The work here was motivated by some basic question of diffeomorphism equivariance that arose in this context.

Contents and results:

Let $C_k(M;X)$ denote the space of k unordered, distinct particles in a compact smooth manifold M with labels in a pointed space X. For a closed submanifold $M_0 \subset M$, let $C(M,M_0;X)$ denote the space of configurations of particles in M which vanish in M_0 or at the base-point of X.

The goal of this paper is two-fold. First we want to revisit the foundations of the subject and provide a natural and equivaraint scanning map that relates the configuration spaces $C(M,M_0;X)$ to mapping spaces or section spaces more generally. The study of these maps goes back to May [May72] and Segal [Seg73] in the case when $M = \mathbb{R}^n$, and to McDuff [McD75] and Bödigheimer

[Böd87] for general manifolds. The diffeomorphism group of M acts naturally on both the configuration spaces and the section spaces. However, the standard scanning maps, which involve choosing a metric on M and are defined by collapsing ε-balls around the particles, do not commute with these actions.

Our approach here is to replace a configuration by its space of tubular neighbourhoods. As the space of tubular neighbourhoods is contractible this construction does not change the homotopy type of $C_k(M;X)$ and, less obviously, also not of $C(M,M_0;X)$. This will be proved in section 2. In section 3 the scanning map on these enlarged configuration spaces is defined by simply collapsing M onto the configuration of tubular neighbourhoods. This construction is equivariant under the action of the diffeomorphism group; see Theorem 3.8.

Our second goal is to revisit the classical splitting theorems for function spaces going back to Snaith [Sna74] when M is a Euclidean space and generalised by Bödigheimer [Böd87] to arbitrary manifolds. Using the above results we construct these splittings equivariantly under the action of the diffeomorphism group; the main result in this direction is Theorem 4.8. In particular this gives stable splittings of the corresponding homotopy orbit spaces, something that for actions of compact Lie groups and a restricted class of manifolds was previously shown by Bödigheimer and Madsen [BM88] by different methods that do not extend to the non-compact setting.

The key to the splitting theorem is the construction in section 4.3 of diffeomorphism equivariant power set maps for configuration spaces. This uses the Barratt-Eccles [BE74a] model for the free infinite loop space functor.

In the final section, for connected M with non-empty boundary, the splitting methods are applied to show that the inclusion $b : C_k(M;X) \to C_{k+1}(M;X)$, which is well-known to be stably split injective, is indeed equivariantly so. As an immediate consequence we prove that a natural homomorphism

$$\bar{b} : \mathrm{Diff}(M \smallsetminus \boldsymbol{k}; \partial M) \to \mathrm{Diff}(M \smallsetminus \boldsymbol{k+1}; \partial M)$$

of diffeomorphisms of M fixing a set of k points to those fixing a set of $k+1$ points induces a split injection in homology on classifying spaces; this is the content of Theorem 4.15.

Much of the literature restricts itself to configurations with labels in a constant space X. We emphasise that more generally we consider here configuration spaces with twisted coefficients, that is where X is replaced by a fibre bundle π over M and the label space may vary with the points in M. On the one hand we will need this in the application we have in mind [Til] and on the other hand it allows us to replace sections spaces with mapping spaces; see Example 4.10.

Future work and extensions:

In forthcoming work of the second author [Til], using the results established here, the map $\bar{b}$ and generalisations thereof will be shown to also induce isomorphisms in homology in a range growing with k.

In another direction, the methods of this paper can be extended to treat configurations of submanifolds as considered by Palmer [Pal] and provide equivariant stable split injections for the stable homology isomorphisms in that setting.

2 Tubular configuration spaces and twisted labels

We define tubular configuration spaces and show that they are homotopic to the usual configuration spaces.

2.1 The definition of tubular configuration spaces

Let M be a smooth compact manifold. The configurations space of k ordered particles in M is the subspace of the k-fold Cartesian product of M

$$\widetilde{C}_k(M) := \{(m_1, \dots, m_k) \in M^k : m_i \neq m_j \,\text{if}\, i \neq j\}.$$

Equivalently, $\widetilde{C}_k(M)$ is the embedding space $\mathrm{Emb}(\boldsymbol{k}, M)$, where $\boldsymbol{k}$ denotes the 0-manifold with k points. The symmetric group Σ_k acts freely and the configuration space of k unordered particles in M is the orbit space $C_k(M)$. When $k = 0$ there is only one configuration, the empty configuration, and $\widetilde{C}_0(M) = C_0(M) = *$.

Let $M_0 \subset M$ be a (possibly empty) compact submanifold. The configuration space of particles in M modulo M_0 is then defined as

$$C(M, M_0) := \left(\coprod_{k=0}^{\infty} C_k(M) \right) \Big/ \sim$$

where $(m_1, \dots, m_k) \sim (m_1, \dots, m_{k-1})$ if $m_k \in M_0$. We think of this relation as particles vanishing in M_0.

To define the tubular configuration spaces of particles in M we replace a configuration $\boldsymbol{m} = (m_1, \dots, m_k)$ by the space of tubular neighbourhoods of its particles considered as a 0-dimensional submanifold of M.

Let W be a manifold without boundary containing M as a codimension zero submanifold. More precisely, if M has empty boundary let $W = M$ and otherwise let $W = M \cup (\partial M \times [0, 1))$ be M with an open collar attached. Similarly we define M^+ as $M^+ = M = W$ when M has no boundary and as $M^+ = M \cup (\partial M \times [0, 1/2))$ otherwise. Such manifolds M^+ and W are required so that particles on the boundary of M admit tubular neighbourhoods.

Let $P \subset M$ be a neat submanifold, let ν be its normal bundle and identify P with the image of the zero section in ν. By a tubular neighbourhood of P in M we mean an embedding $f : \nu \to M$ which restricts to the identity on the zero-section and for which the composition

$$\nu \hookrightarrow \nu \oplus TP = T\nu|_P \xrightarrow{df} TM|_P \longrightarrow \nu \qquad \text{(id)}$$

is the identity on ν. We call this last property (id). Denote the space of tubular neighbourhoods of P by $\mathrm{Tub}(P)$ and topologise it as a subspace of the embedding space $\mathrm{Emb}(\nu, M)$ with the C^∞ topology.

For an ordered or unordered configuration $\boldsymbol{m}$ of M let $\mathrm{Tub}(\boldsymbol{m})$ denote the space of tubular neighbourhoods of $\boldsymbol{m}$ considered as a 0-dimensional submanifold of M^+. Define the tubular configuration space of k ordered particles in M as the disjoint union

$$\widetilde{E}_k(M) := \coprod_{\boldsymbol{m} \in \widetilde{C}_k(M)} \mathrm{Tub}(\boldsymbol{m})$$

equipped with an appropriate topology which restricts to the ordinary C^∞ topology on the fibres. We defer a description of the topology to Section 2.2. We refer to a tubular neighbourhood $f \in \widetilde{E}_k(M)$ as a tubular configuration and we often write it as a collection of its components $(f_1, \dots, f_k)$, with $f_i : T_{m_i}M \to M^+$. The symmetric group on k points acts freely on $\widetilde{E}_k(M)$ and we define the tubular configuration space of k unordered particles as the orbit space $E_k(M)$.

Analogous to the ordinary configuration spaces, we define the tubular configuration space of particles in M modulo M_0 as

$$E(M, M_0) := \left(\coprod_{k=0}^{\infty} E_k(M) \right) \Big/ \sim$$

where $(f_1, \dots, f_k) \sim (f_1, \dots, f_{k-1})$ if $f_k(0) \in M_0$. Components of a tubular configuration vanish if their midpoint is in M_0.

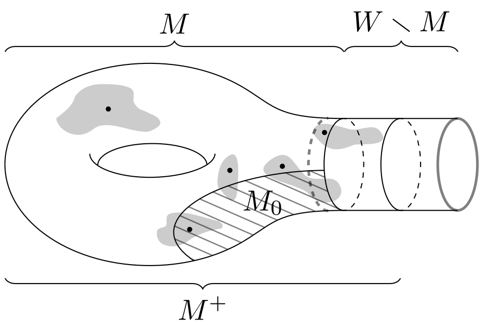


Figure 2.1: *The image of a tubular configuration of five particles on a punctured torus. The underlying particles are marked as black dots.*

Remark 2.1. As an alternative approach, consider configuration spaces with neighbourhoods of each particle without an identification with the normal bundle. We think of this as configurations of submanifolds diffeomorphic to a finite union of open disks with one marked point. Note that for a configuration $\boldsymbol{m}$ of k particles in M, there are isomorphisms

$$\mathrm{Tub}(\boldsymbol{m}) \cong \mathrm{WeakTub}(\boldsymbol{m}) / \textstyle\bigoplus_i \mathrm{GL}_n(\mathbb{R})$$

and

$$\mathrm{Tub}(\boldsymbol{m}) / \textstyle\bigoplus_i \mathrm{Diff}(T_{m_i}M, \mathrm{id}) \cong \mathrm{WeakTub}(\boldsymbol{m}) / \textstyle\bigoplus_i \mathrm{Diff}(T_{m_i}M, 0)$$

where $\mathrm{WeakTub}(\boldsymbol{m})$ is the space of weak tubular neighbourhoods, that is the space of embeddings of the normal bundle of $\boldsymbol{m}$ into M^+ which restrict to the identity on the zero section but do not satisfy property (id), $\mathrm{Diff}(T_{m_i}M, 0)$ is the group of diffeomorphisms which fix 0 and $\mathrm{Diff}(T_{m_i}M, \mathrm{id}) \subset \mathrm{Diff}(T_{m_i}M, 0)$ are the diffeomorphisms whose derivatives at zero are the identity. The second space is the space of embedded discs with given midpoints. We note that

$$\mathrm{Diff}(T_{m_i}M, 0) \simeq \mathrm{GL}_n(\mathbb{R}) \text{ and } \mathrm{Diff}(T_{m_i}M, \mathrm{id}) \simeq *.$$

Hence, the space of embedded discs (with fixed midpoint) is homotopic to the space of tubular neighbourhoods, in other words the two spaces above are homotopic.

2.2 The parc C^∞ topology

We generalise the compact-open topology for spaces of partial maps with closed domain as defined in [BB78] to a C^∞ topology for spaces of smooth partial maps. This generalisation allows us to topologise tubular configuration spaces as subspaces of certain smooth partial mapping spaces.

Let X and Y be topological spaces. A partial map $f : X \to Y$ is a map $A \to Y$ for some subspace $A \subseteq X$. We call A the domain of f and denote it by $\mathcal{D}(f)$. A parc map or partial map with closed domain is a partial map f such that $\mathcal{D}(f)$ is closed in X. Let $P_c(X, Y)$ denote the set of parc maps $X \to Y$. The parc mapping space is $P_c(X, Y)$ equipped with the parc compact-open topology which has sub-base the sets

$$(K, U) := \{f \in P_c(X, Y) : f(K \cap \mathcal{D}(f)) \subseteq U\}$$

for all compact sets $K \subseteq X$ and open sets $U \subseteq Y$.

Similarly there is a paro mapping space $P_o(X, Y)$ for partial maps with open domain [AAB80]. $P_o(X, Y)$ is equipped with the paro compact-open topology with sub-base the sets

$$(K, U) := \{f \in P_o(X, Y) : K \subseteq \mathcal{D}(f),\ f(K) \subseteq U\}.$$

Remark 2.2. Given a fixed closed (open) set $A \in X$, the topology on the subset of parc (paro) maps $X \to Y$ which are defined precisely on A coincides with the ordinary compact-open topology on the mapping space $\mathrm{Map}(A, Y)$.

Let M and N be C^r manifolds with $r < \infty$ and denote the set of partial C^r maps $M \to N$ with closed domain by $C^r P_c(M, N)$. We equip this set with a generalisation of the C^r topology as follows. Let $f \in C^r P_c(M, N)$ be a parc C^r map and let (ϕ, U) and (ψ, V) be charts for M and N respectively. Let $K \subseteq U$ be a compact set such that $f(K \cap \mathcal{D}(f)) \subseteq V$ and let $0 < \varepsilon \leq \infty$. Define a parc sub-basic neighbourhood

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \varepsilon)$$

to be the set of parc C^r maps $g : M \to N$ such that $g(K \cap \mathcal{D}(g)) \subseteq V$ and

$$\|D^k(\psi f \phi^{-1})(x) - D^k(\psi g \phi^{-1})(x)\| < \varepsilon$$

for all $x \in \phi(K)$ and $k = 0, \dots, r$. The set of all such neighbourhoods form a sub-base for the parc C^r topology on $C^r P_c(M, N)$. For C^∞ manifolds M and N, $C^\infty P_c(M, N)$ is the space of parc C^∞ maps equipped with the parc C^∞ topology. This is simply the union of the topologies induced by the inclusions $C^\infty P_c(M, N) \hookrightarrow C^r P_c(M, N)$ for r finite.

Remark 2.3. It follows immediately from the definition that the subspace of $C^\infty P_c(M, N)$ consisting of maps defined on a smooth closed submanifold A is the smooth mapping space $C^\infty(A, N)$ equipped with the ordinary C^∞ topology.

Remark 2.4. The parc smooth mapping space is functorial in each argument. Let M, N and Q be smooth manifolds, then smooth maps $\phi : N \to Q$ and $\psi : M \to Q$ induce the following continuous maps.

$$\begin{aligned} \phi_* : C^\infty P_c(M, N) &\longrightarrow C^\infty P_c(M, Q) \\ f &\longmapsto \phi \circ f \end{aligned} \qquad \begin{aligned} \psi^* : C^\infty P_c(M, N) &\longrightarrow C^\infty P_c(Q, N) \\ f &\longmapsto f \circ \psi|_{\psi^{-1}(\mathcal{D}(f))} \end{aligned}$$

2.3 The topology of tubular configuration spaces

Returning to configuration spaces, let $M \subset M^+ \subset W$ be as defined in section 2.1, and define the tubular configuration space of k unordered particles in M as the subspace

$$E_k(M) := \{f \in C^\infty P_c(TM, M^+) : \mathcal{D}(f) = \coprod_i T_{m_i} M \text{ and } f \in \mathrm{Tub}(\boldsymbol{m}) \text{ for some } \boldsymbol{m} \in C_k(M)\}.$$

This agrees set-wise with the previous definition. Moreover, for any $\boldsymbol{m} \in C_k(M)$ the topology on the subspace $\mathrm{Tub}(\boldsymbol{m}) \subset E_k(M)$ is compatible with the ordinary C^∞ topology by Remark 2.3.

Lemma 2.5. *The projection $p : E_k(M) \to C_k(M)$, $f \mapsto (f_1(0), \ldots, f_k(0))$ is continuous.*

Proof. Let $r : W \to M$ be the retract collapsing the collar onto ∂M and let $j : M \hookrightarrow TM$ be the zero section. Note that $C_k(M)$ can be identified with the space of partial (smooth) identity maps $M \to M$ with domain a finite subset of size k. Then p is the restriction of the induced map

$$C^\infty P_c(TM, M^+) \xrightarrow{(r|_{M^+})_*} C^\infty P_c(TM, M) \xrightarrow{j^*} C^\infty P_c(M, M)$$

to $E_k(M)$, which is continuous by Remark 2.4. $\square$

Proposition 2.6. *$p : E_k(M) \to C_k(M)$ is a fibre bundle.*

Proof. Let $\boldsymbol{m} = (m_1, \ldots, m_k)$ be a configuration in $C_k(M)$. We construct a local trivialisation for p around $\boldsymbol{m}$. Let $h : TM|_{\boldsymbol{m}} \to M^+$ be a tubular neighbourhood of $\boldsymbol{m}$ and let DM and $\mathring{D}M$ be the closed and open disk subbundles of TM for some fixed metric. Then the restriction of h to $DM|_{\boldsymbol{m}}$ is a closed tubular neighbourhood. Choose a continuous family of diffeomorphisms of $h(DM|_{\boldsymbol{m}})$ fixing the boundary

$$\tau : \Gamma(\mathring{D}M|_{\boldsymbol{m}}) \longrightarrow \mathrm{Diff}(h(DM|_{\boldsymbol{m}}), \partial)$$

parameterised by the space of sections of $\mathring{D}M|_{\boldsymbol{m}}$ and such that for each section $s \in \Gamma(\mathring{D}M|_{\boldsymbol{m}})$, $\tau_s \circ h \circ s = \mathrm{id}_{\boldsymbol{m}}$ where $\tau_s := \tau(s)$. We assume the diffeomorphisms are extended to M^+ and W by fixing $W \setminus h(DM|_{\boldsymbol{m}})$.

There is a homeomorphism

$$\begin{aligned} \phi : \Gamma(\mathring{D}M|_{\boldsymbol{m}}) &\longrightarrow \left(\prod_i h(D_{m_i}M)\right) \Big/ \Sigma_k =: V \subseteq C_k(M) \\ s &\longmapsto (h \circ s(m_1), \ldots, h \circ s(m_k)) \end{aligned}$$

whose image is an open neighbourhood of $\boldsymbol{m}$ in $C_k(M)$. Intuitively, h defines an open ball around each particle in $\boldsymbol{m}$ and V is the set of all configurations with precisely one particle in each open ball. For each configuration $\boldsymbol{n} \in V$, $\sigma_{\boldsymbol{n}} := \tau \circ \phi^{-1}(\boldsymbol{n})$ is the diffeomorphism of W which moves the particles in $\boldsymbol{n}$ onto the particles in $\boldsymbol{m}$.

Local trivialisations over V are then given by

$$\begin{aligned} E_k(M)|_V &\longmapsto V \times \mathrm{Tub}(\boldsymbol{m}) \\ \mathrm{Tub}(\boldsymbol{n}) \ni f &\longmapsto \left(\boldsymbol{n}, \sigma_{\boldsymbol{n}} \circ f \circ (d\sigma_{\boldsymbol{n}})^{-1}\right). \end{aligned}$$

Here $(d\sigma_{\boldsymbol{n}})^{-1}$ maps the normal bundle of $\boldsymbol{m}$ to the normal bundle of $\boldsymbol{n}$, f maps this to a neighbourhood of $\boldsymbol{n}$ and $\sigma_{\boldsymbol{n}}$ maps this neighbourhood to a neighbourhood of $\boldsymbol{m}$, thus the composition is indeed a tubular neighbourhood of $\boldsymbol{m}$. $\square$

For each $k \geq 0$ there is a covering map $q : \widetilde{C}_k(M) \to C_k(M)$. We define the tubular configuration space of k ordered particles in M as the pullback $\widetilde{E}_k(M) := q^* E_k(M)$ and let $\tilde{p} : \widetilde{E}_k(M) \to \widetilde{C}_k(M)$ be the corresponding fibre bundle.

The projection p has a right inverse σ. To define σ choose a Riemannian metric on W. For a configuration $\boldsymbol{m}$ let ε_1 be the smallest distance in W between any two of its particles and let ε_2 be the greatest value such that for each i, $\exp_{m_i} : B_{\varepsilon_2}(T_{m_i}M, 0) \to B_{\varepsilon_2}(M^+, m_i)$ is a diffeomorphism. Let $\varepsilon_{\boldsymbol{m}} = \min\{\varepsilon_1, \varepsilon_2\}$, then $\boldsymbol{m} \mapsto \varepsilon_{\boldsymbol{m}}$ is a continuous map $\widetilde{C}_k(M) \to \mathbb{R}_{>0}$. For each i define $f_i : T_{m_i}M \to M^+$ by $v \mapsto \exp_{m_i}\left(\frac{2\varepsilon_{\boldsymbol{m}} v}{\pi|v|} \arctan|v|\right)$, then $(m_1, \ldots, m_k) \mapsto (f_1, \ldots, f_k)$ is a section $\tilde{\sigma} : \widetilde{C}_k(M) \to \widetilde{E}_k(M)$. Moreover, this section is Σ_k-equivariant and descends to a section $\sigma : C_k(M) \to E_k(M)$.

Corollary 2.7. *The projections p and $\tilde{p}$ are homotopy equivalences with homotopy inverses given by the global sections σ and $\tilde{\sigma}$.*

Proof. We have seen that the projections p and $\tilde{p}$ give tubular configuration spaces the structure of fibre bundles. The fibre over any configuration $\boldsymbol{m}$ is the space of tubular neighbourhoods $\mathrm{Tub}(\boldsymbol{m})$. The space of tubular neighbourhoods of any compact submanifold is contractible, see for example [God08]. The contractions of each fibre determine homotopies $\sigma \circ p \sim \mathrm{id}_{E_k(M)}$ and $\tilde{\sigma} \circ \tilde{p} \sim \mathrm{id}_{\widetilde{E}_k(M)}$. □

2.4 Twisted labels and homotopy equivalences

It is common to add local data to configurations in the form of labels in a parameter space. In this paper we consider an extension of this notion which allows the parameter space to vary as the fibre of a fibre bundle over the underlying manifold. We say the configurations have *twisted labels.*

Let M be a smooth compact manifold and let $\pi : Y \to M$ be a fibre bundle and a zero section $o : M \to Y$. Furthermore, assume that for each $m \in M$, the fibre Y_m over m is well-pointed with base-point $o(m)$. We define the configuration space of k ordered particles in M with twisted labels in π as

$$\widetilde{C}_k(M; \pi) := \{(\boldsymbol{m}, x) \in \widetilde{C}_k(M) \times P_c(M, Y) : \mathcal{D}(x) = \boldsymbol{m},\ x(m_i) \in Y_{m_i}\}$$

or equivalently $\widetilde{C}_k(M; \pi) := \{(\boldsymbol{m}; \boldsymbol{x}) \in \widetilde{C}(M) \times Y^k : \pi(x_k) = m_k\}$. We prefer here the definition in terms of partial maps as it motivates the definition of tubular configuration spaces with twisted labels below.

Example 2.8. An important example is when Y is the trivial bundle $M \times X$ where X is well-pointed with base-point $*$ and the zero section is $o(m) = (m, *)$. In this case we write $\widetilde{C}_k(M; X)$ for the configuration space with labels in X.

As in the unlabelled case, we define $C_k(M; \pi)$, the configuration space of k unordered particles with twisted labels in π, as the orbit space under the obvious Σ_k-action and $C_0(M; \pi) = *$. Now let $M_0 \subset M$ be a compact submanifold, then the configuration space of particles in M modulo M_0 with twisted labels in π is

$$C(M, M_0; \pi) := \left(\coprod_{k=0}^{\infty} C_k(M; \pi) \right) \Big/ \sim$$

where $(m_1, \ldots, m_k; x_1, \ldots, x_k) \sim (m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$ if $m_k \in M_0$ or $x_k = o|_{m_k}$. Here x_i denotes the ith component of x, i.e. $x_i = x|_{m_i}$.

Let $M \subset M^+ \subset W$ be as in section 2.1 and let (Y, π, o) be extended over W as the pullback along the retract $r : W \to M$ which maps the collar to ∂M. The tubular configuration space of k ordered particles in M with twisted labels in π is

$$\widetilde{E}_k(M;\pi) := \{(f,s) \in \widetilde{E}_k(M) \times P_o(M^+, Y) : \mathcal{D}(s) = \mathrm{im}(f),\ \pi s = \mathrm{id}_{\mathrm{im}(f)}\}$$

and the unordered tubular configuration space is the orbit space $E_k(M;\pi)$.

The following propositions are proved using similar techniques as in the proof of Proposition 2.6 and Corollary 2.7.

Proposition 2.9. *The projection $q : E_k(M;\pi) \to E_k(M)$, $(f;s) \mapsto f$ is a fibre bundle with fibre $q^{-1}(f) = \Gamma(Y|_{\mathrm{im}(f)})$, the space of sections of Y over the image of f.* □

Proposition 2.10. *The projection $p : E_k(M;\pi) \to C_k(M;\pi)$, $(f;s) \mapsto (p(f); s|_{p(f)})$ is a fibre bundle with contractible fibres.* □

Analogous to the ordinary configuration spaces, we define the tubular configurations space of particles in M modulo M_0 with twisted labels in Y to be

$$E(M, M_0;\pi) := \left(\coprod_{k=0}^{\infty} E_k(M;\pi) \right) \Big/ \approx$$

where $(f_1, \ldots, f_k; s_1, \ldots, s_k) \approx (f_1, \ldots, f_{k-1}; s_1, \ldots, s_{k-1})$ if $f_k(0) \in M_0$ or $s_k = o|_{\mathrm{im}(f_k)}$. We emphasise that we require the entire section s_k to agree with the zero section rather than just $s_k(m_k) = o(m_k)$. Our choice of definition here is motivated by the construction of the scanning map in the next section. Note that whereas a particle in a configuration vanishes if its label agrees with the zero section, a component of a tubular configuration vanishes only if the entire section over the image of that component agrees with the zero section. A tubular configuration does not necessarily vanish when the underlying particle in the configuration vanishes. This fact complicates the proof of the next result.

Proposition 2.11. *There is a well defined weak homotopy equivalence $p : E(M, M_0;\pi) \to C(M, M_0;\pi)$ induced by the projections $p : E_k(M;\pi) \to C_k(M;\pi)$.*

Proof. Define $p : E(M, M_0;\pi) \to C(M, M_0;\pi)$ by $p[(f;s)] := [p(f;s)]$ for any representative of the class. This is well defined since if a component of a configuration in $E(M, M_0;\pi)$ vanishes, the underlying particle in $C(M, M_0;\pi)$ does. The number of particles in configurations and tubular configurations induce filtrations

$$* = C^0(M, M_0;\pi) \subset C^1(M, M_0;\pi) \subset \ldots \subset C(M, M_0;\pi), \quad C^n(M, M_0;\pi) := \left(\coprod_{k=0}^{n} C_k(M;\pi) \right) \Big/ \sim$$

and

$$* = E^0(M, M_0;\pi) \subset E^1(M, M_0;\pi) \subset \ldots \subset E(M, M_0;\pi), \quad E^n(M, M_0;\pi) := \left(\coprod_{k=0}^{n} E_k(M;\pi) \right) \Big/ \approx$$

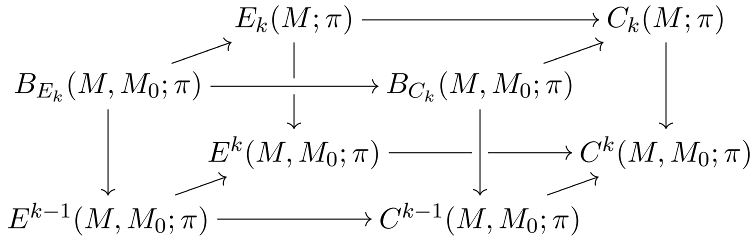
and p respects the filtrations. Topologically the configuration spaces $C(M, M_0;\pi)$ and $E(M, M_0;\pi)$ are the colimits of the filtration sequences. For each k define the subspaces of configurations in $C_k(M;\pi)$ and $E_k(M;\pi)$ in which at least one particle or component vanishes under the equivalence relation $\sim$ or $\approx$ respectively. More precisely, define

$$B_{C_k}(M, M_0;\pi) := \{(\boldsymbol{m}, \boldsymbol{x}) \in C_k(M;\pi) : m_k \in M_0 \text{ or } x_k = o|_{m_k}\}$$

and

$$B_{E_k}(M, M_0;\pi) := \{(f,s) \in E_k(M;\pi) : f_k(0) \in M_0 \text{ or } s_k = o|_{\mathrm{im}(f_k)}\}.$$

For each $k \geq 1$ we have a commutative diagram



in which all the maps from left to right are induced by p, the vertical maps send configurations to their equivalence classes, and the maps from front to back are inclusions. The upper maps from front to back are cofibrations by our condition on the section o, and the left and right faces are homotopy pushout squares. Thus if p induces weak homotopy equivalences on the two upper maps and the front map, it also will induce a weak homotopy equivalence on the lower back map.

By the previous proposition the projections $p : E_k(M;\pi) \to C_k(M;\pi)$ are weak homotopy equivalences for each k. Note that $E^0(M, M_0;\pi) = * = C^0(M, M_0;\pi)$. Thus, if p induces weak homotopy equivalences $B_{E_k}(M, M_0;\pi) \to B_{C_k}(M, M_0;\pi)$ for each k we can proceed by induction on k to show that $E^k(M, M_0;\pi) \to C^k(M, M_0;\pi)$ is a weak homotopy equivalence and hence obtain the result.

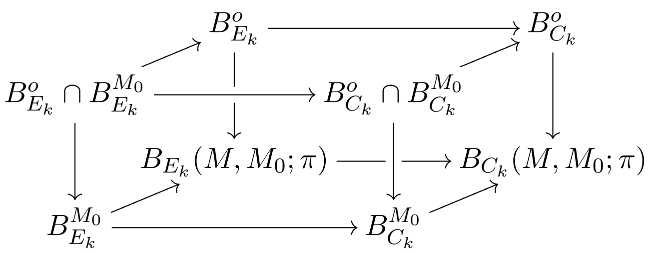
To see that $B_{E_k}(M, M_0;\pi) \to B_{C_k}(M, M_0;\pi)$ is a homotopy equivalence, define subspaces

$$B^o_{C_k} := \{(\boldsymbol{m};\boldsymbol{x}) : x_k = o|_{m_k}\},\ B^{M_0}_{C_k} := \{(\boldsymbol{m};\boldsymbol{x}) : m_k \in M_0\} \subset B_{C_k}(M, M_0;\pi)$$

and

$$B^o_{E_k} := \{(f;s) : s_k = o|_{\mathrm{im}(f)}\},\ B^{M_0}_{E_k} := \{(f;s) : f_k(0) \in M_0\} \subset B_{E_k}(M, M_0;\pi).$$

There is a commutative diagram



where the maps from left to right are induced by p and the left and right faces are homotopy pushout squares. Note that $B^o_{E_k} \subset E_k(M;\pi)|_{B^o_{C_k}}$ are subbundles of $E_k(M;\pi) \to E_k(M)$ with fibres $\Gamma(X|_{\mathrm{im}(f)\setminus\mathrm{im}(f_k)})$ and $\{s \in \Gamma(X|_{\mathrm{im}(f)}) : s_k|_{f_k(0)} = o|_{f_k(0)}\}$ respectively over f. The fibres are homotopy equivalent so $B^o_{E_k} \to B^o_{C_k}$ is a weak homotopy equivalence. The front maps from left to right are weak homotopy equivalences since $B^{M_0}_{E_k} = E_k(M;\pi)|_{B^{M_0}_{C_k}}$ and $B^o_{E_k} \cap B^{M_0}_{E_k} = B^o_{E_k}|_{B^{M_0}_{C_k}}$ as bundles over $C_k(M;\pi)$. So $B_{E_k}(M, M_0;\pi) \to B_{C_k}(M, M_0;\pi)$ is a weak homotopy equivalence. □

Remark 2.12. If Y has the homotopy type of a CW-complex then the spaces $C_k(M;\pi)$ and $C(M, M_0;\pi)$, and their filtration subspaces and quotients have the homotopy types of CW-complexes [McD75] and hence the same holds for tubular configuration spaces. The map in the previous proposition is thus in fact a homotopy equivalence.

3 The scanning map

We define the scanning map for tubular configuration spaces, show that it induces a homotopy equivalence under the usual assumptions and that it is equivariant with respect to the group of those diffeomorphisms of M which can be lifted to an isomorphism of the bundle $\pi : Y \to M$.

3.1 Definition and homotopy equivalence

Let $M_0 \subset M \subset M^+ \subset W$ and (Y, π, o) be as defined in the previous section, and let $n = \dim(M)$. Denote by $\dot{T}W$ the fibrewise one point compactification of the tangent bundle of W and let $\dot{T}_z W$ denote the fibre of $\dot{T}W$ over a point $z \in W$. Let $\dot{\tau}_\pi := \dot{T}W \wedge_W Y$ be the fibrewise smash product of $\dot{T}W$ with the bundle Y, where the base-point in each fibre of Y is determined by the zero-section o. Then, if X denotes a typical fibre of π, $\dot{\tau}_\pi$ is a $\Sigma^n X$-bundle over W with a canonical section $* : W \to \dot{\tau}_\pi$. Let $\Gamma(W \setminus M_0, W \setminus M; \pi)$ denote the space of sections of $\dot{\tau}_\pi$ which are defined outside of M_0 and which agree with $*$ outside of M (and hence on ∂M). In this section we define a scanning map $\gamma : E(M, M_0; \pi) \to \Gamma(W \setminus M_0, W \setminus M; \pi)$.

We begin by constructing a sequence of maps $\gamma_k : E_k(M; \pi) \to \Gamma(W \setminus M_0, W \setminus M; \pi)$. Intuitively, our scanning map will send a tubular configuration $(f; s) \in E_k(M; \pi)$ to a section defined by s on the Y component and by a modification of f^{-1} over W on the $\dot{T}W$ component, sending points outside of $\mathrm{im}(f)$ to the compactification points in the appropriate fibres.

Lemma 3.1. *[Lee03] Let V be a finite dimensional vector space. Given a vector $v \in V$, there is a canonical linear isomorphism $V \to T_v V$. Moreover, for any finite dimensional vector space U and any linear map $L : V \to U$ the following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_v V \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle d_v L} \\ W & \xrightarrow[\cong]{} & T_{Lv} W \end{array}$$

□

For any $(f, s) \in E_k(M; \pi)$ and $z \in \mathrm{im}(f)$, $f^{-1}(z) \in T_{m_i} M$ for some i. For each z and each f let $\phi_{f,z} : T_{m_i} M \to T_{f^{-1}(z)}(T_{m_i} M)$ be the canonical isomorphism. $\phi_{f,z}$ varies continuously in f and z. Let $(f, s) \in E_k(M; \pi)$. Having removed any components (f_i, s_i) with $m_i \in M_0$ if necessary, we define a section of $\dot{\tau}_\pi$ by

$$z \mapsto \begin{cases} \big(d_{f^{-1}(z)} f \circ \phi_{f,z} \circ f^{-1}(z), s(z)\big) & \text{if } z \in \mathrm{im}(f) \cap (M^+ \setminus M_0) \\ *_z & \text{otherwise.} \end{cases}$$

Although the above construction gives a well-defined map, it is not continuous. The problem is that the contribution of f_i in the section of $\dot{\tau}_\pi$ suddenly vanishes as m_i reaches M_0. We introduce the following modification of the above map for components (f_i, s_i) with m_i in a collar neighbourhood of ∂M_0 in M which makes sure that for a fixed $z \in \mathrm{im}(f_i)$ its image under the section goes to the point at infinity in $\dot{T}_z W$ as m_i gets close to the boundary of M_0 in M. To this end we fix a collar and choose a metric on W such that the chosen collar has width 1. Let $\delta(m_i)$ be the distance of m_i to M_0. We now multiply the above formula by $\exp(\frac{1}{\delta(m_i)})$ when $z \in \mathrm{im}(f_i)$ and m_i is in the collar. Note that we only use the metric on the collar. The resulting map γ^+ is now continuous.

Note that the target of γ^+ is a section that may not vanish outside M as our tubular discs may have image in the larger M^+. Let $\rho : \Gamma(W \setminus M_0, W \setminus M^+; \pi) \to \Gamma(W \setminus M_0, W \setminus M; \pi)$ be the homotopy equivalence induced by a diffeomorphism of W that maps $\partial M \cup [0, 1/2)$ into a collar of ∂M in M and leaves M away from the collar pointwise fixed. A homotopy inverse is given by the inclusion $\Gamma(W \setminus M_0, W \setminus M; \pi) \to \Gamma(W \setminus M_0, W \setminus M^+; \pi)$ Define

$$\gamma := \rho \circ \gamma^+ : E(M, M_0; \pi) \to \Gamma(W \setminus M_0, W \setminus M; \pi).$$

Remark 3.2. The composition

$$C(M, M_0; \pi) \xrightarrow{\sigma} E(M, M_0; \pi) \xrightarrow{\gamma} \Gamma(W \setminus M_0, W \setminus M; \pi)$$

is homotopic to the scanning map of McDuff [McD75] when $\pi : Y \to M$ is a trivial X-bundle. If we compose with a reflection $(t_1, \dots, t_n; x) \mapsto (-t_1, \dots, -t_{n-1}, -t_n; x)$ in each fibre of $\dot{\tau}_\pi$ it is homotopic to the scanning map of Bödigheimer and Madsen [BM88] when π is trivial and the extension due to Hesselholt [Hes92] for arbitrary π.

Theorem 3.3. *The scanning map $\gamma : E(M, M_0; \pi) \to \Gamma(W \setminus M_0, W \setminus M; \pi)$ is a weak homotopy equivalence if the pair (M, M_0) or the typical fibre X of π is 0-connected. Moreover, if X has the homotopy type of a CW-complex then it is a homotopy equivalence.*

Proof. By Remark 3.2 we have a homotopy commutative diagram

$$\begin{array}{ccc} E(M, M_0; \pi) & \xrightarrow{\gamma} & \Gamma(W \setminus M_0, W \setminus M; \pi) \\ \simeq \uparrow & \nearrow \simeq & \\ C(M, M_0; \pi) & & \end{array}$$

factoring the ordinary scanning map, which is a weak homotopy equivalence; see [Hes92] for the case with twisted coefficients. By proposition 2.11, the left hand map is a weak homotopy equivalence. Hence, γ is one too. If Y has the homotopy type of CW-complex so do all the three spaces in the diagram and hence the weak homotopy equivalences are homotopy equivalences. □

3.2 Equivariance

Let $\mathrm{Diff}(M, \partial M)$ be the group of diffeomorphisms of M which fix a collar of the boundary pointwise. There is an isomorphism $\mathrm{Diff}(M, \partial M) \to \mathrm{Diff}(W, W \setminus M)$ extending diffeomorphisms by the identity on $W \setminus M$. We will not distinguish between these groups and we understand the action of a diffeomorphism in $\mathrm{Diff}(M, \partial M)$ on W to mean the action of its extension over $W \setminus M$.

Now we define actions on configuration spaces as follows. Let $\phi \in \mathrm{Diff}(M, \partial M)$, then $\phi \cdot (m_1, \dots, m_k) := (\phi(m_1), \dots, \phi(m_k))$ for $(m_1, \dots, m_k) \in \widetilde{C}_k(M)$ or $C_k(M)$ and $\phi \cdot f := \phi \circ f \circ d\phi^{-1}$ for $f \in \widetilde{E}_k(M)$ or $E_k(M)$. With these actions the projections p and $\widetilde{p}$ are $\mathrm{Diff}(M, \partial M)$-equivariant non-equivariant homotopy equivalences.

This action of the diffeomorphism group extends to configuration spaces with labels when $Y = M \times X$ is a trivial bundle. However, in general diffeomorphisms of M do not lift to automorphisms of the labelling bundle Y. Instead we need to consider the automorphism group of Y and the induced diffeomorphisms on the base space M.

Let $\pi_1 : Y_1 \to M$ and $\pi_2 : Y_2 \to M$ be fibre bundles over M. We consider bundle isomorphisms $\alpha : Y_1 \to Y_2$ that induce diffeomorphisms α_M on the base and denote isomorphic bundles by $Y_1 \cong Y_2$. If the isomorphism induces the identity on the base we call it an equivalence and denote equivalent bundles by $Y_1 \equiv Y_2$.

Lemma 3.4. *Let* $\mathrm{Aut}(\pi)$ *denote the group of automorphisms of* $\pi : Y \to M$ *that restrict to diffeomorphisms of* M*. Then the homomorphism* $\mathrm{Aut}(\pi) \to \mathrm{Diff}(M)$*,* $\alpha \mapsto \alpha_M$ *has image*

$$\mathrm{Diff}(M;\pi) := \{\beta \in \mathrm{Diff}(M) : (\beta_M)^*Y \equiv Y\}.$$

Proof. Let $\beta \in \mathrm{Diff}(M;\pi)$ and $\tilde{\beta} : \beta^*Y \to Y$ be the map completing the pullback square. Let $f : Y \to \beta^*Y$ be a bundle equivalence, then $\widetilde{\beta \circ f}$ is an automorphism of Y whose image under ρ is β. Conversely let $\alpha \in \mathrm{Aut}(\pi)$, and let $\widetilde{(\alpha_M)} : (\alpha_M)^*Y \to Y$ be the map completing the pullback square, then $\alpha^{-1} \circ \widetilde{(\alpha_M)} : (\alpha_M)^*Y \to Y$ is an equivalence. □

Remark 3.5. In general, $\mathrm{Diff}(M;\pi) \subsetneq \mathrm{Diff}(M)$. For example, consider the Hopf bundle over S^2 and the antipodal map on the base. The Chern class of the Hopf bundle is $+1$ but the Chern class of the pullback is -1 so they cannot be equivalent. However, when Y is trivial or some other natural bundle such as a tangent bundle, $\mathrm{Diff}(M;\pi) = \mathrm{Diff}(M)$ and there is a lift $\mathrm{Diff}(M) \to \mathrm{Aut}(\pi)$. For any bundle, $\mathrm{Diff}(M;\pi)$ contains $\mathrm{Diff}_0(M)$, the connected component of the identity, since homotopic maps induce equivalent pullbacks.

Let $\mathrm{Aut}^o(M, M_0 \cup \partial M; \pi) \subset \mathrm{Aut}(\pi)$ denote the subgroup of bundle automorphisms of Y which preserve the zero section and restrict to diffeomorphisms in $\mathrm{Diff}(M, M_0 \cup \partial M)$. Here we assume as is standard that the diffeomorphisms fix a collar in M of $M_0 \cup \partial M$. Let $\alpha \in \mathrm{Aut}^o(M, M_0 \cup \partial M; \pi)$, then we define actions as follows:

- $\alpha \cdot (\boldsymbol{m}; \boldsymbol{x}) := (\alpha_M(\boldsymbol{m}); \alpha \circ \boldsymbol{x} \circ \alpha_M^{-1})$ for $(\boldsymbol{m}; \boldsymbol{x}) \in C_k(M;\pi)$
- $\alpha \cdot (f; s) := (\alpha_M \circ f \circ d\alpha_M^{-1}; \alpha \circ s \circ \alpha_M^{-1})$ for $(f; s) \in E_k(M;\pi)$
- $\alpha \cdot \sigma := (d\alpha_M \wedge_W \alpha) \circ \sigma \circ \alpha_M^{-1}$ for $\sigma \in \Gamma(W \smallsetminus M_0, W \smallsetminus M^+; \pi)$.

The action on an equivalence class in $C(M, M_0; \pi)$ or $E(M, M_0; \pi)$ is given by the equivalence class of the image of any representative under the action. This is well-defined as α preserves the zero section.

Proposition 3.6. *The projection* $p : E(M, M_0; \pi) \to C(M, M_0; \pi)$ *is* $\mathrm{Aut}^o(M, M_0 \cup \partial M; \pi)$*-equivariant.*

Proof. It follows immediately from the definitions that the projections $E_k(M;\pi) \to C_k(M;\pi)$ are equivariant. We obtain the result by noting that the actions are well behaved with respect to equivalence classes. □

Remark 3.7. Although the projection is equivariant, the homotopy inverse $\sigma : C(M, M_0; \pi) \to E(M, M_0; \pi)$ defined by the exponential map and open ε-disks around each particle depends on choosing a metric and is not preserved under diffeomorphism.

Theorem 3.8. *The scanning map* $\gamma : E(M, M_0; \pi) \to \Gamma(W \smallsetminus M_0, W \smallsetminus M; \pi)$ *is* $\mathrm{Aut}^o(M, M_0 \cup \partial M; \pi)$*-equivariant.*

Proof. Consider the scanning map $\gamma_k^+ : E_k(M;\pi) \to \Gamma(W \smallsetminus M_0, W \smallsetminus M^+; \pi)$ for some $k \geq 0$. Choose a point $z \in W \smallsetminus M_0$, an automorphism $\alpha \in \mathrm{Aut}^o(M, M_0 \cup \partial M; \pi)$ and a tubular configuration $(f;s)$ with underlying configuration $(\boldsymbol{m};\boldsymbol{x})$. We want to show

$$(\alpha \cdot \gamma_k^+(f;s))(z) = \gamma_k^+(\alpha \cdot (f;s))(z).$$

Suppose $\alpha_M^{-1}(z) \in \mathrm{im}(f_i)$ for some i. By the definition of the action on $\Gamma(W \setminus M_0, W \setminus M^+;\pi)$ we have

$$(\alpha \cdot \gamma_k^+(f;s))(z) = \Big(d_{\alpha_M^{-1}(z)}\alpha_M \circ d_{f_i^{-1}\circ\alpha_M^{-1}(z)} f_i \circ \phi_{f,\alpha_M^{-1}(z)} \circ f_i^{-1} \circ \alpha_M^{-1}(z), \alpha \circ s \circ \alpha_M^{-1}(z)\Big).$$

We calculate $\gamma_k^+(\alpha \cdot (f;s))(z)$ componentwise:

(i) $(\alpha \cdot f_i)^{-1} = d_{m_i}\alpha_M \circ f_i^{-1} \circ \alpha_M^{-1}$

(ii) by Lemma 3.1 there is a commutative diagram

$$\begin{array}{ccc}
T_{m_i}M & \xrightarrow{\ \phi_{f,\alpha_M^{-1}(z)}\ } & T_{f_i^{-1}\circ\alpha_M^{-1}(z)}T_{m_i}M \\
{\scriptstyle d_{m_i}\alpha_M}\big\downarrow & & \big\downarrow{\scriptstyle d_{f_i^{-1}\circ\alpha_M^{-1}(z)}d_{m_i}\alpha_M} \\
T_{\alpha_M(m_i)}M & \xrightarrow[\ \phi_{\alpha\cdot f,z}\ ]{} & T_{(\alpha\cdot f_i)^{-1}(z)}T_{\alpha_M(m_i)}M
\end{array}$$

so we have $\phi_{\alpha\cdot f,z} = d_{f_i^{-1}\circ\alpha_M^{-1}(z)}d_{m_i}\alpha_M \circ \phi_{f,\alpha_M^{-1}(z)} \circ (d_{m_i}\alpha_M)^{-1}$

(iii) $d_{(\alpha\cdot f_i)^{-1}(z)}(\alpha \cdot f_i) = d_{\alpha_M^{-1}(z)}\alpha_M \circ d_{f_i^{-1}\circ\alpha_M^{-1}(z)} f_i \circ \Big(d_{f_i^{-1}\circ\alpha_M^{-1}(z)}d_{m_i}\alpha_M\Big)^{-1}$.

Putting these together we have

$$\gamma_k(\alpha \cdot (f;s))(z) = \Big(d_{\alpha_M^{-1}(z)}\alpha_M \circ d_{f_i^{-1}\circ\alpha_M^{-1}(z)} f_i \circ \phi_{f,\alpha_M^{-1}(z)} \circ f_i^{-1} \circ \alpha_M^{-1}(z), \alpha \circ s \circ \alpha_M^{-1}(z)\Big).$$

In our computations above we have implicitly assumed that none of the factors of $(f;s)$ have midpoint in the collar of ∂M_0 in M. We leave it to the reader to check that the multiplicative factor $\exp(\frac{1}{\delta(m_i)})$ does not interfere with the equivariance argument as our diffeomorphisms fix the collar by assumption.

The homotopy $\Gamma(W \setminus M_0, W \setminus M^+;\pi) \to \Gamma(W \setminus M_0, W \setminus M;\pi)$ is invariant under α, again since automorphisms in $\mathrm{Aut}^o(M, M_0 \cup \partial M;\pi)$ fix the chosen collar. To complete the proof recall that the scanning map γ is defined by $\rho \circ \gamma_k^+$ for any representative of a class and the actions respect the equivalence relation. □

Combining Theorems 3.8 and 3.3 gives the following corollary.

Corollary 3.9. *Let (M, M_0) or a typical fibre X of π be 0-connected. If a topological group G acts on M and π via a homomorphism $G \to \mathrm{Aut}^o(M, M_0 \cup \partial M;\pi)$, then there are weak homotopy equivalences of the reduced Borel constructions*

$$EG \ltimes_G C(M, M_0;\pi) \simeq EG \ltimes_G E(M, M_0;\pi) \simeq EG \ltimes_G \Gamma(W \setminus M_0, W \setminus M;\pi)$$

which are homotopy equivalences if Y has the homotopy type of a CW-complex.

Remark 3.10. Let $Y = M \times X$ be a trivial bundle and let G be a compact Lie group acting on M through isometries and on X fixing the base-point. Then the ordinary scanning map

$$\gamma \circ \sigma : C(M, M_0; X) \longrightarrow \Gamma(W \setminus M_0, W \setminus M; X)$$

is G-equivariant.

4 Stable splittings

In this section we construct Snaith type splittings via diffeomorphism equivariant maps. As a result we generalise and strengthen the results of Bödigheimer and Madsen [BM88]. As an application we also derive an equivariant stable split injection for configuration spaces and diffeomorphisms of manifolds with marked points. This generalises some results in [BT01].

Constructions and assumptions: Let $M_0 \subset M \subset M^+ \subset W$ and $\pi : Y \to X$ be as before. For each $k \geq 1$ let $C^k = C^k(M, M_0; \pi)$ be the filtrations of $C = C(M, M_0; \pi)$. For each $k \geq 1$ let $D^k = D^k(M, M_0; \pi) := C^k/C^{k-1}$ be the filtration quotient and let $V := \bigvee_{i\geq 1} D^i$ be the wedge sum with kth filtration $V^k := \bigvee_{i=1}^k D^i$.

Throughout subsections 4.1-4.4 we will assume that (M, M_0) or the typical fibre X of π is 0-connected. Then also C, C^k, D^k, V, V^k are all path connected as well. We also note that if X has the homotopy type of a CW-complex then so do all these spaces.

Throughout this section we let G be a topological group which acts on M and π via a homomorphism $G \to \mathrm{Aut}^o(M, M_0 \cup \partial M; \pi)$. We will suppress this homomorphism in our notation. Note that the G action on C induces an action on C^k, D^k and hence on V^k.

4.1 Equivariant splittings in homology

Before we construct equivariant stable splittings for mapping spaces we prove the following weaker statement in homology in virtue of the simplicity of its proof.

Theorem 4.1. *There exist a $\pi_0 G$-equivariant isomorphism*

$$\widetilde{H}_*(C(M, M_0; \pi)) \xrightarrow{\cong} \bigoplus_{k\geq 1} \widetilde{H}_*(D^k).$$

Proof. Given a pointed space A we denote the infinite symmetric product of A by $\mathrm{SP}(A)$. Elements of $\mathrm{SP}(A)$ are finite formal sums $\sum k_i a_i$ where $a_i \in A$ and $k_i \in \mathbb{N}$ for each i with the one relation that makes the base point the zero of the monoid, see [DT58]. Let ξ_α be a configuration in C and write it as a formal sum $\xi_\alpha = \sum_{i\in\alpha} m_i x_i$ where α is a finite set. Given a subset $\beta \subseteq \alpha$ of size k there is a subconfiguration $\xi_\beta := \sum_{i\in\beta} m_i x_i \in C^k$. Let $\bar{\xi}_\beta$ denote its image under the composition $C^k \to D^k \hookrightarrow V$. Define a power set map

$$\sigma : C \longrightarrow \mathrm{SP}(V)$$
$$\xi_\alpha \longmapsto \sum_{\beta\subseteq\alpha} \xi_\beta$$

and extend it to a map $\sigma : \mathrm{SP}(C) \to \mathrm{SP}(V)$ by $\sum k_\alpha \xi_\alpha \mapsto \sum k_\alpha \sigma(\xi_\alpha)$. This is a pointed G-map that respects the filtrations and restricts to a pointed G-map $\mathrm{SP}(C^k) \to \mathrm{SP}(V^k)$.

The infinite symmetric product functor sends cofibrations to quasifibrations so for each $k \geq 1$

we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{SP}(C^{k-1}) & \xrightarrow{\sigma} & \mathrm{SP}(V^{k-1}) \\
\downarrow & & \downarrow \\
\mathrm{SP}(C^{k}) & \xrightarrow{\sigma} & \mathrm{SP}(V^{k}) \\
\downarrow & & \downarrow \\
\mathrm{SP}(C^{k}/C^{k-1}) & \xrightarrow{\sigma=\mathrm{id}} & \mathrm{SP}(V^{k}/V^{k-1})
\end{array}$$

in which the vertical sequences are quasifibrations and the lower horizontal map is the identity $\mathrm{SP}(D^k) \to \mathrm{SP}(D^k)$. Starting with $C^1 = V^1$ we proceed by induction on k to see that σ induces isomorphisms

$$\pi_* \mathrm{SP}(C(M, M_0; \pi)) \xrightarrow{\cong} \pi_* \mathrm{SP}(\bigvee_{k \geq 1} D^k).$$

By the Dold-Thom theorem [DT58] this is precisely the homology isomorphism we require. To finish the proof note that the connected component of the identity $G_0 \subset G$ acts trivially on homology so the isomorphism is equivariant under the actions of $\pi_0 G = G/G_0$. □

Combining this with the results in section 3 we have proved the following corollary.

Corollary 4.2. *There exists a $\pi_0 G$-equivariant isomorphism*

$$\widetilde{H}_*(\Gamma(W \smallsetminus M_0, W \smallsetminus M; \pi)) \xrightarrow{\cong} \bigoplus_{k \geq 1} \widetilde{H}_*(D^k). \quad \square$$

The above proof serves as a model for the proofs of the stronger theorems in the following sections. The main difficulty in each case is to construct a suitable G-equivariant power set map.

4.2 Stable splittings of reduced Borel constructions

Bödigheimer and Madsen's power set map for Borel constructions in [BM88] relies crucially on the fact that the groups they consider are compact of Lie type. In particular they use that one can embed the configuration space $C(M)$ G-equivariantly into a finite dimensional G-representation. Instead we use here suitably chosen models for EG which allow us to define the power map on Borel constructions for any G acting on M smoothly and on π. This provides a geometric model of the power set map.

Theorem 4.3. *There is a weak homotopy equivalence of suspension spectra*

$$\Sigma^\infty (EG \ltimes_G C(M, M_0; \pi)) \longrightarrow \Sigma^\infty \bigvee_{k \geq 1} EG \ltimes_G D^k.$$

Proof. As a consequence of Whitney's embedding theorem, the embedding space $\mathrm{Emb}(M; \mathbb{R}^\infty)$ is contractible. It admits a free $\mathrm{Diff}(M)$-action. Choose a model for EG and replace it with $EG \times \mathrm{Emb}(M; \mathbb{R}^\infty)$ equipped with the diagonal action where $g \cdot h := h \circ (g^{-1})$ for $g \in G$ and an embedding $h : M \to \mathbb{R}^\infty$. This is again a model for EG and we denote elements of this EG by pairs (u, h).

Given an embedding $h : M \to \mathbb{R}^\infty$ we define an associated embedding $\widehat{h} : \coprod C_k(M) \to \mathbb{R}^\infty$ as follows. For each $k \geq 1$ and any N, $C_k(\mathbb{R}^N)$ is a finite dimensional, smooth manifold. Thus there is an embedding $C_k(\mathbb{R}^N) \hookrightarrow \mathbb{R}^{L_N}$ for some L_N. Considering $C_k(\mathbb{R}^N)$ as a submanifold of $C_k(\mathbb{R}^{N+1})$ we can extend this embedding to an embedding of $C_k(\mathbb{R}^{N+1}) \hookrightarrow \mathbb{R}^{L_{N+1}}$ for some $L_{N+1} \geq L_N$. Proceeding like this, in the limit we will have constructed an embedding

$$\mu_k : C_k(\mathbb{R}^\infty) \hookrightarrow \mathbb{R}^\infty.$$

Choosing an injection $\coprod_{k\geq 1} \mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$ we can construct from μ_k an injection

$$\mu : \coprod_{k\geq 1} C_k(\mathbb{R}^\infty) \hookrightarrow \mathbb{R}^\infty.$$

Given now any embedding $h : M \hookrightarrow \mathbb{R}^\infty$, the embedding $\widehat{h}$ is defined as the composition

$$\widehat{h} : \coprod_k C_k(M) \xrightarrow{C_k h} \coprod_k C_k(\mathbb{R}^\infty) \xrightarrow{\mu} \mathbb{R}^\infty.$$

The action of G on M cannot be extended to an action on $\mathbb{R}^\infty$, and in particular $\widehat{h}$ cannot be made G-equivariant. Nevertheless, the construction is G-equivariant in the sense that

$$g.\widehat{h} = \widehat{h} \circ g^{-1} = \widehat{h \circ g^{-1}} = \widehat{g.h}.$$

We can now define the power-set map in this setting. Choose a configuration $\xi_\alpha = \sum_{i\in\alpha} m_i x_i \in C$ and define subconfigurations $\xi_\beta \in C^k$ and $\bar{\xi}_\beta \in V$ as in the previous section. Let $m_\beta := \sum_{i\in\beta} m_i \in C_k(M)$ be the associated unlabelled subconfiguration and define

$$\begin{aligned} \sigma : EG \ltimes_G C(M, M_0; \pi) &\longrightarrow C(\mathbb{R}^\infty; EG \ltimes_G V) \\ (u, h, \xi_\alpha) &\longmapsto \sum_{\beta\subseteq\alpha} \widehat{h}(m_\beta)(u, h, \bar{\xi}_\beta) \end{aligned}$$

where the target is the configuration space of unordered particles in $\mathbb{R}^\infty$ with labels in the trivial $EG \ltimes_G V$-bundle. The formula for σ gives a well defined map $EG \times C \to C(\mathbb{R}^\infty; EG \times V)$. To see that it is well defined on the half smash product note that the base-point in C is the empty configuration $\xi_\emptyset$ which has no subconfigurations so σ maps $(u, h, \xi_\emptyset)$ to the empty configuration in the target for any $(u, h) \in EG$. To check that σ is G-equivariant recall that $g \cdot \xi_\alpha = \sum g(m_i)\big(g \circ x_i \circ g^{-1}\big)$. The G-action on the target is trivial on $\mathbb{R}^\infty$ and given by the diagonal action on the labels. Hence,

$$\begin{aligned} \sigma(g \cdot (u, h, \xi_\alpha)) &= \sum_{\beta\subseteq\alpha} \widehat{h \circ g^{-1}}(\sum_{i\in\beta} g(m_i))\big(g \cdot u, h \circ g^{-1}, \overline{g \cdot \xi_\beta}\big) \\ g \cdot \sigma(u, h, \xi_\alpha) &= \sum_{\beta\subseteq\alpha} \widehat{h}(m_\beta)(g \cdot u, h \circ g^{-1}, g \cdot \bar{\xi}_\beta) \end{aligned}$$

By definition $g \cdot \bar{\xi}_\beta = \overline{g \cdot \xi_\beta}$ and

$$\widehat{h \circ g^{-1}}(\sum_{i\in\beta} g(m_i)) = \mu(\sum_{i\in\beta} h \circ g^{-1} \circ g(m_i)) = \mu(\sum_{i\in\beta} h(m_i)) = \widehat{h}(m_\beta).$$

So σ is well defined.

Composing the power set map with the scanning map yields the map

$$EG \ltimes_G C \xrightarrow{\sigma} C(\mathbb{R}^\infty; EG \ltimes_G V) \longrightarrow \Omega^\infty\Sigma^\infty(EG \ltimes_G V)$$

which respects the filtrations of C and V. The cofibration sequences of the filtrations of C and V induce fibration sequences of suspension spectra which fit into commutative diagrams

$$\begin{array}{ccc}
\Sigma^\infty\big(EG \ltimes_G C^{k-1}\big) & \xrightarrow{\quad p \quad} & \Sigma^\infty\big(EG \ltimes V^{k-1}\big) \\
\downarrow & & \downarrow \\
\Sigma^\infty\big(EG \ltimes_G C^{k}\big) & \xrightarrow{\quad p \quad} & \Sigma^\infty\big(EG \ltimes V^{k}\big) \\
\downarrow & & \downarrow \\
\Sigma^\infty\big(EG \ltimes_G C^{k}/C^{k-1}\big) & \xrightarrow{p\simeq \mathrm{id}} & \Sigma^\infty\big(EG \ltimes V^{k}/V^{k-1}\big)
\end{array}$$

for each $k \geq 1$ and where p is the adjoint of the composition above. The lower horizontal map is homotopic to the identity since it is the adjoint of the inclusion $EG \ltimes_G D^k \hookrightarrow \Omega^\infty\Sigma^\infty\big(EG \ltimes_G D^k\big)$. Starting with $C^1 = V^1$ we proceed by induction on k to obtain a weak homotopy equivalence of suspension spectra and the result follows from the observation that $EG \ltimes_G \bigvee_{k\geq 1} D^k = \bigvee_{k\geq 1} EG \ltimes_G D^k$. □

4.3 Equivariant stable splittings

The purpose of this section is to construct a G-equivariant power-set map which induces stable G-equivariant splittings for configuration spaces. It will be completely natural and avoid having to choose an embedding μ as in the proof of Theorem 4.3. It also gives a stronger result.

While the stable splitting in the previous section made use of the configuration space $C(\mathbb{R}^\infty; A)$ as a model for the free infinite loop space $Q(A) := \varinjlim \Omega^n\Sigma^n A$, here we will make use of the Γ^+-construction of Barratt and Eccles. We will also need to replace the Σ_k-orbits of $\widetilde{C}_k(M;\pi)$ by the homotopy Σ_k-orbits. Recall the following constructions from [BE74a].

For a discrete group H, let $W.H$ denote its homogeneous bar construction and let $EH := |W.H|$ be its geometric realisation. Group homomorphisms $H \to K$ induce continuous maps $EH \to EK$, and in particular the inclusion $\Sigma_{k-1} \hookrightarrow \Sigma_k$ induces an inclusion map $E\Sigma_{k-1} \hookrightarrow E\Sigma_k$ with a right inverse induced by the reduction map $R : \Sigma_k \to \Sigma_{k-1}$ as defined in [BE74a]. Given a well-pointed space A, the Γ^+ construction on A is defined as

$$\Gamma^+(A) := \left(\coprod_{k\geq 0} E\Sigma_k \times_{\Sigma_k} A^k\right) \Big/ \sim$$

where $(w; a_1, \ldots, a_k) \sim (R(w); a_1, \ldots, a_{k-1})$ if $a_k = *$. Γ^+ is an endofunctor on the category of well-pointed topological spaces. For any space A there is an inclusion map $\iota_A : A \hookrightarrow \Gamma^+(A)$ identifying A with $\Sigma_1 \times_{\Sigma_1} A \subset \Gamma^+(A)$ and a structure map $h_A : \Gamma^+\Gamma^+(A) \to \Gamma^+(A)$ making the triple (Γ^+, ι, h) into a monad. The spaces $\Gamma^+(A)$ are the free Γ^+-algebras.

We define the Borel configuration space of k unordered particles in M with twisted labels in π as

$$\mathscr{C}_k(M;\pi) := E\Sigma_k \times_{\Sigma_k} \widetilde{C}_k(M;\pi).$$

The Borel configuration space with particles vanishing in M_0 is then

$$\mathscr{C}(M, M_0; \pi) := \left(\coprod_{k \geq 0} \mathscr{C}_k(M; \pi) \right) \Big/ \sim$$

where $(w; m_1, \ldots, m_k; x_1, \ldots, x_k) \sim (R(w); m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$ if $m_k \in M_0$ or $x_k = o|_{m_k}$. The Borel configuration spaces admit filtrations $\mathscr{C}^k := \coprod_{i=0}^k \mathscr{C}_k / \sim$ analogous to the ordinary configuration spaces. We make use of the filtrations quotients $\mathscr{D}^k := \mathscr{C}^k / \mathscr{C}^{k-1}$. Consider the sequence of spaces $\widetilde{D}^k = \widetilde{C}^k(M; \pi) / \equiv$ where $(m_1, \ldots, m_k; x_1, \ldots, x_k) \equiv *$ if $m_i \in M_0$ or $x_i = o|_{m_i}$ for some i. These spaces are the filtration quotients of the ordered configuration space with vanishing particles. The filtration quotients of the Borel configuration spaces can then be identified as $\mathscr{D}^k = E\Sigma_k \ltimes_{\Sigma_k} \widetilde{D}^k$.

Proposition 4.4. *The projection $P : \mathscr{C}(M, M_0; \pi) \to C(M, M_0; \pi)$ and the restriction to the filtration quotients $P : \mathscr{D}^k \to D^k$ are weak homotopy equivalences.*

Proof. The projection $P : E\Sigma_k \times \widetilde{C}_k(M; \pi) \to \widetilde{C}_k(M; \pi)$ is a Σ_k-equivariant homotopy equivalence. As Σ_k acts freely on the source and target, P induces a homotopy equivalence on orbit spaces $\mathscr{C}_k(M; \pi) \xrightarrow{\simeq} C_k(M; \pi)$. Together they give a well defined map $P : \mathscr{C}(M, M_0; \pi) \to C(M, M_0; \pi)$ of filtered spaces.

Let $\widetilde{B}_{C_k}(M, M_0; \pi) \subset \widetilde{C}_k(M; \pi)$ be the subspace of ordered configurations such that for some i either $m_i \in M_0$ or $x_i = o|_{m_i}$. Recall from the proof of Proposition 2.11 that the filtration $C^k(M, M_0; \pi)$ is obtained as the pushout of the diagram $C^{k-1}(M, M_0; \pi) \leftarrow B_{C_k}(M, M_0; \pi) \to C_k(M; \pi)$ where $B_{C_k}(M, M_0; \pi) = \widetilde{B}_{C_k}(M, M_0; \pi)/\Sigma_k$. Similarly the filtrations of the Borel configuration spaces are obtained from the previous filtration as the pushout along a subspace $B_{\mathscr{C}_k}(M, M_0; \pi) := E\Sigma_k \times_{\Sigma_k} \widetilde{B}_{C_k}(M, M_0; \pi)$. The projection π induces a homotopy equivalence $B_{\mathscr{C}_k}(M, M_0; \pi) \xrightarrow{\simeq} B_{C_k}(M, M_0; \pi)$ and a map of homotopy pushout squares. The result then follows by induction as in the proof of Proposition 2.11. The result for the filtration quotients is proved similarly. ☐

Remark 4.5. The maps in the previous proposition are G-equivariant.

Proposition 4.6. *There is a G-equivariant power-set map $\sigma : \mathscr{C}(M, M_0; \pi) \to \Gamma^+(V)$ of filtered spaces.*

Proof. Fix $k \geq 0$. Let α be an ordered set of cardinality k, for each $0 \leq i \leq k$ let $\mathscr{P}_i(\alpha)$ be the set of (unordered) subsets of α of size i. As α is ordered $\mathscr{P}_i(\alpha)$ has an induced lexicographical ordering and we may list its elements as: $\beta_1, \ldots, \beta_{\binom{k}{i}}$.

Let $\xi_\alpha = \sum_{j \in \alpha} m_j x_j$ be an ordered configuration in $\widetilde{C}_k(M; \pi)$. For each $\beta \in \mathscr{P}_i(\alpha)$ let $\xi_\beta = \sum_{i \in \beta} m_j x_j \in C_i(M; \pi)$ be an unordered subconfiguration of ξ_α and let $\bar{\xi}_\beta$ be its image under the composition $C_k \to C^k \to D^k \to V$. For each i define a map

$$\begin{aligned} \widetilde{C}_k(M; \pi) &\longrightarrow V^{\binom{k}{i}} \\ \xi_\alpha &\longmapsto \left(\bar{\xi}_{\beta_1}, \ldots, \bar{\xi}_{\beta_{\binom{k}{i}}} \right). \end{aligned}$$

Σ_k naturally acts on $\mathscr{P}_i(\alpha)$ since permuting the elements in α maps a subset of size i to another subset of size i. This action defines a homomorphism $\Sigma_k \to \Sigma_{\binom{k}{i}}$. With this action, the above map is Σ_k-equivariant. For each k the direct sum of these homomorphisms defines a homomorphism $\phi_k : \Sigma_k \to \Sigma_{\binom{k}{0}} \oplus \Sigma_{\binom{k}{1}} \oplus \cdots \oplus \Sigma_{\binom{k}{k}} \subset \Sigma_{2^k}$. Together with the maps $\widetilde{C}_k(M;\pi) \to V^{\binom{k}{i}}$ this defines

$$\sigma_k : \mathscr{C}_k(M;\pi) = E\Sigma_k \times_{\Sigma_k} \widetilde{C}_k(M;\pi) \longrightarrow E\Sigma_{2^k} \times_{\Sigma_{2^k}} V^{2^k}.$$

We will now show that the σ_k's fit together to define a map $\sigma : \mathscr{C}(M;\pi) \to \Gamma^+(V)$. To see that σ respects the equivalence relations on each side, choose two equivalent Borel configurations. We may assume they are of the form $(w; m_1, \ldots, m_k; x_1, \ldots, x_k)$ and its reduction $(R(w); m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$. Since $x_k = o|_{m_k}$ or $m_k \in M_0$, any subconfiguration containing $(m_k; x_k)$ is mapped to the base-point in the appropriate copy of $V \subset V^{2^k}$. The defining relation of Γ^+ identifies a point in V^n with one in V^{n-1} if a coordinate lies in the base-point of V. Precisely 2^{k-1} subconfigurations contain the particle $(m_k; x_k)$. So noting that the square

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{\phi_k} & \Sigma_{2^k} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R^{2^{k-1}}} \\ \Sigma_{k-1} & \xrightarrow[\phi_{k-1}]{} & \Sigma_{2^{k-1}} \end{array}$$

commutes up to an isomorphism $\Sigma_{2k} \to \Sigma_{2k}$ and iterating the relation 2^{k-1} times we have

$$\sigma_k(w; m_1, \ldots, m_k; x_1, \ldots, x_k) \sim \sigma_{k-1}(R(w); m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}).$$

To see that σ is a map of filtered spaces, note that by definition σ maps $\mathscr{C}^k$ to $\Gamma^+(V_k) \subset \Gamma^+(V)$ where $V_k := \bigvee_{i=1}^k D^k$. Finally, the construction is natural with respect to any inclusions of the underlying fibration π into another fibration, and in particular it is G-equivariant. □

Theorem 4.7. *There is a stable homotopy equivalence*

$$\mathscr{C}(M, M_0;\pi) \overset{s}{\simeq} \bigvee_{k\geq 1} D^k(M, M_0;\pi)$$

induced by a G-equivariant map $\bar{\sigma} : \Gamma^+(\mathscr{C}) \to \Gamma^+(V)$.

Proof. Let $\bar{\sigma}$ denote the composition $\Gamma^+(\mathscr{C}(M, M_0;\pi)) \xrightarrow{\Gamma^+\sigma} \Gamma^+\Gamma^+(V) \xrightarrow{h_V} \Gamma^+(V)$ and note that it restricts to a map of the filtrations and filtration quotients. $\bar{\sigma}$ is G-equivariant as both the Γ^+-construction on a map and the construction h_V are natural. We note now that the image of $\mathscr{D}^k$ under σ is contained in $E\Sigma_1 \times_{\Sigma_1} D^k = D^k$ and hence σ restricted to $\mathscr{D}^k$ factors as $\mathscr{D}^k \to D^k \xhookrightarrow{\iota_{D^k}} \Gamma^+(D^k)$ where the first map is the weak homotopy equivalence in Proposition 4.4. Thus on the filtration quotients $\bar{\sigma}$ factors as the composition

$$\Gamma^+(\mathscr{D}^k) \xrightarrow{\Gamma^+ P} \Gamma^+(D^k) \xrightarrow{\Gamma^+\iota_{D^k}} \Gamma^+\Gamma^+(D^k) \xrightarrow{h_{D^k}} \Gamma^+(D^k).$$

The first map is a homotopy equivalence since Γ^+ is a homotopy functor. Also, the composition $h_{D^k} \circ \Gamma^+\iota_{D^k}$ is the identity by virtue of Γ^+ being a monad. It follows that $\bar{\sigma} : \Gamma^+(\mathscr{D}^k) \to \Gamma^+(D^k)$ is a homotopy equivalence.

The Γ^+ construction converts cofibrations to fibrations so for each k we can form a commutative diagram in which the vertical sequences are fibrations.

$$\begin{array}{ccc}
\Gamma^+(\mathscr{C}^{k-1}) & \xrightarrow{\bar\sigma} & \Gamma^+(V^{k-1}) \\
\downarrow & & \downarrow \\
\Gamma^+(\mathscr{C}^{k}) & \xrightarrow{\bar\sigma} & \Gamma^+(V^{k}) \\
\downarrow & & \downarrow \\
\Gamma^+(\mathscr{D}^{k}) & \xrightarrow{\bar\sigma} & \Gamma^+(D^{k})
\end{array}$$

By the above discussion the bottom arrow is always a homotopy equivalence. Since $\mathscr{C}^1 = \mathscr{D}^1$ and $C^1 = D^1 = V^1$ we proceed by induction on k to obtain a homotopy equivalence $\Gamma^+(\mathscr{C}) \simeq \Gamma^+(V)$.

By the results of [BE74b], a connected (or group-like) Γ^+-algebra is an infinite loop space and a morphism of such Γ^+-algebras is a map of infinite loop spaces. Moreover, the free Γ^+-algebras correspond to free infinite loop spaces if the underlying space is path connected and has the homotopy type of a CW-complex. By our assumptions on (M, M_0) and X all the spaces considered are path connected and have the homotopy types of CW-complexes. Furthermore, by definition $\bar\sigma$ and its restrictions are morphisms of Γ^+-algebras. Thus the statement of the theorem follows. □

4.4 Induced splittings for mapping spaces

Combining the stable splitting of Theorem 4.7 with the fact that the scanning map is equivariant, Theorem 3.11, we immediately deduce the following result.

Theorem 4.8. *There is a zigzag of G-equivariant maps*

$$\Gamma(W \smallsetminus M_0, W \smallsetminus M; \pi) \longleftarrow E(M, M_0; \pi) \longrightarrow C(M, M_0; \pi) \longleftarrow \mathscr{C}(M, M_0; \pi) \longrightarrow \bigvee_{k\geq 1} D^k(M, M_0; \pi)$$

each of which is a stable homotopy equivalence. The zigzag induces a weak homotopy equivalence

$$\Omega^\infty\Sigma^\infty(EG \ltimes_G \Gamma(W \smallsetminus M_0, W \smallsetminus M; \pi)) \simeq \prod_{k\geq 1} \Omega^\infty\Sigma^\infty\Big(EG \ltimes_G D^k(M, M_0; \pi)\Big).$$

□

We now shift emphasis from configuration spaces to mapping spaces and adopt the notation $\bar D^k$ for the kth filtration quotient of

$$C(M \smallsetminus M_0, \partial M \smallsetminus M_0; \pi).$$

When M is parallelisable and π is a trivial bundle, the section space in the above theorem is simply a mapping space. But even when M is not parallelisable, we can use configuration spaces with twisted coefficients in the normal bundle of M to 'untwist' the section spaces so that they become again mapping spaces.

Corollary 4.9. *For N large enough, there is a homotopy equivalence*

$$\Omega^\infty\Sigma^\infty\big(EG \ltimes_G \mathrm{Map}(M, M_0; \Sigma^N X, *)\big) \simeq \prod_{k\geq 1} \Omega^\infty\Sigma^\infty\Big(EG \ltimes_G \bar D^k\Big).$$

Proof. Let ν be a bundle such that $TM \oplus \nu$ is a trivial bundle. (To construct such a ν note that M is compact and can be embedded in $\mathbb{R}^N$ for some N large enough. ν may be taken to be the normal bundle of the embedding.) Let $\pi : Y = \dot{\nu} \wedge_M (M \times X) \to M$ be the fibre bundle obtained by taking the fibrewise smash product of X with the fibrewise one-point compactification of ν, equipped with the canonical section at infinity. Note that

$$\tau_\pi|_M = \dot{T}M \wedge_M \dot{\nu} \wedge_M (M \times X) \cong \Sigma^N X \times M$$

The result is now a special case of the previous theorem. □

Example 4.10. If M is stably parallelisable, then for some $N \in \mathbb{N}$ there exists a trivial vector bundle ν such that $TM \oplus \nu$ is the trivial N-plane bundle on M. As ν is trivial, it admits an action of the full diffeomorphism group of M. Then there is a homotopy equivalence

$$\Omega^\infty \Sigma^\infty \big(E \operatorname{Diff}(M) \ltimes_{\operatorname{Diff}(M)} \operatorname{Map}(M; \Sigma^N X)\big) \simeq \prod_{k \geq 1} \Omega^\infty \Sigma^\infty \Big(E \operatorname{Diff}(M) \ltimes_{\operatorname{Diff}(M)} \bar{D}^k\Big).$$

When M is parallelisable we may take N equal to the dimension of M. For a compact Lie group G and assuming further that M is G-parallelisable, this recovers the main theorem of [BM88].

Example 4.11. When M is the disc in $\mathbb{R}^n$ and M_0 empty, we see that the Snaith stable splitting of $\Omega^n \Sigma^n(X)$ is equivariant under the action of $\operatorname{Diff}(\mathbb{R}^n)$.

Example 4.12. When M is the n-sphere $S^n \subset \mathbb{R}^{n+1}$ and M_0 empty, then the normal bundle ν is the one dimensional trivial bundle. Thus for $\pi : Y = \dot{\nu} \wedge_M (M \times X) \simeq M \times \Sigma X \to M$

$$C(S^n, \emptyset; \pi) \simeq \operatorname{Map}(S^n; \Sigma^{n+1} X).$$

This provides a model for the free higher loop space and a stable splitting, both of which are $\operatorname{Diff}(S^n)$-equivariant.

4.5 Stable injectivity for configurations and diffeomorphisms

Let M be as before, $M_0 = \emptyset$ and $\partial M \neq \emptyset$. Assume further that $\pi : Y \to M$ has typical fibre X which is path connected, and put $\pi_+ = \pi \amalg \mathrm{id}_M : Y \amalg M \to M$ be the X_+-bundle associated to π. Note that in this situation neither (M, M_0) nor X_+ are connected. We have

$$C(M, M_0; \pi_+) = C(M; \pi_+) = \coprod_{k \geq 0} C_k(M; \pi),$$

and though the scanning map is well-defined, it is no longer a (weak) homotopy equivalence. Similarly, the power set maps are still well-defined. But as the filtration quotient D^k is just C_k the splitting theorems are trivially true. Nevertheless, we will now apply the power set maps to analyse the relation between the spaces $C_k(M; \pi)$ as k grows.

We first define inclusion maps that allow us to think of the configurations of k points as a subspace of the configuration space of $k+1$ points. Recall $M \subset M^+ = M \cup \partial M \times [0, 1/2)$ and π can be extended to M^+ by pulling back along the natural projection $M^+ \to M$. Fix points $z_0 \in M^+ \smallsetminus M$ and $x_0 \in \pi^{-1}(z_0)$. We define a stabilisation map

$$b : C_k(M; \pi) \longrightarrow C_{k+1}(M^+; \pi) \longrightarrow C_{k+1}(M; \pi)$$

where the first map adds the particle (z_0, x_0) to any configuration and the second map is an injective homotopy equivalence induced by a map that isotopes M^+ into M mapping $\partial M \times [0, 1/2)$ into a collar of ∂M in M while at the same time leaving the complement of the collar fixed. As we assume that our diffeomorphisms always fix a collar of ∂M the following result follows immediately from these definitions.

Proposition 4.13. *The inclusion $b : C_k(M;\pi) \to C_{k+1}(M;\pi)$ is G-equivariant.* □

The fact that the map b is stably split injective is well-known. The point of the next result is that this can be done G-equivariantly and that therefore we have stable splittings of Borel constructions.

Theorem 4.14. *The inclusion maps $b : C_k(M;\pi) \to C_{k+1}(M;\pi)$ are G-equivariantly, stably split injective. In particular, the induced maps*

$$EG \times_G C_k(M;\pi) \longrightarrow EG \times_G C_{k+1}(M;\pi)$$

are stably split injective.

Proof. Let $C^k = C_k(M;\pi)$. Then via the map b we consider C^{k-1} to be a subspace of C^k and $C = \lim_k C_k(M;\pi)$ is a filtered space with filtration quotient $D^k = C^k/C^{k-1}$. Let V^k and V be the wedge products of these. Since the inclusion map b is G-equivariant, G acts naturally on D^k and hence V^k and V.

As in section 4.3 we can also define Borel configuration spaces $\mathscr{C}^k$, $\mathscr{C}$, etc. and a power set map

$$\sigma_k : \mathscr{C}^k = E\Sigma_k \times_{\Sigma_k} \widetilde{C}_k(M;\pi) \longrightarrow E\Sigma_{2^k} \times_{\Sigma_{2^k}} V^{2^k}$$

in this setting. As the action of Σ_k on $\widetilde{C}_k$ is free, $\mathscr{C}^k$ is (G-equivariantly) homotopy equivalent to C^k and hence $\mathscr{C}$ to C. The arguments used in the proofs of Proposition 4.6 and Theorem 4.7 go through verbatim to show that the σ_k fit together to give a well-defined map of free Γ^+-spaces

$$\bar{\sigma} : \Gamma^+(\mathscr{C}) \longrightarrow \Gamma^+(V)$$

which is G-equivariant, filtration preserving and a homotopy equivalence. As both $\mathscr{C}$ and V are connected, $\bar{\sigma}$ is homotopy equivalent to a map of free infinite loop spaces. This proves the fact that $\mathscr{C}$ and V are stably homotopy equivalent.

Collapsing D^k to a point defines a G-equivariant splitting of the inclusion $V^k \to V^{k+1}$. Via $\bar{\sigma}$ we conclude that stably $\mathscr{C}^k \to \mathscr{C}^{k+1}$ and hence $b : C^k \to C^{k+1}$ are also split injective. □

Let $M \setminus \boldsymbol{k}$ denote the manifold M with a set of k points deleted from the interior of M (away from the collar of the boundary). Using the same maps on the underlying manifolds as in the definition of b we can define a homomorphism of diffeomorphism groups

$$\bar{b} : \mathrm{Diff}(M \setminus \boldsymbol{k}, \partial M) \longrightarrow \mathrm{Diff}(M^+ \setminus \boldsymbol{k+1}, \partial M \times [0, 1/2)) \longrightarrow \mathrm{Diff}(M \setminus \boldsymbol{k+1}; \partial M).$$

Theorem 4.15. *The map $\bar{b} : B\,\mathrm{Diff}(M \setminus \boldsymbol{k}, \partial M) \longrightarrow B\,\mathrm{Diff}(M \setminus \boldsymbol{k+1}, \partial M)$ is stably split injective. In particular the induced map in homology is a split injection in all degrees.* □

Proof. Fix k distinct points $\boldsymbol{k} = \{z_1, \dots z_k\}$ in the interior of M. Simultaneous evaluation of diffeomorphisms on these points defines a fibration

$$\mathrm{Diff}(M \setminus \boldsymbol{k}, \partial M) \longrightarrow \mathrm{Diff}(M, \partial M) \longrightarrow C_k(M)$$

and hence a fibration of classifying spaces

$$C_k(M) \longrightarrow B\,\mathrm{Diff}(M \smallsetminus \boldsymbol{k}, \partial M) \longrightarrow B\,\mathrm{Diff}(M, \partial M).$$

Here we may take as a model for the total space of the latter fibration the Borel construction on $C_k(M)$ by the following general construction: given a topological group H with a closed subgroup K the fibration $K \to H \to H/K$ gives rise to a fibration $H/K \to EH \times_H H/K \simeq EH/K \to EH/H = BH$ where EH is a contractible space with a proper, free H-action; the total space of the latter fibration is a model for BK.

With this identification, the result is now a special case of Theorem 4.14. □

Remark 4.16. When M is an orientable surface $F^k_{g,n}$ of genus $g \geq 1$ with $n \geq 1$ boundary components and k punctures this specialises to a result in [BT01]. The proof in [BT01] uses the geometric construction of the power set map. Unfortunately, as defined there it is not equivariant. The construction needs to be replaced by the one in section 4.2 here.

In a forthcoming paper [Til] we will expand on the work here and show amongst other things that the maps $\bar{b}$ are furthermore isomorphims in homology in degrees $\leq k/2$ thus establishing a homology stability criteria for arbitrary manifolds and punctures.

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