An Extension Theorem to Rough Paths.

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Abstract

We show that any continuous path of finite p-variation can be lifted to a geometric q-rough path, $q > p$.

1 Introduction

Let

$$\begin{array}{rcl} x : [0,1] & \to & \mathbb{R}^n \\ t & \to & (x_1(t), \ldots, x_n(t)) \end{array}$$

be a continuous function of bounded variation, and $V_1, \ldots, V_n$ some smooth functions from $\mathbb{R}^d$ into itself (vector fields). Then there exists a (unique) solution to the control differential equation

$$\left\{ \begin{array}{l} dy(t) = \sum_{i=1}^{n} V_i(y(t))dx_i(t) \\ y(0) = y_0. \end{array} \right. \tag{1}$$

But without the smoothness assumption on x (which is for example almost surely not satisfied by Brownian motion), classical theory fails to give a meaning to the above equation. Rough path theory [11, 12, 10] gives a meaning to equation (1), assuming that x is a continuous path of finite p-variation lifted to a "p-rough path". A informal way of seeing a p-rough path is the following one: one assigns values to the iterated integrals

$$\int_{s<s_{i_1}<\ldots<s_{i_k}<t} dx\left(s_{i_1}\right)\ldots dx\left(s_{i_k}\right),$$

$1 \leq k \leq [p]$, $0 \leq s \leq t \leq 1$. Such integrals are not derived from x using classical integration theory, as x is not smooth enough; a rough path is a continuous path x together with choices for the above objects, The choices have to be internally consistent, respecting the algebraic relationship that iterated integrals of smooth paths and satisfying a particular "p-variation" constraint.

Rough path's theory gives much more than a meaning to equations of the type of equation (1). From the lift of x to a p-rough path, one can solve equation

(1) and obtain y lifted to a p-rough path. Moreover, the map from the lift of x to a p-rough path to the lift of y to a p-rough path is continuous, in an appropriate topology. Canonical lifts from continuous paths of finite p-variation to p-rough paths have been constructed for Brownian motion [12, 9], fractional Brownian motion with Hurst parameter greater than 1/4 [3, 12], free Brownian motion [2, 16], and a large class of random paths on fractals [1, 7]. This lift of a path x in $\mathbb{R}^n$ with finite p-variation to $G^{[p]}(\mathbb{R}^n)$ might not exist for every path (the theorems refer to almost every path), and if it does, it will never be unique (although the lift to $G^m(\mathbb{R}^n)$ from a p-rough path in $G^{[p]}(\mathbb{R}^n)$ will always exist and will be unique if $m > [p]$).

A very natural question is the following: can a given continuous path of finite p be lifted to a p-rough path? We will see that any path of finite p-variation can be lifted to a q-rough path, for $q > p$. Actually, any V-valued path of finite p-variation can be lifted to a $G^{[p]}(V)$-valued path of finite p-variation, whenever p is not an integer. This is not true for integer valued p; a counter-example for $p = 2$ was provided in [16].

The theorem we prove is actually more general. We are going to show that if x is a path of finite p-variation with values in a group $G^{[p]}(\mathbb{R}^n)/K$, where K is a normal subgroup of $G^{[p]}(\mathbb{R}^n)$, then one can lift x to a path of finite p-variation with values in $G^{[p]}(\mathbb{R}^n)$ (providing that p is not an integer greater than or equal to 2).

Given a signal x of finite p-variation with values in $G^{[p]}(\mathbb{R}^n)/K$, there always exists a non-unique lift of x to X as a q-rough path, $q > p$. The last part of the paper answers the following question: which solutions of differential equations depend explicitely on the lift X of x? In general, it is obvious that it does depend on the lift. We will see that if the Lie algebra generated by the vector fields $(V_i)_{1\leq i\leq n}$ in (1) is isomorphic to the Lie algebra of a quotient of $G^{[p]}(\mathbb{R}^n)/K$, the first level of the rough path solution of equation (1) does not depend on the lift, while, in general, the higher levels will depend on this lift.

2 Algebraic Preliminaries

2.1 Carnot Groups

If G is a simply connected nilpotent Lie group with Lie algebra $\mathcal{G}$, then the Lie group exponential map $\exp : \mathcal{G} \rightarrow G$ is a diffeomorphism [14, 15]. In this case we let $\ln : G \rightarrow \mathcal{G}$ denote the inverse of the exponential function. We start with a couple definitions.

Definition 1 *A Carnot group*[1] *is a connected nilpotent Lie group G, such that its Lie algebra $\mathcal{G}$ can be written as*

$$\mathcal{G} = W_1 \oplus \ldots \oplus W_n,$$

[1] In most definitions of a Carnot group, $\mathcal{G}$ is assumed to be finite dimensional. We do not make such an assumption here.

where for all i, $W_{i+1} = [W_1, W_i]$. *For an element* $g = \exp(w_1 + \ldots + w_n) \in G$, *with* $w_i \in W_i$, *we let, for* $t \in \mathbb{R}$,

$$\delta_t g = \exp(tw_1 + \ldots + t^n w_n).$$

δ *is called the dilation operator.*

Definition 2 *A (symmetric sub-additive) homogeneous norm [4] on a Carnot group* G *is a function* $\|.\|_G : G \to \mathbb{R}^+$ *such that*
(i) $\|g\|_G$ *if and only if* $g = 1$,
(ii) $\|\delta_t g\|_G = |t| \, \|g\|_G$,
(iii) *for all* $g, h \in G$, $\|g \otimes h\|_G \leq \|g\|_G + \|h\|_G$,
(iv) *for all* g, $\|g\|_G = \left\|g^{-1}\right\|_G$.
Such a norm define a left invariant distance on the group by $d_G(g,h) = \left\|h^{-1} \otimes g\right\|_G$. *We will say that* $(G, \|.\|_G)$ *is a normed Carnot group.*

If G is a fixed Carnot group with finite dimensional Lie algebra, all homogeneous norms on G are equivalent. The Carnot-Caratheodory norm is an example of a homogeneous norm on a Carnot group [6].

Let G be a normed Carnot group with Lie algebra $\mathcal{G}$, K a Lie subgroup of G, with Lie algebra $\mathcal{K}$. If K is a closed normal Lie subgroup of G, or equivalently if $\mathcal{K}$ is closed ideal of $\mathcal{G}$, then G/K is then a Carnot group with Lie algebra $\mathcal{G}/\mathcal{K}$ [14]. If G is equipped with a homogeneous norm $\|.\|_G$, then we equip G/K with the quotient homogeneous norm on G/K

$$\begin{array}{rcl} \|.\|_{G/K} : G/K & \to & \mathbb{R} \\ gK & \to & \inf_{k \in K} \|g \otimes k\|_G . \end{array}$$

We will denote by $\pi_{G,G/K}$ the canonical homomorphism from G onto G/K. Sometimes, it will be more convenient to write gK for $\pi_{G,G/K}(g)$.

Proposition 3 *Let* $(G, \|.\|_G)$ *be a normed Carnot group,* K *a closed normal Lie subgroup of* G. *There exists an injection* $i_{G/K,G} : G/K \to G$ *such that*
(i) $\pi_{G,G/K} \circ i_{G/K,G}$ *is the identity map of* G/K,
(ii) *For all* $t \in \mathbb{R}^+$, $gK \in G/K$, $\delta_t\left(i_{G/K,G}(gK)\right) = i_{G/K,G}\left(\delta_t(gK)\right)$,
(iii) *For all* $gK \in G/K$, $\|gK\|_{G/K} \leq \left\|i_{G/K,G}(gK)\right\|_G \leq 2\,\|gK\|_{G/K}$.

Proof. Let $B_{G/K}(\exp(0), 1)$ be the unit ball of $\left(G/K, \|.\|_{G/K}\right)$ centered around the neutral element of G/K. By definition of the homogeneous norm on G/K, for all $g \in \pi^{-1}_{G,G/K}(B_{G/K}(\exp(0),1))$, there exists $k \in K$ such that

$$\|gK\|_{G/K} \leq \|g \otimes k\|_G \leq \|gK\|_{G/K} + 1.$$

Hence for all $g \in \pi^{-1}_{G,G/K}(B_{G/K}(\exp(0),1))$, the set

$$M_g = \{g \otimes k \text{ such that } k \in K \text{ and } 1 \leq \|g \otimes k\|_G \leq 2\}$$

is non-empty. $i_{G/K,G}$ on $B_{G/K}(\exp(0),1)$ is just any function which at $gK \in B_{G/K}(\exp(0),1)$ associates an element of $\cup_{m\in\pi^{-1}_{G,G/K}(gK)} M_m$; such function exists by definition of the axiom of choice. We then extend $i_{G/K,G}$ to G/K with the help of the formula $i_{G/K,G}(\delta_t gK) = \delta_t i_{G/K,G}(gK)$. ■

2.2 Free Nilpotent Groups

We now introduce a fundamental example of a Carnot group.

We fix (for the rest of the paper) a normed vector space $(V, \|.\|_1)$. We let $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra over V. $T(V)$ equipped with standard addition $+$, tensor multiplication $\otimes$ and scalar product is an associative algebra. $T^{(n)}(V)$, the quotient algebra of $T(V)$ by the ideal $\bigoplus_{m=n+1}^{\infty} V^{\otimes m}$, inherits this algebra structure. One can define on $T^{(n)}(V)$ a Lie bracket by the formula

$$[a,b] = a\otimes b - b\otimes a,$$

which makes $T^{(n)}(V)$ into a Lie algebra. We let $\mathcal{G}^{(n)}(V)$ be the Lie subalgebra of $T^{(n)}(V)$ generated by elements in V. Note that

$$\mathcal{G}^{(n)}(V) \simeq \bigoplus_{i=1}^{n} V_i,$$

where

$$V_1 = V \text{ and } V_{i+1} = [V, V_i]. \tag{2}$$

$\mathcal{G}^{(n)}(V)$ is the free nilpotent Lie algebra of step n [11, 12, 13]. The exponential, logarithm and inverse function are defined on $T^{(n)}(V)$ by mean of their power series. We denote by $G^{(n)}(V) = \exp\left(\mathcal{G}^{(n)}(V)\right)$. By the Baker-Campbell-Hausdorff formula, $\left(G^{(n)}(V), \otimes\right)$ is a connected nilpotent Lie group, called the free nilpotent Lie group of step n over V. By construction, $\left(G^{(n)}(V), \otimes\right)$ is a Carnot group, with Lie algebra $\mathcal{G}^{(n)}(V)$.

We are now going to equip $G^{(n)}(V)$ with a homogeneous norm. We first let $\|.\|_i$ be some norms on $V^{\otimes i}$ such that for all $(a_i, a_j) \in V^{\otimes i}\times V^{\otimes j}$, $\|a_i\otimes a_j\|_{i+j} \leq \|a_i\|_i + \|a_j\|_j$. To simplify notations, we will write $\|.\|$ for all these norms. Now define

$$\|g\|_{G^{(n)}(V)} = \max_{i=1,\ldots,n} \left(i!\,\|g_i\|\right)^{1/i},$$

where $g = 1 + g_1 + \ldots + g_n$, $g_i \in V^{\otimes i}$, is an element of the group $G^{(n)}(V)$. Then $\|.\|_{G^{(n)}(V)}$ defines a homogeneous norm on $G^{(n)}(V)$. We also let

$$d_{G^{(n)}(V)}(g,h) = \left\|h^{-1}\otimes g\right\|_{G^{(n)}(V)}.$$

Proposition 4 *Let $g = \exp\left(l_1 + \ldots + l_n\right)$, with $l_i \in V_i$. Then,*

$$c_n \max_{i=1,\ldots,n} \|l_i\|^{1/i} \leq \|g\|_{G^{(n)}(V)} \leq C_n \max_{i=1,\ldots,n} \|l_i\|^{1/i},$$

for some constants c_n and C_n which depends only on n.

Proof. Let us fix $i \in \{1, \ldots, n\}$ and write $g = 1 + g_1 + \ldots + g_n$, with $g_i \in V^{\otimes i}$. By definition of the exponential function,

$$g_k = \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{j_1,\ldots,j_i \\ j_1+\ldots+j_i=k}} l_{j_1} \otimes \ldots \otimes l_{j_i}.$$

Hence,

$$\begin{aligned} \left(k!\, \|g_k\|_k\right)^{1/k} &\leq \left(\sum_{i=1}^{k} \frac{k!}{i!} \sum_{\substack{j_1,\ldots,j_i \\ j_1+\ldots+j_i=k}} \|l_{j_1}\| \ldots \|l_{j_i}\| \right)^{1/k} \\ &\leq \left(k!\, (\exp k - 1)\right)^{1/k} \max_{i=1,\ldots,n} \|l_i\|^{1/i}. \end{aligned}$$

Similarly, by definition of the logarithm function,

$$l_k = \sum_{i=1}^{k} \frac{(-1)^i}{i} \sum_{\substack{j_1,\ldots,j_i \\ j_1+\ldots+j_i=k}} g_{j_1} \otimes \ldots \otimes g_{j_i},$$

which gives that for all $1 \leq k \leq n$,

$$\|l_k\|^{1/k} \leq c_n^{-1} \|g\|_{G^{(n)}(V)}$$

for a constant $c_n > 0$. ■

Corollary 5 *Let $K = \exp(\mathcal{K})$ be a closed normal subgroup of $G^{(n)}(V)$. Then, if $g = \exp(l_1 + \ldots + l_n)$ with $l_i \in V_i$,*

$$c_n \leq \frac{\|gK\|_{G^{(n)}(V)/K}}{\max_{i=1,\ldots,n} \left(\inf_{k_i \in \mathcal{K} \cap V_i} \|l_i + k_i\|\right)^{1/i}} \leq C_n,$$

Corollary 6 *Let $C(G^{(n)}(V))$ be the centre of $G^{(n)}(V)$ and θ the canonical isomorphism between $G^{(n-1)}(V)$ and $G^{(n)}(V)/C(G^{(n)}(V))$. Then the homogeneous norm $\|.\|_{G^{(n-1)}(V)}$ and $\|\theta(.)\|_{G^{(n)}(V)/C(G^{(n)}(V))}$ are equivalent. We will therefore not distinguish between them.*

3 Geometric Rough Paths

In this paper, by E-valued path, we mean a function from $[0,1]$ into E.

3.1 On p-variation

Definition 7 *Let (E, d) be a metric space. A (E, d)-valued path x is said to have finite p-variation if for all subdivisions $D = (0 \leq t_0 \leq \ldots \leq t_n \leq 1)$ of the interval $[0,1]$, $\sum_{i=0}^{n-1} d(x_{t_i}, x_{t_{i+1}})^p < \infty$.*

Note that x is continuous and of finite regular p-variation if and only if for all $s \leq t$, $d(x_s, x_t) \leq \omega(s,t)$, where

$$
\begin{array}{ll}
\text{(i)} & \omega : \{(s,t), 0 \leq s \leq t \leq 1\} \rightarrow \mathbb{R}^+ \text{ is continuous.} \\
\text{(ii)} & \omega \text{ is super-additive, i.e. } \forall\, s < t < u,\ \omega(s,t) + \omega(t,u) \leq \omega(t,u). \\
\text{(iii)} & \omega(t,t) = 0 \text{ for all } t \in [0,1]
\end{array}
\qquad (3)
$$

We will say in such case that x has finite p-variation controlled by ω.

We are going to show that a continuous (E,d)-valued path of finite p-variation is, up to reparametrisation of time, $\frac{1}{p}$-Hölder continuous. If ω satisfies (3), then $(s,t) \rightarrow \omega(0,1)\left(\frac{\omega(0,t)}{\omega(0,1)} - \frac{\omega(0,s)}{\omega(0,1)}\right)$ is a continuous additive map, equal to zero on the diagonal, and $\omega(0,t) - \omega(0,s) \geq \omega(s,t)$ (by the super-additivity of ω). Therefore, a path is of finite p-variation if and only if there exists an non-decreasing and continuous surjection γ from $[0,1]$ onto $[0,1]$ and a positive constant C such that

$$\text{for all } s \leq t,\ d(x_s, x_t)^p \leq C\left|\gamma(t) - \gamma(s)\right|.$$

For such a γ, we define

$$
\begin{array}{rcl}
\gamma^{-1} : [0,1] & \rightarrow & [0,1] \\
t & \rightarrow & \inf\{u, \gamma(u) = t\}.
\end{array}
$$

The following is straightforward to check.

Lemma 8 *Let x be a continuous (E,d)-valued path of finite p-variation controlled by $(s,t) \rightarrow C\left|\gamma(t) - \gamma(s)\right|$, where γ is a continuous increasing surjection from $[0,1]$ onto $[0,1]$. Define*

$$
\begin{array}{rcl}
y : [0,1] & \rightarrow & E \\
t & \rightarrow & y_{\gamma^{-1}(t)}.
\end{array}
$$

Then, y is a $1/p$-Hölder (E,d)-valued path
Reciprocally, if y is a $1/p$-Hölder (E,d)-valued path then

$$
\begin{array}{rcl}
x : [0,1] & \rightarrow & E \\
t & \rightarrow & \mathbf{x}_{\gamma(t)}
\end{array}
$$

is a continuous path of finite p-variation controlled by $(s,t) \rightarrow C\left|\gamma(t) - \gamma(s)\right|$.

When x is a path with values in a group $(G, \otimes)$, we will use the *notation* $x_{s,t} = x_s^{-1} \otimes x_t$.

Definition 9 *A geometric p-rough path in V is the closure in the p-variation metric of the canonical lift of smooth V-valued paths to $G^{[p]}(V)$-valued paths.*

In particular, a geometric p-rough path is a $G^{[p]}(V)$-valued path of finite p-variation. The set of $G^{[p]}(V)$-valued path of finite p-variation strictly contains the set of geometric p-rough path, and is strictly included in the set of geometric q-rough path, $q > p$. We refer to [5] for a precise discussion on the subject. We will deal exclusively with $G^{[p]}(V)$-valued path of finite p-variation, and use the fact that such paths are geometric q-rough path, $q > p$.

4 The Extension Theorem

We first need an important lemma.

Lemma 10 *Let* $(G, \|.\|_G)$ *be a normed Carnot group with Lie algebra*

$$\mathcal{G} = W_1 \oplus W_2 \oplus \ldots \oplus W_n,$$

Define K *to be a closed subgroup of* $\exp(W_n)$*, which gives us a normed Carnot group* $\left(G/K, \|.\|_{G/K}\right)$*. Let* x *be a* $1/p$*-Hölder* $\left(G/K, \|.\|_{G/K}\right)$*-valued path. Then, if* $p > n$*, there exists a* $1/p$*-Hölder* $(G, \|.\|_G)$*-valued path* $\widetilde{x}$ *such that* $\pi_{G,G/K}(\widetilde{x}) = x$*.*

Proof. Observe first that K is a subgroup of the center of G. In particular, K is a closed normal subgroup of G. To construct our path $\widetilde{x}$, we are first going to construct its increments $\widetilde{x}_{s,t}$ when $s, t \in D_m = \left\{\frac{j}{2^m}, j \in \left\{0, \ldots, 2^{m-1}\right\}\right\}$ with $t - s = 2^{-m}$, doing this for all m. $\widetilde{x}_{s,t}$ will be constructed in such a way that $\|\widetilde{x}_{s,t}\|_G \leq C\,|t-s|^{1/p}$ for a given $C < \infty$. Multiplying the increments, we will then have defined x on all dyadics, and we will (classically) check that for all s and t dyadics, $\|\widetilde{x}_{s,t}\|_G \leq C'\,|t-s|^{1/p}$, for a given $C' < \infty$. This will allows us to extend by continuity the definition of $\widetilde{x}$ over the whole interval $[0, 1]$.

So we define recursively on m some elements $k_{\frac{j}{2^m},\frac{j+1}{2^m}}$, $j \in \{0, \ldots, 2^m - 1\}$, $m \in \mathbb{N}$, in the aim of defining the elements $\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}}$ with the formula

$$\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}} = i_{G/K,G}\left(x_{\frac{j}{2^m},\frac{j+1}{2^m}}\right) \otimes k_{\frac{j}{2^m},\frac{j+1}{2^m}}$$

where $i_{G/K,G}$ is the injection of proposition 3. This will ensure that $\pi_{G,G/K}(\widetilde{x}) = x$. First, we let, $k_{0,1} = \exp(0)$. Then, we assume that $k_{\frac{j}{2^m},\frac{j+1}{2^m}}$ (and hence $\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}}$) has been constructed for all $0 \leq j \leq 2^m - 1$ and a fixed m, and we let

$$\begin{aligned}
\delta_{2^{1/n}}\big(k_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}}\big) &= \delta_{2^{1/n}}\big(k_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}}\big) \\
&= \widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}} \otimes i_{G/K,G}\left(x^{-1}_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}}\right) \otimes i_{G/K,G}\left(x^{-1}_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}}\right).
\end{aligned}$$

We easily check that $\pi_{G,G/K}\left(k_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}}\right) = \exp(0)$, i.e. that $k_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}} = k_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}} \in K$. As elements of K commute with elements of G, and with the help of the formula $\delta_{2^{1/n}}(j) = j^{\otimes 2}$ for $j \in K$, we check that this choice for $k_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}}$ and $k_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}}$ gives

$$\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}} = \widetilde{x}_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}} \otimes \widetilde{x}_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}}.$$

We then define $a_m = 2^{m/p} \sup_{j \in \{0,\ldots,2^{m-1}\}} \left\|k_{\frac{j}{2^m},\frac{j+1}{2^m}}\right\|_G$. By the assumption that x is $1/p$-Hölder and by the definition of $i_{G/K,G}$,

$$\left\|i_{G/K,G}\left(x_{\frac{j}{2^m},\frac{j+1}{2^m}}\right)\right\|_G \leq 2\left\|x_{\frac{j}{2^m},\frac{j+1}{2^m}}\right\|_{G/K} \leq 2C2^{-m/p}.$$

Hence, from the previous inequality, we obtain that

$$2^{1/n}2^{-(m+1)/p}a_{m+1} \le a_m 2^{-m/p} + 2^{-(m+1)/p}2^{2+1/p}C,$$

i.e.

$$a_{m+1} \le 2^{1/p-1/n}a_m + 2^{2+1/p-1/n}C.$$

As $n < p$, we have $2^{1/p-1/n} < 1$ which forces the sequence a_m to be bounded. So we have constructed for every $m \ge 0$, $j \in \{0, \dots, 2^m - 1\}$ some elements $\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}} = i_{G/K,G}\left(x_{\frac{j}{2^m},\frac{j+1}{2^m}}\right) \otimes k_{\frac{j}{2^m},\frac{j+1}{2^m}} \in G$, such that

$$\begin{aligned} \left\|\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}}\right\|_G &\le 2\left\|x_{\frac{j}{2^m},\frac{j+1}{2^m}}\right\|_{G/K} + \left\|k_{\frac{j}{2^m},\frac{j+1}{2^m}}\right\|_G \\ &\le \left(2C + \sup_m a_m\right)2^{-m/p} = C_{p,n}2^{-m/p}. \end{aligned} \tag{4}$$

Remember also that for all dyadic $\frac{j}{2^m}$,

$$\widetilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}} = \widetilde{x}_{\frac{2j}{2^{m+1}},\frac{2j+1}{2^{m+1}}} \otimes \widetilde{x}_{\frac{2j+1}{2^{m+1}},\frac{2j+2}{2^{m+1}}}.$$

That allows us to define

$$\widetilde{x}_{\frac{i}{2^m},\frac{j}{2^m}} = \bigotimes_{l=i}^{j-1}\widetilde{x}_{\frac{l}{2^m},\frac{l+1}{2^m}}.$$

From this point, the proof is similar to the end of Kolmogorov-Centsov criteria's proof [8]. We fix $r \in \mathbb{N}$, and show by induction on m that for all $s, t \in D_m$ such that $0 < t - s < 2^{-r}$,

$$\|\widetilde{x}_{s,t}\|_G \le 2C_{p,n}\sum_{j=r+1}^{m} 2^{-j/p}. \tag{5}$$

When $m = r+1$, necessarily, (s,t) is of the form $\left(\frac{j}{2^m}, \frac{j+1}{2^m}\right)$, $j \in \{0, \dots, 2^m - 1\}$, and so (5) is exactly formula (4). Suppose now that formula (5) is valid for $m = r+1, \dots, M-1$. Take $s, t \in D_M$ such that $0 < t - s < 2^{-r}$, and consider $t_1 = \max\{u \in D_{M-1}; u \le t\}$ and $s_1 = \max\{u \in D_{M-1}; u \ge s\}$. Notice that $\|\widetilde{x}_{s,s_1}\|_G$ and $\|\widetilde{x}_{t_1,t}\|_G$ are both bounded by $C_{p,n}2^{-M/p}$, and, by the induction assumption, that

$$\|\widetilde{x}_{s_1,t_1}\|_G \le 2C_{p,n}\sum_{j=r+1}^{M-1} 2^{-j/p}.$$

Therefore, as

$$\widetilde{x}_{s,t} = \widetilde{x}_{s,s_1} \otimes \widetilde{x}_{s_1,t_1} \otimes \widetilde{x}_{t_1,t},$$

$$
\begin{aligned}
\|\widetilde{x}_{s,t}\|_G &\leq 2C_{p,n}2^{-M/p} + 2C_{p,n}\sum_{j=r+1}^{M-1} 2^{-j/p} \\
&= 2C_{p,n}\sum_{j=r+1}^{M} 2^{-j/p},
\end{aligned}
$$

which concludes the induction.
Now let us consider $(s,t) \in \bigcup_{m\geq 0} D_m$, and let n be the natural number such that $2^{-(r+1)} < t-s < 2^{-r}$. From the induction, we obtain

$$
\begin{aligned}
\|\widetilde{x}_{s,t}\| &\leq 2C_{p,n}\sum_{j=r+1}^{\infty} 2^{-j/p} \leq \widetilde{C}_{p,n}2^{-(r+1)/p} \\
&\leq \widetilde{C}_{p,n}\left|t-s\right|^{1/p}.
\end{aligned}
\tag{6}
$$

We finally define $\widetilde{x}_{s,t}$ for $0 \leq s \leq t \leq 1$ by

$$
\widetilde{x}_{s,t} = \lim_{r\to\infty} \widetilde{x}_{\frac{[2^r s]}{2^r},\frac{[2^r t]}{2^r}}.
$$

From (6), the limit exists and $\widetilde{x}_{s,t}$ satisfies $\|\widetilde{x}_{s,t}\| \leq \widetilde{C}_{p,n}\left|t-s\right|^{1/p}$. ■

We are now ready for our main theorem.

Theorem 11 *We fix $p \in [1,+\infty)$. Let K be a closed normal subgroup of $G^{([p])}(V)$. If x is a $\left(G^{([p])}(V)/K, \|.\|_{G^{([p])}(V)/K}\right)$ continuous path of finite p-variation, with $p \notin \mathbb{N}\backslash\{0,1\}$, then there exists a continuous $\left(G^{([p])}(V), \|.\|_{G^{([p])}(V)}\right)$-valued path $\widetilde{x}$ of finite p variation such that*

$$
\pi_{G^{([p])}(V),G^{([p])}(V)/K}(\widetilde{x}) = x.
$$

Proof. As noticed in section 3.1, we assume without loss of generalities that x is $1/p$-Hölder. We denote by $\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_{[p]}$ the Lie algebra of K, with $\mathcal{K}_i \subset V_i$. We define for $k = 1, \ldots, n$,

$$
H^{(k)} = G^{(k)}(V)/\exp(\mathcal{K}_k)
$$

and

$$
\begin{aligned}
M^{(k)} &= G^{(k)}(V)/\exp\left(\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_k\right) \\
&\simeq H^{(k)}/\exp\left(\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_{k-1}\right).
\end{aligned}
$$

We are going to construct recursively some $H^{(k)}$-valued paths $y^{(k)}$ and $G^{(k)}$-valued paths $x^{(k)}$ which are $1/p$-Hölder and such that

$$
\pi_{H^{(k)},M^{(k)}}\left(y^{(k)}\right) = \pi_{M^{([p])},M^{(k)}}\left(x\right), \tag{7}
$$

$$
\pi_{G^{(k)},M^{(k)}}\left(x^{(k)}\right) = \pi_{M^{([p])},M^{(k)}}\left(x\right). \tag{8}
$$

Using lemma 10, $x^{(k)}$ is easily constructed from $y^{(k)}$, so we only need to construct the paths $y^{(k)}$.

For $k=1$, $y^{(1)} = \pi_{M^{([p])},M^{(1)}}(x)$ is a $H^{(1)} = G^{(1)}(V)/\exp(\mathcal{K}_1)$ valued path, which is $1/p$-Hölder.

We now assume that we have a $G^{(k)}$-valued path $x^{(k)}$ which is $1/p$-Hölder and which satisfies equality (8).

$$Z^{(k+1)} = \left\{(g,m) \in G^{(k)} \times M^{(k+1)} \text{ such that } \pi_{G^{(k)},M^{(k)}}(g) = \pi_{M^{(k+1)},M^{(k)}}(m)\right\}$$

with the product

$$(g_1,m_1) \otimes (g_2,m_2) = (g_1 \otimes g_2, m_1 \otimes m_2)$$

is a group. The application

$$\begin{aligned} \Psi \quad &: \quad H^{(k+1)} \to Z^{(k+1)} \\ g\exp(\mathcal{K}_{k+1}) \quad &\to \quad (g\exp(V_{k+1}), g\exp(\mathcal{K}_2 \oplus \ldots \oplus \mathcal{K}_k)). \end{aligned}$$

is easily seen to be an isomorphism. Ψ can also be described by the following formula: if $\ell_k \in V_1 \oplus \ldots \oplus V_k$ and $l^{k+1} \in V_{k+1}$,

$$\begin{aligned} &\Psi^{-1}(\exp(\ell_k)\exp(V_{k+1}), \exp(\ell_k + l^{k+1})\exp(\mathcal{K}_2 \oplus \ldots \oplus \mathcal{K}_{k+1})) \\ &\qquad = \exp(\ell_k)\exp(l^{k+1})\exp(\mathcal{K}_{k+1}). \end{aligned}$$

Hence, using proposition 4 corollary 5, we see that there exists a constant C_{k+1} such that for all $(g,m) \in Z^{(k+1)}$,

$$\left\|\Psi^{-1}(g,m)\right\|_{H^{(k+1)}} \le C\left(\|g\|_{G^{(k)}} + \|m\|_{M^{(k+1)}}\right).$$

For all $t \in [0,1]$, $(x_t^{(k)}, \pi_{M^{([p])},M^{(k+1)}}(x_t)) \in Z^{(k+1)}$, hence we can define

$$y_t^{(k+1)} = \Psi^{-1}\left(x_t^{(k)}, \pi_{M^{([p])},M^{(k+1)}}(x_t)\right).$$

Note first that $y^{(k+1)}$ satisfies the equality (7). Because Ψ^{-1} is an isomorphism, $y_{s,t}^{(k+1)} = \Psi^{-1}\left(x_{s,t}^{(k)}, \pi_{M^{([p])},M^{(k+1)}}(x_{s,t})\right)$ and hence

$$\begin{aligned} \left\|y_{s,t}^{(k+1)}\right\|_{H^{(k+1)}} &\le C\left(\left\|x_{s,t}^{(k)}\right\|_{G^{(k)}} + \left\|\pi_{M^{([p])},M^{(k+1)}}(x_{s,t})\right\|_{M^{(k+1)}}\right) \\ &\le C\left(\left\|x_{s,t}^{(k)}\right\|_{G^{(k)}} + \|x_{s,t}\|_{M^{([p])}}\right). \end{aligned}$$

By hypothesis and induction hypothesis, $\left\|x_{s,t}^{(k)}\right\|_{G^{(k)}} + \|x_{s,t}\|_{M^{([p])}} \le C|t-s|^{1/p}$, hence $\left\|y_{s,t}^{(k+1)}\right\|_{H^{(k+1)}} \le C|t-s|^{1/p}$.

Using the induction step until we reach the level $[p]$, we obtain a $G^{([p])}(V)$-valued path $x^{([p])}$ which is $1/p$-Hölder and such that

$$\pi_{G^{([p])},M^{([p])}}\left(x^{([p])}\right) = \pi_{M^{([p])},M^{([p])}}(x) = x.$$

■

We ought to make a couple comments on our main theorem.

Remark 12 *Note that we could have considered a continuous path of finite p-variation with values in a quotient space of* $G^{(n)}(V)$, *with* $n > [p]$. *If* $K^{(n)}$ *is a closed normal subgroup of* $G^{(n)}(V)$ *and* x *is a* $1/p$*-Hölder path with values in* $\left(G^{(n)}(V)/K^{(n)}, \|.\|_{G^{(n)}(V)/K^{(n)}}\right)$, *with* $p \notin \mathbb{N}\backslash\{0,1\}$, *then there exists a* $1/p$*-Hölder* $G^{(n)}(V)$*-valued path* $\widetilde{x}$ *such that*

$$\pi_{G^{(n)}(V),G^{(n)}(V)/K^{(n)}}(\widetilde{x}) = x.$$

To prove this, first let $K^{([p])} = \pi_{G^{(n)},G^{([p])}}(K^{(n)})$. *The canonical projection of* x *into* $G^{[p]}(V)/K^{([p])}$ *is a* $1/p$*-Hölder path. Hence, by the previous theorem, there exists a* $1/p$*-Hölder* $G^{[p]}(V)$*-valued path* $x^{([p])}$ *such that*

$$\pi_{G^{([p])}(V),G^{[p]}(V)/K^{([p])}}(x^{([p])}) = \pi_{G^{(n)}(V)/K^{(n)},G^{[p]}(V)/K^{([p])}}(x).$$

Then, by the first main theorem in [11], $x^{([p])}$ *can be (uniquely) extended to a* $1/p$*-Hölder* $\left(G^{(n)}(V), \|.\|_{G^{(n)}(V)}\right)$*-valued path* $x^{(n)}$. *By the uniqueness statement,* $x^{(n)}$ *must satisfy*

$$\pi_{G^{(n)}(V),G^{(n)}(V)/K^{(n)}}(x^{(n)}) = x.$$

Remark 13 *As already pointed out in [11, 12], if* $p \geq 2$ *and if there exists one* $\left(G^{([p])}(V), \|.\|_{G^{([p])}(V)}\right)$*-valued path* $\widetilde{x}$ *of finite* p *variation such that*

$$\pi_{G^{([p])}(V),G^{([p])}(V)/K}(\widetilde{x}) = x,$$

then there exists uncountably many such paths.

Remark 14 *The condition* $p \notin \mathbb{N}\backslash\{0,1\}$ *is necessary. In [16], a counter example was given for the free Brownian motion, which is a path of finite* 2 *variation (being* $\frac{1}{2}$*-Hölder) living in a non-commutative* L^2*-space. It was proven that, when* $L^2 \otimes L^2$ *is equipped with the projective tensor product, there does not exist a path of finite* 2*-variation with values in the nilpotent group of step* 2 *over* L^2.

Remark 15 *If* p *is a natural number greater than or equal to* 2, *keeping the notation of the previous theorem, we can find, for any fixed* $q > p$, *a geometric* q*-rough path* $\widetilde{x}$ *such that*

$$\pi_{G^{(p)}(V),G^{(p)}(V)/K}(\widetilde{x}) = x.$$

This is obtained just by noticing that a path of finite p*-variation has finite* $(p+\varepsilon)$*-variation, for all* $\varepsilon > 0$.

We end up with a corollary, which was the motivation of this paper.

Corollary 16 *If* $p \in [1,\infty)\backslash\{2,3,\ldots\}$, *a continuous* V*-valued path of finite* p*-variation can be lifted to a path of finite* p*-variation, with values in* $G^{[p]}(V)$ *For any* p, *a continuous path of finite* p*-variation can be lifted to a geometric* q*-rough path,* $q > p$.

Proof. Apply theorem 11 to $K = \exp\left(\bigoplus_{i=2}^{[p]} V_i\right)$ and use the previous remark. ■

That means, in particular, that one can always define a notion of solution to differential equations controlled by a continuous path of finite p-variation, whatever the p is.

5 Rough Differential Equations for which the Extension does not Matter

We fix a real $p \geq 1$. $\mathcal{X}^{k+\varepsilon}(\mathbb{R}^d)$ denotes the class of k times differentiables vector fields with the k^{th}-derivatives being ε-Holder and with all the first k-derivatives being bounded. We consider $A_1, \cdots, A_m$ some elements of $\mathcal{X}^{\gamma}(\mathbb{R}^d)$, with $\gamma > p$. We fix a basis $e_1, \cdots, e_m$ of $\mathbb{R}^m$, and extend the linear application

$$\begin{array}{rcl} \mathbb{R}^m & \to & \mathcal{X}^{\gamma}(\mathbb{R}^d) \\ e_i & \to & A_i. \end{array}$$

to an algebra homomorphism $\digamma_{[p]}^{A}$ from $T^{([p])}(\mathbb{R}^d)$ into the space of continuous differential operators. Note that $\digamma_{[p]}^{A}$ restricted to the free Lie algebra $\mathcal{G}^{([p])}(\mathbb{R}^m)$, i.e. $\left(\digamma_{[p]}^{A}\right)_{|\mathcal{G}^{([p])}(\mathbb{R}^m)}$, is a Lie homomorphism into $\mathcal{X}^0(\mathbb{R}^d)$.

Recall that if $\mathbf{x}$ is a p-geometric rough path, a solution of the differential equation

$$\begin{array}{rcl} d\mathbf{y}_t & = & A(y_t)d\mathbf{x}_t \\ y_0 & = & a \end{array}$$

is an extension of $\mathbf{x}$ to $\mathbf{z} \in G\Omega(\mathbb{R}^{m+d})$ that projects onto $(\mathbf{x}, \mathbf{y})$, $(x_0, y_0) = (0, a)$, and such that

$$\mathbf{z}_{s,t} = \int_s^t h(z_u)d\mathbf{z}_u,$$

where

$$\begin{array}{rcl} h & : & R^m \oplus R^d \to Hom\left(R^m \oplus R^d, R^m \oplus R^d\right) \\ (x,y) & \to & \left((dX, dY) \to (dX, A(y)dX)\right). \end{array}$$

The map $\mathbf{x} \to \mathbf{z}$ is called the Ito map, and denoted $I_A : G\Omega(\mathbb{R}^m) \to G\Omega(\mathbb{R}^m \oplus \mathbb{R}^d)$. We also let π denote the projection from $G^{([p])}(\mathbb{R}^m \oplus \mathbb{R}^d)$ onto $\mathbb{R}^d$.

Theorem 17 *Let x be a $1/p$-Holder path in $G^{([p])}(\mathbb{R}^m)/K$, where K is a normal subgroup of $G^{([p])}(\mathbb{R}^m)$ with Lie algebra $\mathcal{K}$ and $\mathbf{x}$ an extension of x to a $1/p$-Holder path in $G^{([p])}(\mathbb{R}^m)$ ($1/(p+\varepsilon)$ if p is an integer). Assume that the kernel of the Lie algebra homomorphism $\left(\digamma_{[p]}^{A}\right)_{|\mathcal{G}^{([p])}(\mathbb{R}^m)}$ contains $\mathcal{K}$. Then $I_A(\mathbf{x})$ is a $1/p$-Holder path in $G^{([p])}(\mathbb{R}^{m+d})$ which, in general depends on the extension of x to a geometric p-rough path $\mathbf{x}$. Nevertheless, $\pi \circ I_A(\mathbf{x})$ depends only on x.*

Proof. To see that $I_A(\mathbf{x})$ depends on general of the extension of x to a p-rough path $\mathbf{x}$, just consider the Ito map which is the identity. $\pi \circ I_A(\mathbf{x})$ is easily seen, using similar techniques as in [11, 12], to be the only $1/p$-Holder path associated to the almost additive functional

$$\begin{aligned} y_{s,t} &= \digamma^{A}_{[p]}(\mathbf{x}_{s,t})(y_s) - y_s \\ y_0 &= a. \end{aligned}$$

We then see that $y_{s,t}$ only depends on x, which proves our result. ■

References

[1] Bass R.F.; Hambly B.M.; Lyons T.J.: Extending the Wong-Zakai theorem to reversible Markov processes. J. Eur. Math. Soc. 4, 237-269 (2002).

[2] Capitaine, M.; Donati-Martin, C.: The Lévy area process for the free Brownian motion. J. Funct. Anal. Vol 179, no. 1, 153–169, (2001).

[3] Coutin L.; Qian Z.: Stochastic analysis, rough path analysis and fractional Brownian motions. Probab. Theory Related Fields 122, no. 1, 108–140 (2002).

[4] Folland, G. B.; Stein, E.M.: Hardy spaces on homogeneous groups. Mathematical Notes, 28, (1982).

[5] Friz, P. Victoir, N: On the notion of Geometric Rough Paths, preprint.

[6] M. Gromov, Carnot-Caratheodory spaces seen from within, in: Sub-Riemannian Geometry, A. Bellaiche, J.-J. Risler eds., Progress in Mathematics, 144, Birchuser, 79-323 (1996).

[7] Hambly, B. M.; Lyons, T. J.: Stochastic area for Brownian motion on the Sierpinski gasket. Ann. Probab. 26, no. 1, 132–148, (1998).

[8] Karatzas, I.; Shreve, E.: Brownian Motion and Stochastic Calculus. Second Edition. (1991).

[9] Ledoux, M.; Lyons, T.; Qian, Z.: Lévy area of Wiener processes in Banach spaces. Ann. Probab. 30, no. 2, 546–578. (2002)

[10] Lejay, A.: Introduction to Rough Paths, Séminaire de probabilités. Lecture Notes in Mathematics, vol. XXXVII (2003) .

[11] Lyons, T.: Differential equations driven by rough signals. Rev. Mat. Iberoamericana 14, no. 2, 215–310, (1998).

[12] Lyons, T.; Qian, Z.: System Control and Rough Paths, Oxford University Press (2002).

[13] Reutenauer C.: Free Lie algebras. London Mathematical Society Monographs. New Series, 7. Oxford Science Publications, (1993).

[14] Serre, J.P.: Lie algebras and Lie groups. Lecture Notes in Mathematics, 1500. (1992).

[15] Varadarajan, V. S.: Lie groups, Lie algebras, and their representations. Graduate Texts in Mathematics, 102. (1984).

[16] Victoir N.B.: Levy area for the free Brownian motion: existence and non-existence. to apper in Journal of Functional Analysis.