On Gauss-Green theorem and boundaries of a class of Hölder domains

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Abstract. The purpose of this paper is to show that, if $\alpha > \frac{1}{3}$ and $\epsilon > 0$, the boundary of an $\alpha-$Hölder domain is a $\frac{1}{\alpha} + \epsilon$ geometric rough path; and as a direct application, we extend the classical Green-Gauss' formula to this class of fractal domains.

1. Introduction

Consider a planar Jordan domain D with a C^1-boundary γ, then the Gauss-Green theorem asserts that one can compare the integral of a $1-$form ω over γ with that of a $2-$form $d\omega$ over D, in particular:

$$\int_\gamma \omega = \iint_D d\omega. \tag{1.1}$$

It is clear that the double integral in (1.1) is robust to the underlying domain D if $\omega \in C^1$; however, unless ω is closed, the line integral in (1.1) is sensitive to the underlying curve γ and so there are real difficulties on giving a local definition for this integral. For instance, one can show that there is a smooth non-closable $1-$form ω such that the integral of ω is not a continuous functional on the path space $C[0,T]$ in $\mathbb{C}$ equipped with uniform norm. Nevertheless, Lyons [11] proved that it is possible to extend the notion of the integral of any smooth enough $1-$form ω to geometric rough paths and hence deduced that the Itô functional associated to ω, namely:

$$I_\omega : \gamma \longmapsto \int_\gamma \omega \tag{1.2}$$

is continuous over the space of paths of finite $p-$variation One can refer to Section 2.3 for the definitions of $p-$variation norm and geometric rough paths; or one can consult the notes prepared by Lejay (2003) and the book written by Lyons and Qian (2002) for a more elaborated introduction.

In this paper, we shall identify the boundary of any $\alpha\,(> 1/3) -$ Hölder planar domain D as a geometric rough path which in turn leads to an independent meaning

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to the line integral in (1.1) in accordance with the theory of rough paths; by approximating the domain D from inside, we also generalize the classical Gauss-Green formula via the theory of rough paths.

1.1. Background. For many practical problems arising in analysis or physics, the geometric objects of interest have non-smooth or even fractal boundaries, and so the Gauss-Bonnet-Chern theorem and/or isoperimetric inequalities fail to be applied directly in the study of their structures. The validity of many important theorems in geometry, including the previous two, relies critically on the Stokes' theorem; therefore, we believe it is interesting to extend Stokes' theorem to domains with irregular boundaries. For domains of locally finite perimeter in any finite dimensional Euclidean space, with the class of Lipschitz domains as a special case, it was first treated by Federer (1945, 1958); the key idea was to look at the measure-theoretic exterior normal along the boundary and the corresponding reduced boundary.

The main recent advance in the planar case was first given by Harrison and Norton (1991). Recall that the box dimension of any compact planar curve γ is defined to be

$$\limsup_{\varepsilon\to 0}\left(-\frac{\log N(\varepsilon)}{\log\varepsilon}\right),$$

where $N(\varepsilon)$ is the minimum number of squares of side ε required to cover γ. Notice that the Hausdorff dimension of an arbitrary curve γ can be shown to be less than the box dimension of γ. Now, consider the space $\mathcal{J}_d$ of planar Jordan curves of box dimension $\leq d < 2$. For any $\alpha > d-1$ and a planar Jordan domain D with the boundary $\gamma \in \mathcal{J}_d$, Harrison and Norton defined the integral of an $\alpha-$Hölder continuous, but not necessary differentiable, $1-$form ω along γ to be the integral of a $2-$form $d\widetilde{\omega}$ over D, where the $1-$form $\widetilde{\omega}$ is a smooth extension of ω over D according to the Whitney decomposition theorem; and they also showed that this definition is independent of the choice of $\widetilde{\omega}$. As an immediate consequence of this definition, they extended the classical Gauss-Green theorem to planar domains with boundaries in $\mathcal{J}_d$.

Moreover, Harrison and Norton also deduced the following two properties for all γ_1, $\gamma_2 \in \mathcal{J}_d$:

(1) If γ_1 and γ_2 are disjoint, then

$$\int_{\gamma_1\sqcup\gamma_2}\omega = \int_{\gamma_1}\omega + \int_{\gamma_2}\omega, \tag{1.3}$$

where $\sqcup$ denotes the disjoint union.

(2) Consider a $d-1-$Hölder continuous $1-$form ω; let Y be the region bounded by γ_1 and γ_2 with $\mathcal{Q}$ as a Whitney decomposition, then

$$\left|\int_{\gamma_1}\omega - \int_{\gamma_2}\omega\right| \leq |\omega|_{d-1}\sum_{Q\in\mathcal{Q}}|Q|^d, \tag{1.4}$$

where $|\omega|_{d-1}$ is the $d-1-$Hölder norm of ω.

In summary, their main perspective was the search of the minimum regularity on ω such that the Itô functional (1.2) can be continuously extended from $\mathcal{J}_1$ to $\mathcal{J}_d$ under a suitably chosen measure of closeness between any two curves γ_1 and γ_2 in

$\mathcal{J}_d$; and this measure of closeness is simply the d-sum $\sum_{Q\in\mathcal{Q}}|Q|^d$ over the region bounded by γ_1 and γ_2. However, this measure fails to be a metric in $\mathcal{J}_d$.

Next, we want to compare our work with that of Harrison and Norton. We first consider a class of Weierstrass functions, for any $t\in[0,1]$,

$$W^{\beta}: t\longmapsto\sum_{k=1}^{\infty}\lambda^{-\beta k}\sin\left(\lambda^{k}t\right), \tag{1.5}$$

where $\lambda>0$ and $0<\beta<1$. The box dimension of the graph $W^{\beta}([0,1])$ can be shown to be $2-\beta$; and the function $W^{\beta}(\cdot)$ is $\alpha-$Hölder continuous only for $\alpha<\beta$, and this $W^{\beta}(\cdot)$ also has infinite $p-$variation for $p<1/\beta$. Now, it is clear that one can construct a Jordan curve $l^{\beta}\in\mathcal{J}_{2-\beta}$ by augmenting the planar graph $W^{\beta}([0,1])$ by a non-self-intersecting smooth arc C_{β} such that $W^{\beta}([0,1])\cap C_{\beta}=\left\{W^{\beta}(0),W^{\beta}(1)\right\}$; however, l^{β} fails to be a $p-$geometric rough path for any $p<1/\beta$.

Before we move on, we point out two obvious but important features among continuous paths of finite variation. Consider the Banach space $(\Omega_1,||\cdot||_1)$ of continuous curves of finite variation defined on $[0,T]$ equipped with $1-$variation norm $||\cdot||_1$. Firstly, it can easily be shown that, for any continuous $1-$form ω, the Itô functional (1.2) is continuous over $(\Omega_1,||\cdot||_1)$.

Secondly, for any $\gamma\in\Omega_1$, consider the tensor

$$X(\gamma)_{s,t}=\left(1,X(\gamma)_{s,t}^{1},\ldots,X(\gamma)_{s,t}^{k},\ldots\right) \tag{1.6}$$

(to be formally defined in Section 2.3) of iterated integrals of γ over $[s,t]$, where

$$X(\gamma)_{s,t}^{k}=\int\cdots\int_{s<u_1<\ldots u_k<t}d\gamma_{u_1}\otimes\cdots\otimes d\gamma_{u_k} \tag{1.7}$$

This tensor (1.6) is called the signature of the curve γ over $[s,t]$ and is a continuous functional over the space $(\Omega_1,||\cdot||_1)$. Moreover, by properly partitioning the domain of integration and using the Fubini's theorem, one can also deduce the so-called Chen identity

$$X(\gamma)_{s,t}\otimes X(\gamma)_{t,u}=X(\gamma)_{s,u} \tag{1.8}$$

for $0\le s\le t\le u\le T$. In other words, the mapping $S:(\gamma,*)\longmapsto(X(\gamma),\otimes)$ is a homomorphism, where $*$ is the concatenation of paths such that for any γ_1, $\gamma_2\in\Omega_1$,

$$\gamma_1*\gamma_2=\begin{cases}\gamma_1(2t) & \text{for } 0\le t<T/2\\ \gamma_1(T)-\gamma_2(0)+\gamma_2(2t-T) & \text{for } T/2\le t<T\end{cases}.$$

Furthermore, we say that γ_1 and γ_2 are equivalent, denoted by $\gamma_1\sim\gamma_2$, if there are two reparametrisations τ_1 and τ_2 such that $\gamma_1\circ\tau_1\equiv\gamma_2\circ\tau_2$ on $[0,T]$; according to Chen's works (1957, 1958), the signature (1.6) over $[0,T]$ completely characterizes the underlying path γ up to the equivalence $\sim$. In a nutshell, the concatenation of paths in Ω_1 has an equivalent algebraic analogue, namely the tensor product $\otimes$ over $(\Omega_1,||\cdot||_1)$.

We therefore believe that it is desirable if one can continuously, under a suitable norm, extend both the Itô functional (1.2) and the signature (1.6) simultaneously to a wide class of paths that includes many interesting non-rectifiable curves; and indeed, a positive answer is provided by the theory of rough paths according to which the space of $p-$geometric rough paths equipped with $p-$variation norm can satisfy our concern.

On the other hand, for each $\beta \in (1/3, 1/2]$, there is no apparent meaning for the iterated integral

$$\iint_{0<u_1<u_2<1} dW^{\beta}(u_1)\, dW^{\beta}(u_2) \tag{1.9}$$

in the theory by Harrison and Norton since (1.9) is essentially an integral of a $\beta(\leq 1/2)-$Hölder continuous function against itself and $\beta \not> (2-\beta)-1$. In contrast, for every $\beta \in (1/3, 1/2]$, our Theorem 1 in Subsection 1.2 identifies the graph $W^{\beta}([0,1])$ as a $p-$geometric rough path, for $p > 1/\beta$; in accordance with the theory of rough paths, we can provide a useful meaning to the signature (1.6) of W^{β} over [0.1], in particular to the iterated integral (1.9), and also to any flow controlled by $W^{\beta}(\cdot)$.

Finally, it should be remarked that the class of Jordan curves l^{α} for $\alpha \leq 1/3$ also provides examples that Harrison and Norton can treat easily while our rough path approach is still open.

In conclusion, our work demonstrates an alternative approach, which is also complementary, to that by Harrison and Norton on how to do calculus and geometric analysis on fractals

1.2. Main results. Our first main theorem is the following which characterizes the roughness of the boundaries of a class of $\alpha(>1/3)-$Hölder domains:

THEOREM 1. *For $\frac{1}{3} < \alpha \leq 1$, the boundary of an $\alpha-$Hölder planar domain $\bar{D}$ is a $p-$geometric rough path, for all $p > \frac{1}{\alpha}$.*

In other words, we can approximate the boundary of an $\alpha(>1/3)-$Hölder domain by a sequence of C^1-loops from inside in $\frac{1}{\alpha}+\varepsilon-$variation norm, for any $\varepsilon > 0$. It should be remarked that, unlike Hausdorff or box counting measure, no external measure is required to be defined on the ambient space, only the intrinsic metric are needed to define the $p-$variation norm and hence the roughness p of the domains of interest.

Next, as a direct consequence of our first main theorem, we immediately have:

THEOREM 2 (Gauss-Green). *For $\frac{1}{3} < \alpha \leq 1$, let γ to be the boundary of an $\alpha-$Hölder planar domain $\bar{D}$ and $\omega = \omega_1 dx_1 + \omega_2 dx_2$ be a $1-$form such that for $i = 1, 2$,*

1. *if $\frac{1}{2} < \alpha \leq 1$, $\omega_i \in C_b^1(\mathbb{C})$,*
2. *if $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $\omega_i \in C_b^1(\mathbb{C})$ and all its first order partial derivatives are $\beta-$Hölder continuous with $\beta > \frac{1}{\alpha} - 2$.*

Define $X(\gamma)_{s,t} = \left(1, X(\gamma)^1_{s,t}, X(\gamma)^2_{s,t}\right)$ to be the $p-$geometric rough path, as constructed in the proof of Theorem 1, associated with γ in the sense that

$$X(\gamma)^1_{s,t} = \gamma(t) - \gamma(s) \tag{1.10}$$

for some $p > 1/\alpha$ and $\pi^1(\cdot)$ to be the projection map from.$T^{(2)}(V)$ onto $V^{\otimes 1}$.

Then we have the generalized Green's formula

$$\pi^1\left(\int \omega(X(\gamma))\, dX(\gamma)\right) = \iint_{\bar{D}} d\omega, \tag{1.11}$$

where the integral $\int \omega(X(\gamma))\, dX(\gamma)$ *is the unique* $p-$*rough path* $\widetilde{Y}$ *to be mentioned in Remark 2 following Proposition 3. The value* $\pi^1\left(\int \omega(X(\gamma))\, dX(\gamma)\right)$ *is independent of the choices of both the parametrisation of* γ *and the roughness* p*. On the other hand, the double integral in (1.11) is defined in the usual Lebesgue sense.*

The paper is organized as follows: Section 2 presents some basic definitions and lemmas; and the proofs of our main theorems are presented in Section 3.

2. Preliminary definitions and results

In this section, we prepare some definitions and lemmas that will be needed in the sequel.

2.1. Hölder domains and Hardy-Littlewood Lemma.

Definition 1. *A function* ϕ *is* $\alpha-$*Hőlder continuous with exponent* α $(0 < \alpha \leq 1)$ *on a connected set* $A \subset \mathbb{C}$ *if there is a positive constant* M*, such that for any* z_1*,* $z_1 \in A$*,*

$$|\phi(z_1) - \phi(z_2)| \leq M\, |z_1 - z_2|^{\alpha}\,. \tag{2.1}$$

Furthermore, if $z_1 = e^{i\theta_1}$ *and* $z_2 = e^{i\theta_2}$*, with* $0 \leq \theta_1, \theta_2 < 2\pi$*, then to say that* ϕ *is* $\alpha-$*Hőlder continuous over* $\mathbb{C}$ *is the same as the condition*

$$|\phi(z_1) - \phi(z_2)| \leq M'\, |\arg(z_2 - z_1)|^{\alpha} \leq M'\, |\theta_1 - \theta_2|^{\alpha} \tag{2.2}$$

imposed on ϕ *for some positive* M'*.*

Definition 2. *A simply connected planar domain* $D \subset \mathbb{C}$ *is called an* $\alpha-$*Hőlder domain if it is the conformal image of a unique univalent analytic function* ϕ *over the unit disc* $\mathbb{D}$ *such that* ϕ *is also* $\alpha-$*Hőlder continuous when restricted to* $\partial\mathbb{D}$ *and* $f(\partial\mathbb{D}) = \partial D$*.*

Remark 1. *Notice that the boundary of an* $\alpha-$*Hőlder domain (with* $\alpha < 1$*) need not to be rectifiable, and its boundary may be of infinite length. Therefore, the classical Green's formula has no apparent meaning for this class of 'exotic' domains. We aim to generalize this well-known theorem to these* $\alpha-$*Hőlder domains.*

Next, we introduce a lemma first obtained by Hardy and Littlewood (1932) which characterizes the rate of growth of the first derivative of ϕ of an $\alpha-$Hőlder domain D. For a more recent account of this result and its further applications, see Duren (1970) and Pommerenke (1992).

Lemma 1 (**Hardy-Littlewood**). *Suppose* ϕ *is a univalent analytic function in the interior of the unit disc* $\mathbb{D}$*. Then* ϕ *is continuous in* $\mathbb{D}$ *and* $\phi(e^{i\theta})$ *is* $\alpha(< 1)-$*Hőlder continuous in* θ*, if and only if there is a positive constant* C *such that*

$$\left|\phi'(z)\right| \leq \frac{C}{(1 - |z|)^{1-\alpha}}. \tag{2.3}$$

Proof. One can also consult the book by Duren (1970) on p.74. □

For any $r \leq 1$, we consider $\phi_r(z) = \phi(rz)$ for any $z \in \mathbb{D}$; suppose that $\phi(e^{i\theta})$ is $\alpha-$Hőlder continuous in θ, according to the *Hardy-Littlewood* lemma,

$$\left|\phi'(z)\right| \leq \frac{C}{(1 - |z|)^{1-\alpha}}, \text{ for every } z \in \mathbb{D},$$

for some positive constant C. Differentiating ϕ_r with respect to z, we get $\phi_r{}'(z) = r\phi'(rz)$, and hence

$$\left|\phi_r'(z)\right| \leq \frac{Cr}{(1-r|z|)^{1-\alpha}} \leq \frac{Cr}{(1-|z|)^{1-\alpha}}, \tag{2.4}$$

therefore, applying the converse part of the *Hardy-Littlewood* lemma, we deduce that $\phi_r(e^{i\theta})$ is also $\alpha-$Hőlder continuous in θ, in particular:

$$\left|\phi_r\left(e^{i\theta_1}\right) - \phi_r\left(e^{i\theta_2}\right)\right| \leq Cr\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|\theta_2 - \theta_1|^\alpha, \tag{2.5}$$

or

$$\left|\phi\left(re^{i\theta_1}\right) - \phi\left(re^{i\theta_2}\right)\right| \leq Cr\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|\theta_2 - \theta_1|^\alpha. \tag{2.6}$$

Now, for any two points z_1 and z_2 with $|z_1| = |z_2| = r$, one can express $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ such that $|\theta_1 - \theta_2| \leq \pi$; using the above inequality, we also have

$$|\phi(z_1) - \phi(z_2)| \leq \frac{C\pi^\alpha r^{1-\alpha}}{2^\alpha}\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)|z_2 - z_1|^\alpha. \tag{2.7}$$

Moreover, we also have the increment of such ϕ along any radial direction θ being bounded above by:

$$\begin{aligned} \left|\phi\left(e^{i\theta}\right) - \phi\left(\rho e^{i\theta}\right)\right| &= \lim_{q\uparrow 1}\left|\int_\rho^q \phi'(re^{i\theta})d\left(re^{i\theta}\right)\right| \\ &\leq \int_\rho^{1-} \frac{C}{(1-r)^{1-\alpha}}dr = \frac{C}{\alpha}(1-\rho)^\alpha. \end{aligned} \tag{2.8}$$

2.2. Isoperimetric inequality.

Definition 3. *Let $I = [0,1]$. A loop in a metric space (X,d) is a path $\gamma_\cdot : I \to X$ that begins and ends at the same point, i.e. $\gamma(1) = \gamma(0)$. For instances, for a loop $\gamma_\cdot : I \to \mathbb{C}$ over $\mathbb{C}$, we consider a point $z \in \mathbb{C}$ not lying in the image of $\gamma_\cdot$ and set*

$$g(s) = \frac{\gamma_s - z}{||\gamma_s - z||}. \tag{2.9}$$

Then $g : I \to \mathbb{S}^1$ is a loop in the unit circle. Let $p : \mathbb{R} \to \mathbb{S}^1$ be the standard covering map, i.e. $p(\cdot) = e^{2\pi\cdot}$ and $\tilde{g} : I \to \mathbb{R}$ be a lifting of g to $\mathbb{S}^1$, i.e. $p \circ \tilde{g} = g$. Since g is a loop, the difference $\tilde{g}(1) - \tilde{g}(0)$ can be shown to be an integer, which is called the winding number of $\gamma_\cdot$ with respect to z and is denoted by $\eta(\gamma_\cdot, z)$.

As a remark, the notion of winding number can be shown to be independent of the choice of the lifting of g. In particular, if $\tilde{g}$ is one of such lifting of g, then the uniqueness of liftings implies that any other lifting of g has the form $\tilde{g}(s) + m$ for some integer m. Intuitively, even a formal proof is not so simple, for any Jordan curve $\gamma_\cdot$, $|\eta(\gamma_\cdot, \cdot)| \leq 1$, for each $z \in \gamma_\cdot^c$. In addition, for any rectifiable curve $\gamma_\cdot$, $\eta(\gamma_\cdot, \cdot)$ is a measurable function since $\gamma_\cdot^c$ is a countable union of connected open components.

Definition 4. *With the above notation, let $\Gamma : I \times I \to X$ be a continuous map such that $\Gamma(0, \cdot) = \Gamma(1, \cdot)$, that is to say, for each $t \in I$, the map $\Gamma(\cdot, t)$ is a loop in X. The map Γ is then called a free homotopy between loops $\Gamma(\cdot, 0)$ and $\Gamma(\cdot, 1)$.*

LEMMA 2. *Let $\gamma_{\cdot}$ be a loop in $\mathbb{C}/\{z\}$. If $\gamma_{\cdot}$ is freely homotopic to another loop $\gamma'_{\cdot}$, through loops lying in $\mathbb{C}/\{z\}$, then $\eta(\gamma_{\cdot}, z) = \eta(\gamma'_{\cdot}, z)$.*

PROOF. Suppose Γ to be a free homotopy between $\gamma_{\cdot}$ and $\gamma'_{\cdot}$. Define $G : I \times I \to \mathbb{S}^1$ by

$$G(s,t) = \frac{\Gamma(s,t) - z}{||\Gamma(s,t) - z||}$$

for $(s,t) \in I \times I$. Let $\tilde{G}$ be a lifting of G to $\mathbb{R}$. Then $\tilde{G}(1,t) - \tilde{G}(0,t)$ is an integer for every t. Since $\tilde{G}(1,\cdot) - \tilde{G}(0,\cdot)$ is continuous, hence the image of $\tilde{G}$ is connected and therefore it is a constant. □

Next, we will introduce a generalized version of the isoperimetric inequality - the so-called Pohl-Banchoff inequality, first proven by Pohl and Banchoff in [1], and then Vogt (1981) obtained a simpler proof; and point out the relationships, which first discovered by Rado (1936), between the Levy area for a closed rectifiable curve γ and the integral of winding numbers over the domain bounded by γ. For the readers' convenience as well as their crucial role in the proof of our main theorem, we include a shorter proof for its essential parts in the Appendix.

PROPOSITION 1. *With the previous notation, let $\gamma_{\cdot}$ be a closed rectifiable curve in $\mathbb{C}$ of length l and D be the region enclosed by this curve γ, i.e. $\partial D = \gamma$. Then, one can have*

(1) *the Pohl-Banchoff inequality*

$$2\pi i \iint_{\mathbb{C}} \eta^2(\gamma_{\cdot}, \zeta)\, d\zeta \wedge d\bar{\zeta} = 4\pi \iint_{\mathbb{C}} \eta^2(\gamma_{\cdot}, \zeta)\, \lambda(dA) \le l^2, \tag{2.10}$$

where the equality holds if and only if $\gamma_{\cdot} = z_0 + R\, e^{2n\pi i \cdot}$ for some $n \in \mathbb{Z}$, $z_0 \in \mathbb{C}$, and $R > 0$.

(2)

$$\iint_{\mathbb{C}} \eta(\gamma_{\cdot}, z)\, \lambda(dA) = \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right). \tag{2.11}$$

PROOF. See the Appendix. □

2.3. Some useful terminologies and results in the theory of rough paths. For a detailed introduction to the subject of this subsection, one can refer to the book written by Lyons and Qian (2002) and the lecture notes prepared by Lejay (2003). All the proofs of the results in this subsection can be found in the text by Lyons and Qian (2002)

Consider a separable Banach space $(V, |\cdot|_V)$ and its family of algebraic tensor products $V^{\otimes_a k} \equiv V \otimes_a \cdots \otimes_a V$ (total of k copies) with tensor norms $|\cdot|_k$ which together satisfy the compatibility condition:

(1) $|\cdot|_1 = |\cdot|_V$,
(2) $|v \otimes w|_{k,l} \le |v|_k \cdot |w|_l$, where $v \in V^{\otimes_a k}$ and $w \in V^{\otimes_a l}$.

The completion of the algebraic tensor product $V^{\otimes_a k}$ under the norm $|\cdot|_k$ is denoted by $(V^{\otimes k}, |\cdot|_k)$ or $V^{\otimes k}$ for short. For each $N \in \mathbb{N}$, we define the (truncated) tensor algebra on V, $\left(T^{(N)}(V), \otimes\right)$ or $T^{(N)}(V)$ for short if there is not cause of ambiguity, to be the sum of all tensor products up to order N, i.e.:

$$T^{(N)}(V) \equiv \oplus_{k=0}^{N} V^{\otimes k}, \text{ where } V^{\otimes 0} \equiv \mathbb{R} \text{ and } V^{\otimes 1} \equiv V. \tag{2.12}$$

Its multiplication is taken to be the same as that for polynomials, except that the higher order ($\geq$ degree n) terms are omitted, that is to say that if $\xi = (\xi^0, \xi^1, \ldots, \xi^N)$ and $\eta = (\eta^0, \eta^1, \ldots, \eta^N)$ are elements in $T^{(N)}(V)$, their product $\zeta = (\zeta^0, \zeta^1, \ldots, \zeta^N)$ is simply:

$$\zeta^k = \sum_{i=0}^{k} \xi^i \otimes \eta^{k-i}, \text{ for } k = 1, 2, \ldots, N. \tag{2.13}$$

Finally, the norm $|\cdot|$ on $T^{(N)}(V)$ is defined to be

$$|\xi| = \sum_{k=0}^{N} \left|\xi^k\right|_k. \tag{2.14}$$

For simplicity, we use Δ_T to denote the simplex $\{(s,t) : 0 \leq s < t \leq T\}$.

Definition 5. *A continuous map $X. : \Delta_T \to T^{(N)}(V)$ is said to be a multiplicative functional of degree $N \in \mathbb{N}$ $(n \geq 1)$ if for each $(s,t) \in \Delta_T$,*

$$X_{s,t} = (1, X^1_{s,t}, \ldots, X^N_{s,t}), \tag{2.15}$$

where $X^k_{s,t} \in V^{\otimes k}$ for $k = 2, \ldots, N$ and it also satisfies the Chen identity:

$$X_{s,t} \otimes X_{t,u} = X_{s,u} \tag{2.16}$$

for $(s,t), (t,u), (s,u) \in \Delta_T$. We use $C_0\left(\Delta_T, T^{(N)}(V)\right)$ to denote the space of all multiplicative functionals of degree N.

Definition 6. *A multiplicative functional $X. = (1, X^1_., \ldots, X^N_.)$ in $T^{(N)}(V)$, is said to be of finite $p-$variation if*

$$\sup_D \sum_l \left|X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}} < +\infty, \quad \forall i = 1, \ldots, N. \tag{2.17}$$

where $\sup_D$ runs over all finite partitions $D = \{0 = t_0 \leq \cdots \leq t_r = T\}$ of $[0,T]$. We use $C_{0,p}\left(\Delta_T, T^{(N)}(V)\right) \subset C_0\left(\Delta_T, T^{(N)}(V)\right)$ to denote the space of all multiplicative functionals of finite $p-$variation.

Definition 7. *A multiplicative functional $X.$ of degree N is said to be a $p-$rough path if it is of finite $p-$variation with $[p] \leq N$. We use $\Omega_p(V)$ to denote the space of all $p-$rough paths of degree $[p]$ in the separable Banach space V. Furthermore, if there is a sequence of Lipschitz paths $\{\gamma(n)\}_{n\in\mathbb{N}}$ such that their respective multiplicative functional of degree $[p]$ is:*

$$X(n). = (1, X(n)^1_., \ldots, X(n)^{[p]}_.),$$

where $X^k(n)$ is the $k^{th}-$order iterated integral over the path $\gamma(n)$, i.e.

$$X(n)^k_{s,t} = \int_{s<u_1<\ldots u_k<t} \cdots \int d\gamma(n)_{u_1} \otimes \cdots \otimes d\gamma(n)_{u_k}, \text{ for } k = 1, \ldots, [p]$$

and $X(n)$ converges to X in the sense that:

$$\sup_{1\leq i\leq[p]} \sup_D \left(\sum_l \left|X(n)^i_{t_{l-1},t_l} - X^i_{t_{l-1},t_l}\right|^{\frac{p}{i}}\right)^{\frac{i}{p}} \longrightarrow 0, \text{ as } n \longrightarrow 0, \tag{2.18}$$

we then call $X.$ to be a $p-$geometric rough path. We also use $G\Omega_p(V) \subset \Omega_p(V)$ to denote the space of all $p-$geometric rough paths in the space V.

DEFINITION 8. *A non-negative continuous function* ω *on* Δ_T *is called a control if*

(1) ω *is superadditive, namely:*

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u)$$

for $(s,t),(t,u) \in \Delta_T$,

(2) $\omega(t,t) = 0$ *for all* $t \in [0,T]$.

DEFINITION 9. *For* $p \geq 1$, *a multiplicative functional* $X \in C_0\left(\Delta_T, T^{([p])}(V)\right)$ *is called an almost* $p-$*rough path if:*

(1) $X \in C_{0,p}\left(\Delta_T, T^{([p])}(V)\right)$,

(2) *there is a control* ω *and a constant* $\theta > 1$, *such that*

$$\left|(X_{s,t} \otimes X_{t,u})^k - X_{s,u}^k\right| \leq \omega(s,u)^\theta \tag{2.19}$$

for all $(s,t),(t,u) \in \Delta_T$ *and* $k = 1,\ldots,[p]$.

PROPOSITION 2. *Suppose that* $X \in C_{0,p}\left(\Delta_T, T^{([p])}(V)\right)$ *is an almost rough path, then there is a unique rough path* $\widetilde{X} \in \Omega_p(V)$ *such that*

$$\left|\widetilde{X}_{s,t}^k - X_{s,t}^k\right| \leq \omega(s,t)^\theta \tag{2.20}$$

for all $(s,t) \in \Delta_T$, $k = 1,\ldots,[p]$ *and a constant* $\theta > 1$.

DEFINITION 10. *The p-variation metric* d_p *on* $C_{0,p}\left(\Delta_T, T^{([p])}(V)\right)$ *is defined by*

$$d_p(X,Y) = \max_{1\leq k\leq [p]} \sup_D \left(\sum_i \left|X_{t_{i-1},t_i}^k - Y_{t_{i-1},t_i}^k\right|^{p/k}\right)^{k/p},$$

where $X, Y \in C_{0,p}\left(\Delta_T, T^{([p])}(V)\right)$ *and the induced topology is called* $p-$*variation topology..*

LEMMA 3. $(\Omega_p(V), d_p)$ *is a complete metric space.*

DEFINITION 11. *Fix a* $p \geq 1$. *Consider two separable Banach spaces* V *and* W *together with their Banach tensor product spaces up to degree* $[p]$, *namely:* $V^{\otimes 2},\ldots,V^{\otimes [p]}$ *and* $W^{\otimes 2},\ldots,W^{\otimes [p]}$ *respectively. Suppose that* $p < \gamma \leq [p]+1$, *consider a one-form* $\alpha : V \to L(V,W)$, *a system* $\left(\alpha, V^{\otimes j}, W^{\otimes j} : 1 \leq j \leq [p]\right)$ *is said to be admissible if:*

(1) α *is a Lip*(γ) *one-form in the sense that, for* $j = 1,\ldots,[p]$, *there exist functions i)* $\alpha^j : V \to L\left(V^{\otimes j}, W\right)$ *and ii)* $R_j : V \times V \to L\left(V^{\otimes j}, W\right)$ *such that*
   - (a) $\alpha^1 = \alpha$
   - (b) *For any Lipschitz path* $X_{s,t} = \left(1, X_{s,t}^1, \ldots, X_{s,t}^{[p]}, \ldots\right)$ *in* V,

$$\alpha^j(X_t) = \sum_{i=0}^{[p]-j} \alpha^{j+i}(X_s)\left(X_{s,t}^i\right) + R_j(X_s, X_t), \tag{2.21}$$

$$\alpha^j(X_t) - \alpha^j(X_s) = \int_s^t \alpha^{j+1}(X_u)(dX_u) \tag{2.22}$$

for all $s < t$ and for any $\xi, \eta \in V$, there is a positive M, called a Lipschitz constant of α, such that:

$$\left\|\alpha^1(\xi)\right\| \leq M(1+|\xi|), \ \left\|\alpha^{j+1}(\xi)\right\| \leq M, \text{ for } j = 1, \dots, [p]-1$$

$$\|R_j(\xi,\eta)\| \leq M|\xi-\eta|^{\gamma-j}, \text{ for } j = 1, \dots, [p]$$

(2) *For all $\mathbf{j} = (j_1, \dots j_k)$ such that j_i are non-negative integers and $|\mathbf{j}| = \sum_{i=1}^{k} j_i \leq [p]$, for each $\xi \in V$, the linear operator*

$$\alpha^{j_1}(\xi) \otimes \cdots \otimes \alpha^{j_k}(\xi) : V^{\otimes j_1} \otimes \cdots \otimes V^{\otimes j_k} \to W^{\otimes j_1} \otimes \cdots \otimes W^{\otimes j_k}$$

is bounded with the above mentioned M, where for each $v_i^{j_l} \in V^{\otimes j_l}$

$$\alpha^{j_1}(\xi) \otimes \cdots \otimes \alpha^{j_k}(\xi)\left(\sum_i v_i^{j_1} \otimes \cdots \otimes v_i^{j_k}\right) = \sum_i \alpha^{j_1}(\xi)\left(v_i^{j_1}\right) \otimes \cdots \otimes \alpha^{j_k}(\xi)\left(v_i^{j_k}\right).$$

Condition 1. *Set $[p] = 2$ and $\left(\alpha, V^{\otimes j}, W^{\otimes j} : j = 1, 2\right)$ to be admissible.*

Proposition 3. *Under the Condition 1, for any $X \in \Omega_p(V)$, consider a multiplicative functional $Y \in C_0\left(\Delta_T, T^{(2)}(W)\right)$ defined by:*

$$Y^1_{s,t} = \alpha^1(X_s)\left(X^1_{s,t}\right) + \alpha^2(X_s)\left(X^2_{s,t}\right), \tag{2.23}$$

$$Y^2_{s,t} = \alpha^1(X_s) \otimes \alpha^1(X_s)\left(X^2_{s,t}\right) \tag{2.24}$$

for all $(s,t) \in \Delta_T$. Then Y is an almost $p-$rough path in $T^{(2)}(W)$.

Remark 2. *According to Proposition 2, there is a unique rough path $\widetilde{Y} \in \Omega_p(W)$ associated with this almost $p-$rough path Y. This rough path $\widetilde{Y}$ is called the integral of the one-form α against the rough path X and is denoted by $\int \alpha(X)\,dX$. It is clear that if X is a Lipschitz path, then*

$$\left(\int \alpha(X)\,dX\right)^1_{s,t} = \int_s^t \alpha(X_u)(dX_u), \tag{2.25}$$

$$\left(\int \alpha(X)\,dX\right)^2_{s,t} = \iint_{s<u_1<u_2<t} \alpha(X_{u_1})(dX_{u_1}) \otimes \alpha(X_{u_2})(dX_{u_2}), \tag{2.26}$$

where the integrals on the RHS are in usual Riemann sense.

Definition 12. *The integration operator defined by*

$$\int \alpha : \Omega_p(V) \to \Omega_p(W) \tag{2.27}$$

such that for all $X \in \Omega_P(V)$

$$\left(\int \alpha\right)(X) = \int \alpha(X)\,dX \tag{2.28}$$

is called the Itô functional associated with α.

Proposition 4 (Continuity of Itô funtionals). *Under the Condition 1, the Itô funtional associated with α is a continuous map from $\Omega_p(V)$ to $\Omega_p(W)$ in $p-$variation topology.*

3. The proofs of our main theorems

3.1. Proof of Theorem 1. For the case $p = 1$, this is obvious. For $p < 1$, according to the definition, there is a surjective univalent analytic function $\phi : \overline{\mathbb{D}} \to \bar{D}$, i.e. $\phi\left(\overline{\mathbb{D}}\right) = \bar{D}$, such that $\phi\left(e^{i\theta}\right)$is an $\alpha-$Hölder continuous function in θ. As before, we denote $\phi\left(re^{i\cdot}\right)$ by $\phi_r\left(e^{i\cdot}\right)$; also, let $\Delta = \{(\theta, \varphi) : 0 \le \theta \le \varphi \le 2\pi\}$.
(I): For the increment process: for any $\beta, \gamma > 0$, taking $\eta = \beta + \delta$, we consider the modulus of the difference, for any $(\theta, \varphi) \in \Delta$

$$\begin{aligned} & \left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^\eta \\ \le\ & 2^{\beta+\delta-2}\left(\left|\phi\left(\rho e^{i\theta}\right) - \phi\left(e^{i\theta}\right)\right|^\beta + \left|\phi\left(\rho e^{i\varphi}\right) - \phi\left(e^{i\varphi}\right)\right|^\beta\right) \\ & \cdot\left(\left|\phi\left(\rho e^{i\theta}\right) - \phi\left(\rho e^{i\varphi}\right)\right|^\delta + \left|\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right|^\delta\right); \end{aligned}$$

using inequalities (2.6) and (2.8), we then obtain:

$$\begin{aligned} & \left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^\eta \\ \le\ & 2^{\eta-1}\frac{C\eta}{\alpha^\beta}\left(1+\rho^\gamma\right)\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^\gamma \cdot (1-\rho)^{\alpha\beta} \cdot |\theta - \varphi|^{\alpha\gamma}. \end{aligned}$$

Now, if one chooses β to be an arbitrary positive number ϵ and $\delta = \frac{1}{\alpha}$, we get

$$\begin{aligned} & \left|\phi_\rho\left(e^{i\theta}\right) - \phi_\rho\left(e^{i\varphi}\right) - \left(\phi\left(e^{i\theta}\right) - \phi\left(e^{i\varphi}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\ \le\ & 2^{\left(\frac{1}{\alpha}-1\right)+\epsilon}\frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon}\left(1+\rho^{\frac{1}{\alpha}}\right)\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}} \cdot (1-\rho)^{\alpha\epsilon} \cdot |\varphi - \theta|. \end{aligned}$$

Therefore, for any partition $\mathcal{D} = \{0 = \theta_0 < \cdots < \theta_n = 2\pi\}$ of $[0, 2\pi]$, the partial sum:

$$\begin{aligned} & \sum_{i=0}^{n-1}\left|\phi_\rho\left(e^{i\theta_i}\right) - \phi_\rho\left(e^{i\theta_{i+1}}\right) - \left(\phi\left(e^{i\theta_i}\right) - \phi\left(e^{i\theta_{i+1}}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon} \\ \le\ & 2^{\left(\frac{1}{\alpha}-1\right)+\epsilon}\frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon}\left(1+\rho^{\frac{1}{\alpha}}\right)\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}}(1-\rho)^{\alpha\epsilon} \cdot \sum_{i=0}^{n-1}|\theta_{i+1} - \theta_i| \\ =\ & 2^{\frac{1}{\alpha}+\epsilon}\pi\frac{C^{\frac{1}{\alpha}+\epsilon}}{\alpha^\epsilon}\left(1+\rho^{\frac{1}{\alpha}}\right)\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)^{\frac{1}{\alpha}}(1-\rho)^{\alpha\epsilon}, \end{aligned}$$

which in turn implies that

$$\sup_{\mathcal{D}}\sum_{i=0}^{n-1}\left|\phi_\rho\left(e^{i\theta_i}\right) - \phi_\rho\left(e^{i\theta_{i+1}}\right) - \left(\phi\left(e^{i\theta_i}\right) - \phi\left(e^{i\theta_{i+1}}\right)\right)\right|^{\frac{1}{\alpha}+\epsilon}$$

converges to zero as ρ tends to 1.
(II): For the Levy area process: for simplicity, we identify the image of $\phi\left(e^{i\cdot}\right)$ as $\gamma_\cdot$, and that of $\phi_r\left(e^{i\cdot}\right)$ as $\gamma^r_\cdot$ for $0 \le r \le 1$; in particular, $.\gamma^1_\cdot \equiv \gamma$. Denote the arc of $\gamma^r_\cdot$ from θ to φ by $\gamma^r_{\theta,\varphi}$ and the closed path starting from the point γ^r_θ along $\gamma^r_\cdot$ to γ^r_φ, and then back to γ^r_θ along the chord joining γ^r_θ and γ^r_φ, by $\underrightarrow{\gamma^r_{\theta,\varphi}}$.

In their paper, Jones and Markarov (1995) proved that the Hausdorff dimension (or the Minkowski dimension) of the boundary of the image of the disc under a uniformly α-Hőlder continuous univalent function does not exceed $2 - C\alpha$, where C is a universal constant. As a consequence, the Lebesgue measure of $\gamma_\cdot$ is zero;

indeed, for each $\epsilon > 0$, if the Hausdorff measures $H^{2-C\alpha+\epsilon}(\gamma_\cdot)$ of $\gamma_\cdot$ is M_ϵ, one can find a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with $\delta_n \to 0$, and a sequence of open-balls covers $\{U_{i,n}\}_{(i,n)\in\mathbb{N}\times\mathbb{N}}$ of $\gamma_\cdot$ such that $|U_{\cdot,n}| \le \delta_n$ and $\sum_{i=1}^{\infty}|U_{i,n}|^{2-C\alpha+\epsilon} \le M_\epsilon + \epsilon$. Now, $\lambda(\gamma_\cdot) \le \sum_{i=1}^{\infty}\lambda(U_{i,n}) \le \frac{\pi}{4}\sum_{i=1}^{\infty}|U_{i,n}|^2 \le \frac{\pi}{4}\delta_n^{C\alpha-\epsilon}\sum_{i=1}^{\infty}|U_{i,n}|^{2-C\alpha+\epsilon} \le \frac{\pi}{4}(M_\epsilon+\epsilon)\delta_n^{C\alpha-\epsilon}$; choosing $\epsilon < C\alpha$ and passing n to ∞, one deduce that $\lambda(\gamma_\cdot) = 0$. Besides, since each $\gamma_\cdot^r$ is a rectifiable loop, and therefore its Lebesgue measure is also zero.

Consider a family of sets $\Omega_{\theta,\varphi} = \mathbb{C}/\cup_{r\in\mathbb{Q}\cap[0,1]}\underrightarrow{\gamma_{\theta,\varphi}^r}$ indexed by $(\theta,\varphi)\in\Delta$, note that, as a consequence of the previous arguments, $\lambda\left(\cup_{r\in\mathbb{Q}\cap[0,1]}\underrightarrow{\gamma_{\theta,\varphi}^r}\right) = 0$. For each $(\theta,\varphi)\in\Delta$ and any $z\in\Omega_{\theta,\varphi}$ with $\left\|z-\underrightarrow{\gamma_{\theta,\varphi}}\right\| > \epsilon > 0$, since ϕ is continuous in $\overline{\mathbb{D}}$ and $\gamma_{\theta,\varphi}$ is compact, and so $\underrightarrow{\gamma_{\theta,\varphi}^r}$ uniformly converges to $\underrightarrow{\gamma_{\theta,\varphi}}$, and hence all but finitely many $\underrightarrow{\gamma_{\theta,\varphi}^r}$ lie inside the $\epsilon/2-$neighborhood/sausage of $\underrightarrow{\gamma_{\theta,\varphi}}$ which excludes the point z. Now, for all large enough r, $\underrightarrow{\gamma_{\theta,\varphi}^r}$ is freely homotopic to $\underrightarrow{\gamma_{\theta,\varphi}}$, therefore, according to Lemma 2, we have $\eta\left(\underrightarrow{\gamma_{\theta,\varphi}^r}, z\right) = \eta\left(\underrightarrow{\gamma_{\theta,\varphi}}, z\right)$, and

$$\eta\left(\underrightarrow{\gamma_{\theta,\varphi}}, z\right) = \lim_{r\in\mathbb{Q},\ r\to 1}\eta\left(\underrightarrow{\gamma_{\theta,\varphi}^r}, z\right), \text{ for all } z\in\Omega_{\theta,\varphi} \text{ with } (\theta,\varphi)\in\Delta. \tag{3.1}$$

For instance, for each $(\theta,\varphi)\in\Delta$, $\eta\left(\underrightarrow{\gamma_{\theta,\varphi}},\cdot\right)$, as the a.e. pointwise limit of $\eta\left(\underrightarrow{\gamma_{\theta,\varphi}^r},\cdot\right)$, is also a measurable function.

(i): Pointwise convergence of $A_{\theta,\varphi}^r$

For any $r_1 < r_2 \in \mathbb{Q}$, we now consider the difference of Levy areas $A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}$. Denote $S_{\theta,\varphi}^r$ to be the sector

$$\left\{z\in\mathbb{C} : z = \rho e^{ix},\ where\, 0\le\rho\le r\ and\ \theta\le x\le\varphi\right\}$$

and $D_{\theta,\varphi}^{r,R} = S_{\theta,\varphi}^R / S_{\theta,\varphi}^r$, for any $r\le R$. Define $Q_{\theta,\varphi}^{r,R}$ to be the interior bounded by the quadrilateral with vertices $\phi\left(re^{i\theta}\right), \phi\left(re^{i\varphi}\right), \phi\left(Re^{i\theta}\right)$, and $\phi\left(Re^{i\varphi}\right)$. Also define $W_\theta^{r,R}$ as the interior bounded by the curves $\left\{\phi\left(\rho e^{i\theta}\right) : r\le\rho\le R\right\}$ and the chord joining $\phi\left(re^{i\theta}\right)$ and $\phi\left(R\, e^{i\theta}\right)$. For any $0<\rho<r_1$, we can split the required difference into:

$$\begin{aligned}2\left(A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}\right) &= \left(\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right)} - \int_{\partial Q_{\theta,\varphi}^{r_1,r_2}}\right)(x_s dy_s - y_s dx_s) \\ &\quad - \left(\int_{\partial W_\theta^{r_1,r_2}} + \int_{\partial W_\varphi^{r_1,r_2}}\right)(x_s dy_s - y_s dx_s).\end{aligned}$$

Now, since $\partial Q_{\theta,\varphi}^{r_1,r_2}$ can be decomposed into two triangular paths $(T_1)_{\theta,\varphi}^{r_1,r_2}$ and $(T_2)_{\theta,\varphi}^{r_1,r_2}$ with respective vertices $\phi\left(r_1e^{i\theta}\right), \phi\left(r_1e^{i\varphi}\right), \phi\left(r_2e^{i\varphi}\right)$ and $\phi\left(r_1e^{i\theta}\right), \phi\left(r_2e^{i\theta}\right)$,

$\phi\left(r_2 e^{i\varphi}\right)$, we then have

$$\begin{aligned}\left|\int_{\partial Q_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s\right| &\leq \left|\int_{(T_1)_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s\right| + \left|\int_{(T_1)_{\theta,\varphi}^{r_1,r_2}} x_s dy_s - y_s dx_s\right| \\ &\leq \frac{1}{2}\frac{C^2}{\alpha}\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)(r_1+r_2)(r_2-r_1)^{\alpha}|\varphi-\theta|^{\alpha}.\end{aligned}$$

Using the Pohl-Banchoff inequality in Proposition 1 and the fact that the image $\phi\left(\overline{\mathbb{D}}\right)$ lies in a compact set in $\mathbb{C}$, we also have

$$\begin{aligned}\left|\int_{\partial W_{\cdot}^{r_1,r_2}} x_s dy_s - y_s dx_s\right| &\leq \iint_{\mathbb{C}} \left|\eta\left(\partial W_{\cdot}^{r_1,r_2}, z\right)\right| \lambda(dA) \\ &\leq \left\|\eta\left(\partial W_{\cdot}^{r_1,r_2}, \cdot\right)\right\|_2 \\ &\leq \frac{1}{\sqrt{4\pi}} l\left(\partial W_{\cdot}^{r_1,r_2}\right) \\ &\leq \frac{2}{\sqrt{4\pi}}\frac{C}{\alpha}(r_2-r_1)^{\alpha}.\end{aligned}$$

Therefore, we now have

$$\begin{aligned}2\left|A_{\theta,\varphi}^{r_2} - A_{\theta,\varphi}^{r_1}\right| \leq\ & \left|\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right)} (x_s dy_s - y_s dx_s)\right| \\ &+ \left(\frac{C^2}{\alpha}\left(2\pi + \frac{2}{\alpha}(2\pi)^{\alpha}\right) + \frac{4}{\sqrt{4\pi}}\frac{C}{\alpha}\right)(r_2-r_1)^{\alpha}.\end{aligned}$$

Finally, since, for any $0 < r < R \leq 1$, every $\phi\left(D_{\theta,\varphi}^{r,R}\right)$ is a Jordan domain, therefore $\eta\left(\partial\phi\left(D_{\theta,\varphi}^{r,R}\right), \cdot\right)$ takes value 1 in the interior and zero in the exterior, i.e. $\left|\eta\left(\partial\phi\left(D_{\theta,\varphi}^{r,R}\right), \cdot\right)\right| \leq 1$; in particular, as the image $\phi\left(\overline{\mathbb{D}}\right)$ lies in a compact set in $\mathbb{C}$, $\iint_{\mathbb{C}} \eta\left(\partial D_{\theta,\varphi}^{r,1}, z\right)\lambda(dA)$ is well-defined and bounded for all r. Because $\partial\phi\left(D_{\theta,\varphi}^{\rho,r}\right)$ uniformly converges to $\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right)$, so $\eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right), \cdot\right) = \lim_{r\to 1}\eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r}\right), \cdot\right)$; as a consequence of the well-known Bounded Convergence Theorem, we conclude that the integral

$$\iint_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right), z\right)\lambda(dA) = \lim_{r\to 1}\iint_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right), z\right)\lambda(dA).$$

For instance, for any $\varepsilon > 0$, there is a positive δ, such that whenever $1-\delta < r_1 < r_2 \leq 1$, we have

$$\left|\iint_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right), z\right)\lambda(dA) - \iint_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right), z\right)\lambda(dA)\right| < \varepsilon.$$

By virtue of Proposition 1 again, we obtain

$$\begin{aligned}&\left|\int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_2}\right)} - \int_{\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right)} (x_s dy_s - y_s dx_s)\right| \\ =\ &\left|\iint_{\mathbb{C}} \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,r_1}\right), z\right) - \eta\left(\partial\phi\left(D_{\theta,\varphi}^{\rho,1}\right), z\right)\lambda(dA)\right| \\ <\ &\varepsilon.\end{aligned}$$

Henceforth, for each $(\theta, \varphi) \in \Delta$, the pointwise convergence of $A^r_{\theta,\varphi}$ follows for r over $\mathbb{Q}$. From now on, we denote the limit by $A_{\theta,\varphi} = \lim_{r\in\mathbb{Q},\, r\to 1} A^r_{\theta,\varphi}$, as a function of $(\theta, \varphi) \in \Delta$.

(ii): Equicontinuity of $A^r_{\theta,\varphi}$

Here, we only present the proof that the difference $\left|A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi}\right|$ can be made arbitrarily small, uniformly for all $r \in \mathbb{Q}$, whenever θ_1 and θ_2 are close enough; the proof for the general case is similar but too cumbersome to include here. Suppose $0 \le \theta_1 \le \theta_2 \le \varphi \le 2\pi$, and $x^r_{\cdot} = Re\left(\phi\left(re^{i\cdot}\right)\right)$ and $y^r_{\cdot} = Im\left(\phi\left(re^{i\cdot}\right)\right)$, then a direct calculation gives

$$A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi} = A^r_{\theta_1,\theta_2} + \frac{1}{2}\left(\left(x^r_{\theta_2} - x^r_{\theta_1}\right)\left(y^r_{\varphi} - y^r_{\theta_1}\right) - \left(x^r_{\varphi} - x^r_{\theta_2}\right)\left(y^r_{\theta_2} - y^r_{\theta_1}\right)\right)$$

Now, applying the inequality (2.6),

$$\begin{aligned} & \left|\left(x^r_{\theta_2} - x^r_{\theta_1}\right)\left(y^r_{\varphi} - y^r_{\theta_1}\right) - \left(x^r_{\varphi} - x^r_{\theta_2}\right)\left(y^r_{\theta_2} - y^r_{\theta_1}\right)\right| \\ \le\ & \left|\phi\left(re^{i\varphi}\right) - \phi\left(re^{i\theta_2}\right)\right|\left|\phi\left(re^{i\theta_2}\right) - \phi\left(re^{i\theta_1}\right)\right| \\ \le\ & \left(C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)^2 (2\pi)^{\alpha} \left|\theta_2 - \theta_1\right|^{\alpha}. \end{aligned}$$

For $\rho < 1$, we have for any $z = |z|\, e^{it} \in D^{\rho r, r}_{\theta_1,\theta_2}$, the difference

$$\begin{aligned} \left|\phi(z) - \phi\left(re^{i\theta_1}\right)\right| &\le \left|\phi\left(|z|\, e^{it}\right) - \phi\left(re^{it}\right)\right| + \left|\phi\left(re^{it}\right) - \phi\left(re^{i\theta_1}\right)\right| \\ &\le \frac{C}{\alpha} r^{\alpha}(1-\rho)^{\alpha} + Cr\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha}; \end{aligned}$$

if we now choose $\rho = 1 - \frac{\theta_2 - \theta_1}{2\pi}$, then

$$\left|\phi(z) - \phi\left(re^{i\theta_1}\right)\right| \le C\left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha},$$

that is to say that if $\rho = 1 - \frac{\theta_2 - \theta_1}{2\pi}$, $\phi\left(D^{\rho r, r}_{\theta_1,\theta_2}\right)$ lies in the ball centered at $\phi\left(re^{i\theta_1}\right)$ with radius $C\left((2\pi)^{1-\alpha} + \frac{3}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha}$. In addition, with this value of ρ, the length of the $\phi-$image of a path $\Gamma^{\rho r, r}_{\theta_1,\theta_2}$, where $\Gamma^{\rho r, r}_{\theta_1,\theta_2}$ is a path starts at $re^{i\theta_2}$, running radially to $\rho re^{i\theta_2}$, and then moves along the circular arc with radius ρ to $\rho re^{i\theta_1}$, and then reach $re^{i\theta_2}$ along the radial direction, is

$$\begin{aligned} \left|\phi\left(\Gamma^{\rho r, r}_{\theta_1,\theta_2}\right)\right| &\le 2\frac{C}{\alpha}(r - \rho r)^{\alpha} + C\rho r\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\left|\theta_2 - \theta_1\right|^{\alpha} \\ &\le \left(\frac{2C}{\alpha} + C\left((2\pi)^{1-\alpha} + \frac{2}{\alpha}\right)\right)\left|\theta_2 - \theta_1\right|^{\alpha}. \end{aligned}$$

Denote the chord joining $\phi\left(re^{i\theta_2}\right)$ and $\phi\left(re^{i\theta_1}\right)$ by $\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}$. Again, using the Pohl-Banchoff inequality in Proposition 1 and the fact that $\phi\left(D^{\rho r, r}_{\theta_1,\theta_2}\right)$ is a

Jordan domain, we deduce that

$$\begin{aligned}
\left|A^r_{\theta_1,\theta_2}\right| &\leq \left|\left(\int_{\partial\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right)} - \int_{\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right)\cup\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}}\right)(x_s dy_s - y_s dx_s)\right| \\
&\leq \iint_{\mathbb{C}} \left|\eta\left(\partial\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right), z\right)\right| \lambda(dA) \\
&\quad + \iint_{\mathbb{C}} \left(\eta\left(\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right)\cup\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}, z\right)\right)^2 \lambda(dA) \\
&\leq \iint_{\phi\left(D^{\rho r,r}_{\theta_1,\theta_2}\right)} 1\cdot\lambda(dA) + \frac{1}{4\pi} l\left(\phi\left(\Gamma^{\rho r,r}_{\theta_1,\theta_2}\right)\cup\overline{\phi\left(re^{i\theta_2}\right)\phi\left(re^{i\theta_1}\right)}\right)^2 \\
&\leq C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)\left|\theta_2-\theta_1\right|^{2\alpha}.
\end{aligned}$$

Consequently , we have $\left|A^r_{\theta_1,\varphi} - A^r_{\theta_2,\varphi}\right| \leq M\left|\theta_2-\theta_1\right|^\alpha$ where

$$M = C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)(2\pi)^\alpha + \frac{1}{2}\left(C\left((2\pi)^{1-\alpha}+\frac{2}{\alpha}\right)\right)^2(2\pi)^\alpha.$$

Notice that the last two inequalities hold uniformly for all $r\in\mathbb{Q}$; therefore, by passing to the pointwise limit, we also obtain $\left|A_{\theta_1,\varphi} - A_{\theta_2,\varphi}\right| \leq M\left|\theta_2-\theta_1\right|^\alpha$ and

$$\left|A_{\theta_1,\theta_2}\right| \leq C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)\left|\theta_2-\theta_1\right|^{2\alpha}. \tag{3.2}$$

Furthermore, for any partition $\mathcal{D} = \{0=\theta_0\leq\cdots\leq\theta_n=2\pi\}$ and $\varepsilon\geq 0$, the sums

$$\begin{aligned}
\sum_{i=1}^n \left|A^r_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}+\varepsilon} &\leq \left(C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon} \\
&\quad \cdot\sum_{i=1}^n\left|\theta_{i+1}-\theta_i\right|^{1+2\alpha\varepsilon} \\
&\leq 2\pi\left(C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon} \\
&\quad \cdot\max\left|\theta_{i+1}-\theta_i\right|^{2\alpha\varepsilon}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{i=1}^n \left|A_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}+\varepsilon} &\leq 2\pi\left(C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^\alpha+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}+\varepsilon} \\
&\quad \cdot\max\left|\theta_{i+1}-\theta_i\right|^{2\alpha\varepsilon}.
\end{aligned}$$

To complete the proof of our main theorem, applying the Arzela-Ascoli lemma, there is a subsequence $\{A^{r_k}_{\cdot}\}_{k=1}^\infty$ converges uniformly to $\{A_{\cdot}\}$, where $r_k\in\mathbb{Q}$. The sum

$$\begin{aligned}
&\sum_{i=1}^{n}\left|A^{r_k}_{\theta_i,\theta_{i+1}}-A_{\theta_i,\theta_{i+1}}\right|^{\frac{\frac{1}{\alpha}+\varepsilon}{2}} \\
\leq\ & \max\left|A^{r_k}_{\theta_i,\theta_{i+1}}-A_{\theta_i,\theta_{i+1}}\right|^{\frac{\varepsilon}{2}}\left(\sum_{i=1}^{n}\left|A^{r_k}_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}}+\sum_{i=1}^{n}\left|A_{\theta_i,\theta_{i+1}}\right|^{\frac{1}{2\alpha}}\right) \\
\leq\ & 4\pi\left(C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}}\max\left|A^{r_k}_{\theta_i,\theta_{i+1}}-A_{\theta_i,\theta_{i+1}}\right|^{\frac{\varepsilon}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{\mathcal{D}}\sum_{i=1}^{n}\left|A^{r_k}_{\theta_i,\theta_{i+1}}-A_{\theta_i,\theta_{i+1}}\right|^{\frac{\frac{1}{\alpha}+\varepsilon}{2}} \\
\leq\ & 4\pi\left(C^2\left((2\pi)^{1-\alpha}+\frac{3}{\alpha}\right)^2\left(\pi(2\pi)^{\alpha}+\pi^{-1}\right)\right)^{\frac{1}{2\alpha}}\sup_{(\theta,\varphi)\in\Delta}\left|A^{r_k}_{\theta,\varphi}-A_{\theta,\varphi}\right|^{\frac{\varepsilon}{2}}
\end{aligned}$$

which converges to zero as k tends to infinity.

Hence, for $\alpha > \frac{1}{3}$, the multiplicative functionals $X^{r_k}_{s,t}$, i.e. its tensor of all iterated integrals, over γ^{r_k} converge in $p\left(> \frac{1}{\alpha}\right)-$variation norm to the multiplicative functional $X_{s,t} = \left(1, X^1_{s,t}, X^2_{s,t}\right)$ where

$$X^1_{s,t} = \gamma_{s,t}, \tag{3.3}$$

$$\begin{aligned}
X^2_{s,t} &= \frac{1}{2}\begin{pmatrix} (x_t-x_s)^2 & (x_t-x_s)(y_t-y_s) \\ (x_t-x_s)(y_t-y_s) & (y_t-y_s)^2 \end{pmatrix} \\
&\quad +\frac{1}{2}\begin{pmatrix} 0 & A_{s,t} \\ -A_{s,t} & 0 \end{pmatrix}
\end{aligned} \tag{3.4}$$

for any $(s,t)\in\Delta$. Therefore, $X_{s,t}$ is a $p-$geometric rough path over γ with $\frac{1}{\alpha} < p < 3$ and completes the proof.

3.2. Proof of Theorem 2 (Gauss-Green). For a fixed $p > 1/\alpha$, according to Theorem 1, there is a canonical $p-$geometric rough paths $X(\gamma)_{s,t} = (1, X(\gamma)^1_{s,t}, X(\gamma)^2_{s,t})$ associated with γ, where X^1 and X^2 are given by the formulae (3.3) and (3.4); in addition, there is also a sequence of C^1-paths $\{\gamma^{r_k}\}_{k\in\mathbb{N}}$ such that their respective iterated integrals

$$X\left(\gamma^{r_k}\right)^j_{s,t} = \int_{s<u_1<\ldots u_j<t}\cdots\int d\gamma_{u_1}\otimes\cdots\otimes d\gamma_{u_j}$$

converges to $X^j(\gamma)$ in $p'-$variation topology for $j = 1, 2$ and any $p' > 1/\alpha$. Applying the continuity property of Itô functional in Proposition 4,

$$\int \omega\left(X(\gamma)\right)dX(\gamma) = \lim_{k\to\infty}\int \omega\left(X\left(\gamma^{r_k}\right)\right)dX\left(\gamma^{r_k}\right)$$

using the fact that π^1 is continuous and the formula (2.25), we further have

$$\begin{aligned}\pi^1\left(\int\omega\left(X\left(\gamma\right)\right)dX\left(\gamma\right)\right) &= \lim_{k\to\infty}\pi^1\left(\int\omega\left(X\left(\gamma^{r_k}\right)\right)dX\left(\gamma^{r_k}\right)\right)\\ &= \lim_{k\to\infty}\int_0^{2\pi}\omega\left(\gamma_u^{r_k}\right)\left(d\gamma_u^{r_k}\right)=\lim_{k\to\infty}\oint_{\gamma^{r_k}}\omega\end{aligned}$$

Notice that the value of $\pi^1\left(\int\omega\left(X\left(\gamma\right)\right)dX\left(\gamma\right)\right)$ is independent of the choices of parametrisation of γ and the roughness p since each integral $\int\omega\left(X\left(\gamma^{r_k}\right)\right)dX\left(\gamma^{r_k}\right)$ is also independent of the choice of parametrisation and for all $j=1,2$, $X\left(\gamma^{r_k}\right)_{s,t}^j$ converges to $X^j\left(\gamma\right)$ in $p'-$variation topology for all $p'>1/\alpha$.

On the other hand, using the classical Green's formula for domains with C^1-boundaries, we have

$$\oint_{\gamma^{r_k}}\omega=\iint_{\bar{D}^{r_k}}d\omega \tag{3.5}$$

where $\bar{D}^{r_k}$ is the Jordan domain bounded by γ^{r_k}. Finally, using the fact that the Hausdorff dimension of $\bar{D}$ is strictly less than 2 and γ^{r_k} converges to γ from inside, we also have

$$\iint_{\bar{D}}d\omega=\lim_{k\to\infty}\iint_{\bar{D}^{r_k}}d\omega, \tag{3.6}$$

and therefore the conclusion follows.

4. Appendix: Proof of Proposition 1

We first prove that the results (1) and (2) hold for any polygonal closed curve γ. We prove this by induction on number of vertices, ν. It is obvious that the results are valid for $\nu=2,3$. Indeed, for $\nu=3$, the polygonal closed curve γ is essentially a triangle; furthermore, if this triangle γ is non-degenerated, then γ is clearly a Jordan curve, and $\eta\left(\gamma,\cdot\right)$ will take value of 1 in the interior and 0 in the exterior; applying the classical isoperimetric inequality, the results (1) and (2) follow for $\nu=3$. Suppose that (1) and (2) hold for all polygonal closed curve with $\nu\leq n$. Now, consider a polygonal closed curve γ with vertices at times $0=t_0<t_1<\cdots<t_{n+1}=1$.

Case I: If $\gamma:[0,1]\to\mathbb{C}$ is simple, the result (1) immediately follows by the classical isoperimetric inequality. Add a chord $\gamma\left(t_0\right)\to\gamma\left(t_2\right)$, γ is now composed of two closed paths

$$\begin{aligned}\gamma_1 &: \gamma\left(t_0\right)\to\gamma\left(t_1\right)\to\gamma\left(t_2\right)\to\gamma\left(t_0\right),\\ \gamma_2 &: \gamma\left(t_0\right)\to\gamma\left(t_2\right)\to\cdots\to\gamma\left(t_n\right)\to\gamma\left(t_{n+1}\right)=\gamma\left(t_0\right).\end{aligned}$$

Now, it is clear that $\eta\left(\gamma,\cdot\right)=\eta\left(\gamma_1,\cdot\right)+\eta\left(\gamma_2,\cdot\right)$ and γ_1, γ_2 are both polygonal closed curves with their respective number of vertices not more than n. Using the induction hypothesis, we have

$$\begin{aligned}\iint_{\mathbb{C}} \eta(\gamma_{\cdot}, z)\,\lambda(dA) &= \iint_{\mathbb{C}} \eta(\gamma_1, z)\,\lambda(dA) + \iint_{\mathbb{C}} \eta(\gamma_2, z)\,\lambda(dA)\\ &= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right)\\ &= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right).\end{aligned}$$

Case II: If γ is not simple, since γ is polygonal and rectifiable, there are at most finitely many self-intersecting points, say at times $\{s_1 < \cdots < s_m\}$. Consider the point $\gamma(s_1)$, of course, there must be another crossing time, say s_1' such that $s_1 < s_1'$ and $\gamma(s_1) = \gamma(s_1')$. Furthermore, we assume $t_0 < \cdots < t_{j_1} < s_1 < t_{j_1+1} < \cdots < t_{j_2} < s_1' < t_{j_2+1} < \cdots < t_{n+1}$. Now, γ decomposed into two polygonal paths

$$\begin{aligned}\gamma_1 &: \gamma(t_0) \to \cdots \to \gamma(t_{j_1}) \to \gamma(s_1) \to \gamma(t_{j_2+1}) \to \cdots \to \gamma(t_{n+1}) = \gamma(t_0),\\ \gamma_2 &: \gamma(s_1) \to \gamma(t_{j_1+1}) \to \cdots \to \gamma(t_{j_2}) \to \gamma(s_1).\end{aligned}$$

If there is exactly one t_j in between s_1 and s_1', then $t_{j_1+1} = t_{j_2}$; since $\gamma|_{[t_{j_1}, t_{j_1+1}]}$ and $\gamma|_{[t_{j_1+1}, t_{j_1+2}]}$ are both straight lines, so either

$$\gamma|_{[t_{j_1}, t_{j_1+1}]} \subseteq \gamma|_{[t_{j_1+1}, t_{j_1+2}]} \text{ and } \gamma(t_{j_1}) = \gamma(s_1) = \gamma(s_1'),$$

or

$$\gamma|_{[t_{j_1}, t_{j_1+1}]} \supseteq \gamma|_{[t_{j_1+1}, t_{j_1+2}]} \text{ and } \gamma(s_1) = \gamma(s_1') = \gamma(t_{j_1+2}).$$

For the former case, γ_1 is equivalent to another curve $\gamma_1' : \gamma(t_0) \to \cdots \to \gamma(t_{j_1}) \to \gamma(t_{j_1+2}) \to \cdots \to \gamma(t_{n+1}) = \gamma(t_0)$ which only has n vertices; and γ_2 is essentially a straight line. Therefore, $\eta(\gamma, \cdot) = \eta(\gamma_1', \cdot)$ almost everywhere and

$$4\pi \iint_{\mathbb{C}} \eta^2(\gamma_{\cdot}, \zeta)\,\lambda(dA) = 4\pi \iint_{\mathbb{C}} \eta^2(\gamma_1', \zeta)\,\lambda(dA) \le l(\gamma_3)^2 \le l(\gamma)^2$$

and

$$\begin{aligned}\iint_{\mathbb{C}} \eta(\gamma_{\cdot}, z)\,\lambda(dA) &= \iint_{\mathbb{C}} \eta(\gamma_1', z)\,\lambda(dA) + \iint_{\mathbb{C}} \eta(\gamma_2, z)\,\lambda(dA)\\ &= \frac{1}{2}\left(\int_{\gamma_1'} x_s dy_s - y_s dx_s\right) + 0\\ &= \frac{1}{2}\left(\int_{\gamma_1'} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right)\\ &= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right).\end{aligned}$$

For the later case, the deduction is essentially the same as above.

If there are at least two but at the most $n-1$ $t_j's$ in between s_1 and s_1', both γ_1 and γ_2 have number of vertices not more than n. Since $\eta(\gamma, \cdot) = \eta(\gamma_1, \cdot) + \eta(\gamma_2, \cdot)$, according to the induction hypothesis, we have

$$\begin{aligned}\|\eta(\gamma,\cdot)\|_2 &\triangleq \left(\iint_{\mathbb{C}} \eta^2(\gamma.,\zeta)\,\lambda(dA)\right)^{\frac{1}{2}} \\ &\leq \|\eta(\gamma_1,\cdot)\|_2 + \|\eta(\gamma_2,\cdot)\|_2 \\ &\leq \frac{l(\gamma_1)}{\sqrt{4\pi}} + \frac{l(\gamma_2)}{\sqrt{4\pi}} = \frac{l(\gamma)}{\sqrt{4\pi}}\end{aligned}$$

and

$$\begin{aligned}\iint_{\mathbb{C}} \eta(\gamma.,z)\,\lambda(dA) &= \iint_{\mathbb{C}} \eta(\gamma_1,z)\,\lambda(dA) + \iint_{\mathbb{C}} \eta(\gamma_2,z)\,\lambda(dA) \\ &= \frac{1}{2}\left(\int_{\gamma_1} x_s dy_s - y_s dx_s\right) + \frac{1}{2}\left(\int_{\gamma_2} x_s dy_s - y_s dx_s\right) \\ &= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right).\end{aligned}$$

If there are exactly n $t_j's$ in between s_1 and s_1', the proof of the induction step is essentially the same as that of case (a).

After deducing the validity of the results for cases I and II for any polygonal closed curves, we now return back to the general case. Consider a family of partitions $\mathcal{D}^{(m)} \triangleq \left\{0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = 1\right\}$ of $[0,1]$, for $m = 1,2,\ldots,$ such that $\mathcal{D}^{(m+1)} \subset \mathcal{D}^{(m)}$ and their mesh sizes tend to zero. For any rectifiable closed curve γ, consider the family of polygonal closed curve $\gamma^{(m)}$ formed by the vertices $\left\{\gamma(t) : t \in \mathcal{D}^{(m)}\right\}$. Since γ is compact and continuous, one can easily see that $\gamma^{(m)}$ converges uniformly to γ. Since all γ and $\gamma^{(m)}$ are rectifiable, so $\lambda(\gamma) = \lambda\left(\gamma^{(m)}\right) = \lambda\left(\gamma \cup_{m=1}^{\infty} \gamma^{(m)}\right) = 0$; and hence the set $\Omega \triangleq \mathbb{C}/\gamma \cup_{m=1}^{\infty} \gamma^{(m)}$ has the full measure. For any $z_0 \in \Omega$ with $\|z_0 - \gamma\| > \epsilon > 0$, all but except finitely many $\gamma^{(m)\prime}s$ lie inside the $\epsilon/2-$ neighborhood/sausage of γ which excludes z_0. Now, for all large enough m, $\gamma^{(m)}$ is freely homotopic to γ, therefore we have $\eta\left(\gamma^{(m)}, z_0\right) = \eta(\gamma, z_0)$ by Lemma 2, and hence $\eta(\gamma, z) = \lim_{m\to\infty} \eta\left(\gamma^{(m)}, z\right)$, for all $z \in \Omega$; for instance, $\eta(\gamma,\cdot)$, as the a.e. pointwise limit of $\eta\left(\gamma^{(m)},\cdot\right)$, is also a measurable function. Now, from the previous results, we have for each m

$$4\pi \iint_{\mathbb{C}} \eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA) \leq l\left(\gamma^{(m)}\right)^2.$$

Using the Fatou's lemma, we immediately have

$$\begin{aligned}4\pi \iint_{\mathbb{C}} \eta^2(\gamma,\zeta)\,\lambda(dA) &= 4\pi \iint_{\mathbb{C}} \lim_{m\to\infty}\inf \eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA) \\ &\leq 4\pi \lim_{m\to\infty}\inf \iint_{\mathbb{C}} \eta^2\left(\gamma^{(m)},\zeta\right)\lambda(dA) \\ &\leq \lim_{m\to\infty}\inf\, l\left(\gamma^{(m)}\right)^2 = l(\gamma)^2.\end{aligned}$$

Since all γ and $\gamma^{(m)}$ lie in a compact set, therefore we can apply L^2- convergence theorem to conclude that

$$\begin{aligned} \iint_{\mathbb{C}} \eta(\gamma., z)\lambda(dA) &= \lim_{m\to\infty}\iint_{\mathbb{C}} \eta\left(\gamma^{(m)}, z\right)\lambda(dA) \\ &= \frac{1}{2}\left(\lim_{m\to\infty}\int_{\gamma^{(m)}} x_s dy_s - y_s dx_s\right) \\ &= \frac{1}{2}\left(\int_{\gamma} x_s dy_s - y_s dx_s\right) \end{aligned}$$

where the third equality follows by the definition of Riemann sum.

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