Notes of a Numerical Analyst

A picture worth 2000 words

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When I saw this image at Dave Hewett's home page at UCL, it brought an instant smile to my face. It's just one picture, but it illustrates two famous and important phenomena.

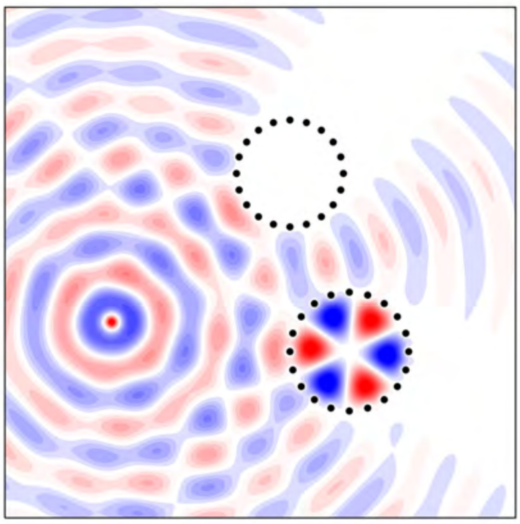

Figure 1. Electromagnetic waves around cages of slightly different radii.

Clearly it's an image of 2D wave scattering (computed by Hewett's coauthor Ian Hewitt). The PDE is the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad (1)$$

for a wave number k, and the solution $u(z)$ represents the spatial dependence of a wave at a fixed time-frequency. The context is electromagnetic radiation, with u representing the out-of-plane component of the electric field. Following a standard mathematical simplification, we take advantage of linearity to let u be complex, and the plot shows its oscillatory real part. The wave is driven by a point source on the left, a Hankel function $H_0(k|z - z_0|)$, and the Sommerfeld radiation condition is imposed at infinity.

The action is in the two cages on the right, which have radii that differ by 10% and very different behaviours. Each black dot is a disk of finite radius, and the boundary condition is $u = 0$ on all the disks. We can think of these as cross-sections of parallel wires in the third dimension, which are all connected and thus at the same fixed potential.

The story in the upper cage is *Faraday shielding.* Obviously the wave has not penetrated much inside, and this effect has been exploited since Faraday's discovery to shield people and instruments from electrostatic and electromagnetic fields. We all have a Faraday cage in our kitchens, namely the microwave oven, whose front door has a metal screen with holes big enough to let light out but small enough to keep microwaves in. As the holes get smaller, or the wires get closer together in our 2D model, the shielding only improves algebraically, not exponentially as has often been supposed [2].

The story in the lower cage is *resonance,* and we can think of this as a model of an AM radio. These thick wires exclude most wave energy, but if the radius is tuned just right, so that k corresponds to an eigenmode of a disk of this radius, then the response can be very great. As the wires get thicker and closer together, the tuning gets ever sharper and the potential amplification ever greater. In the limit where the wires touch, we have perfect shielding and perfect tuning—with infinite amplification, if only it could be excited.

FURTHER READING

[1] D. P. Hewett and I. J. Hewitt, Homogenized boundary conditions and resonance effects in Faraday cages, Proc. Roy. Soc. London A, 472 (2016): 20160062.
[2] L. N. Trefethen, Surprises of the Faraday cage, *SIAM News* 49 (2016).

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