Notes of a Numerical Analyst

Analytic Continuation

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Analytic continuation starts from a paradox: it is perfectly exact, yet impossible. It is exact in that the values of an analytic function in a set $\Omega \subseteq \mathbb{C}$ are determined by its values in any subset $E \subseteq \Omega$. (We assume Ω and E are nonempty, simply-connected continua.) It is impossible in that it is ill-posed: if we know f to accuracy $\varepsilon > 0$ on E, then nothing whatsoever can be inferred about its values at any point $z_0 \in \Omega \backslash \overline{E}$.

Somewhere between these extremes lies a terrain where useful things can be done. Suppose we know that $|f(z)| \leq 1$ in Ω. Then analytic continuation to a point $z_0 \in \Omega \backslash E$ is *well-posed* but with *infinite condition number* in the sense that as $\|f - g\|_E \to 0$, $|f(z_0) - g(z_0)| \to 0$ sublinearly. This goes back to the Hadamard three-circles theorem.

For example, suppose f is analytic with $|f(z)| \leq 1$ in the half-strip $\mathrm{Re}z \geq 0$, $-1 \leq \mathrm{Im}z \leq 1$ and we know it to accuracy ε on the interval $[-i, i]$. Then it is determined to accuracy $\varepsilon^{\alpha(x)}$ at any $x \in [0, \infty)$ with $\alpha(x) \sim (4/\pi)e^{-\pi x/2}$. If you know f to d digits at the end of the strip, you know it to $d/10$ digits at $x \approx 1.47$, $d/100$ digits at $x \approx 2.94$, and so on (since $(2/\pi)\log 10 \approx 1.47$). Figure 1 shows a memorable variation on this estimate.

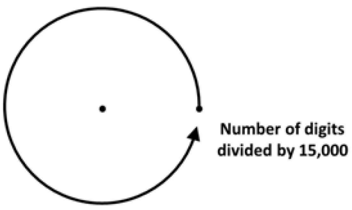

Figure 1. Suppose f can be analytically continued around the origin for $0 < |z| < 2$ and is bounded by 1. If f is known to d digits of accuracy near $z = 1$, then after one circuit around the origin, it is determined to about $d/(\pi/4)\exp(\pi^2)$ digits.

Such results seem impossibly gloomy, yet analytic continuation is established numerical practice based on rational approximations. Traditionally Padé approximation is used, working from Taylor series coefficients, and a more recent alternative is AAA approximation, based on function values as in Figure 2. Ultimately such methods work because functions arising in practice tend to be simpler than worst-case bounds allow.

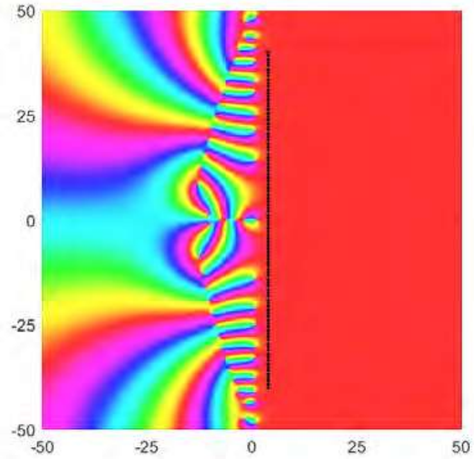

Figure 2. Phase portrait, showing complex arguments by colours, of a numerical analytic continuation of the Riemann zeta function. Here $\zeta(z)$ has been evaluated at 100 points with $\mathrm{Re}z = 4$ and then approximated by a rational function $r(z)$ by the AAA method. The region of good accuracy $r(z) \approx \zeta(z)$ encompasses about 20 zeros on the critical line $\mathrm{Re}z = \frac{1}{2}$.

FURTHER READING

[1] Y. Nakatsukasa, O. Sète, and L. N. Trefethen, *The AAA algorithm for rational approximation,* SIAM J. Sci. Comput., 40 (2018), A1494–A1522.

[2] L. N. Trefethen, *Quantifying the ill-conditioning of analytic continuation,* BIT Numer. Math., 60 (2020), 901–915.

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