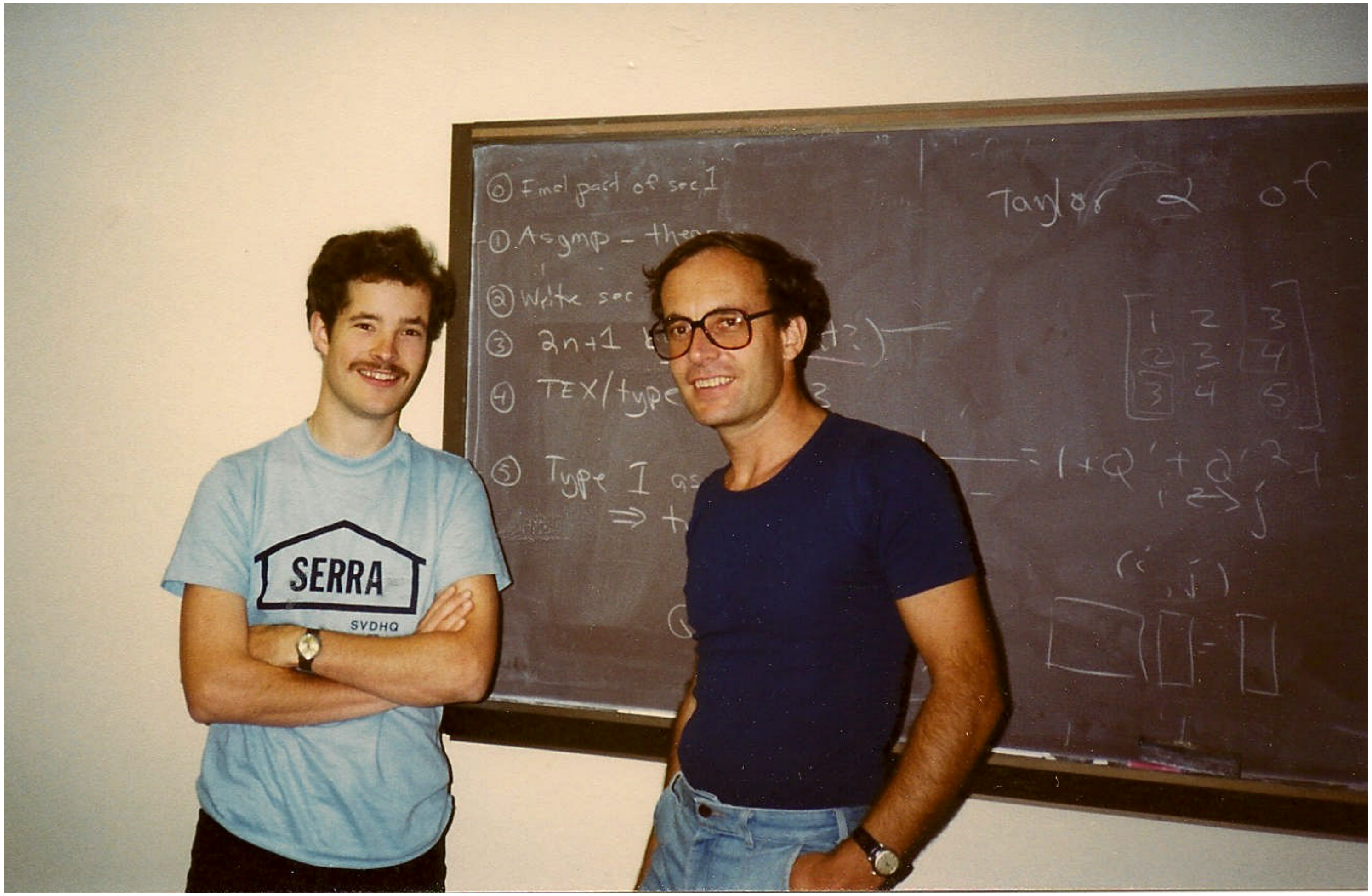
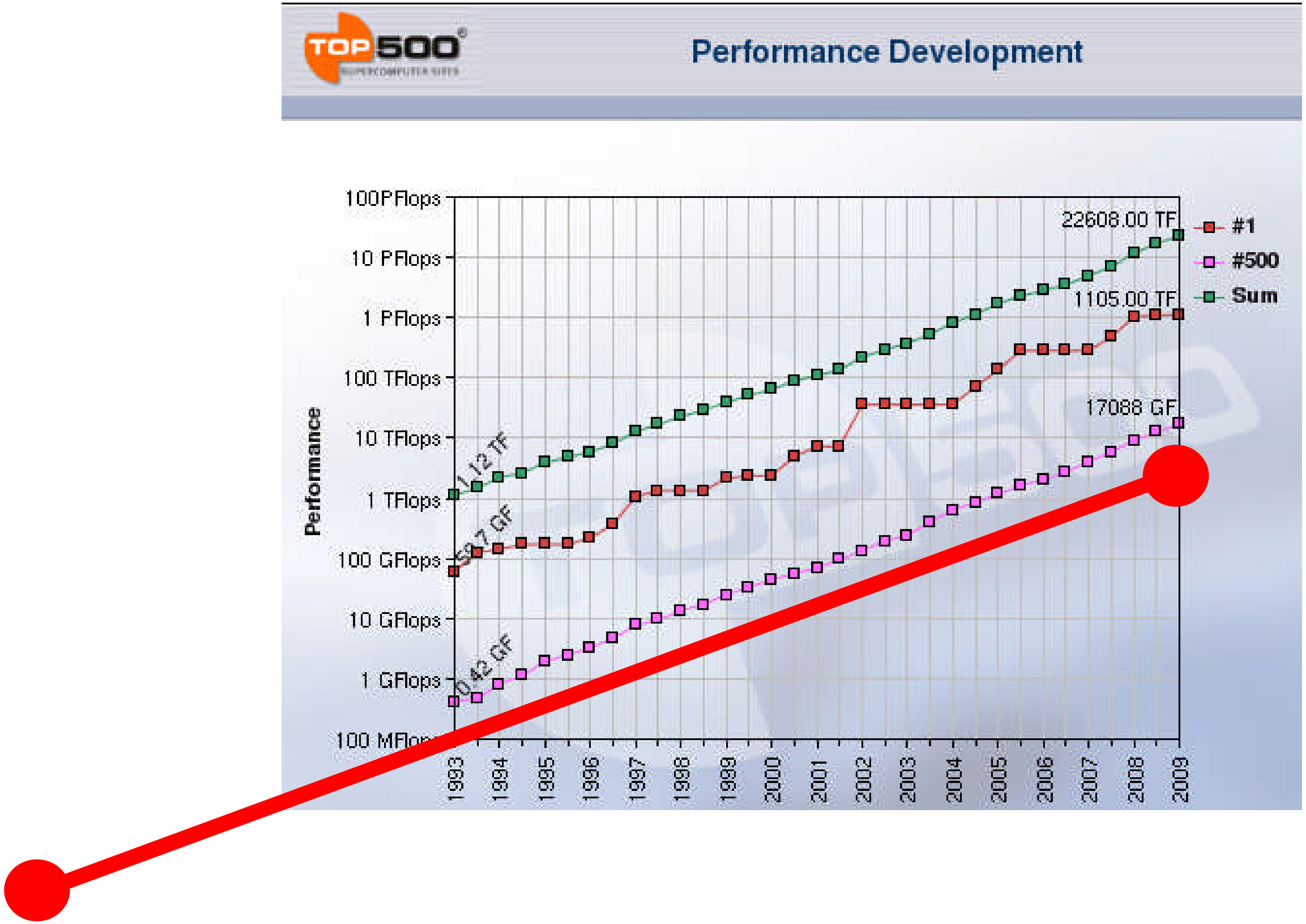
CF Approximation 30 Years Later

Nick Trefethen, Oxford University





Now.
Teraflops.
Matlab.

Then.
Megaflops.
Fortran.

1. Then

0.17737

Harvard 1976-77: senior thesis with Birkhoff on complex approximation.

ETHZ 1979: visiting Henrici for the summer — when I met MHG.

Since Harvard I had known that the degree-2 polynomial best approx. to e^z on the unit disk had error 0.17737. This number was burned into my brain.

At ETHZ I read Carathéodory + Féjer 1911. I had an idea….

One day Walter Gander brought me to the Neutechnikum Buchs.
He had an computer with an *interactive eigenvalue calculator !*

We typed in the Hankel matrix

1/6	1/24	1/120	1/720	1/5040
1/24	1/120	1/720	1/5040	0
1/120	1/720	1/5040	0	0
1/720	1/5040	0	0	0
1/5040	0	0	0	0

We computed the first eigenvalue: 0.17737 !

The CF Method was born, along with the collaboration between LNT & MHG.

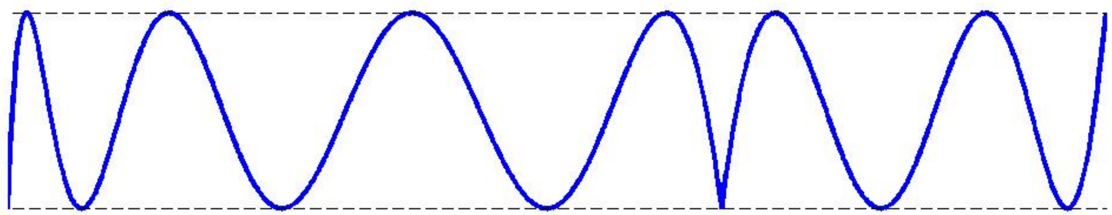
Error curve — real best approximation

Given f real, continuous on [−1,1], m≥0, n≥0. Seek type (m,n) rational BA r* to f .

r* exists and is unique.

Equioscillation theorem: r = r* if and only if f−r equioscillates ≥ m+n+2−δ times, where δ = defect of r (mutual degree deficiency of numerator and denominator).

Proof of "if": if q were better than r, r−q would have too many zeros.



Error curve — complex best approximation

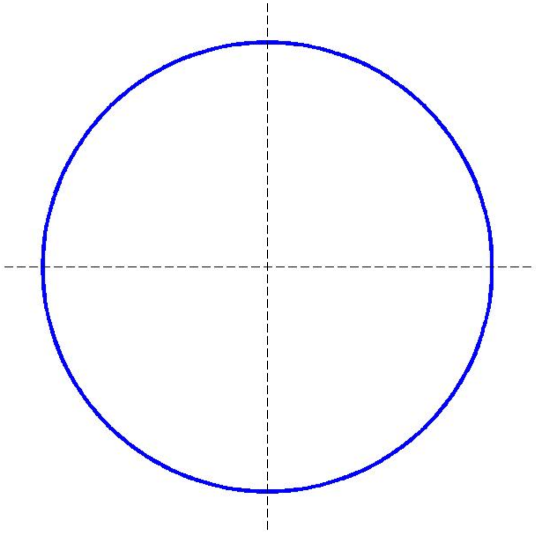
Given f analytic on unit disk, m≥0, n≥0. Seek type (m,n) rational BA r* to f .

r* exists but is not always unique (MHG+LNT, *J. Approx. Theory* 1983).

Circularity theorem: r = r* if f−r maps the unit circle to a circle of winding number ≥ m+n+1−δ. ("Only if" does not hold.)

Proof: if q were better than r, r−q would have too many zeros (Rouché's theorem).

A surprise! Error curves are not usually circular, but they are often nearly circular.



Example: $f(z) = e^z$, m=2, n=0

| (f−r*)(z) | varies between 0.177369 and 0.177376.

For larger m and n, often circular to machine precision.

CF and AAK theorems

Error curves in polynomial or rational approximation are not exactly circular.

But error curves for approx. in certain extended spaces are exactly circular and can be constructed from SVD of infinite Hankel matrix H of Taylor coefficients.

Case n=0: approximate f by p+h, where h is analytic outside unit disk.
C+F 1911. Error = singular value σ_1.

Case m=n>0: likewise with r+h. Adamyan, Arov + Krein ("AAK") 1971.
Error = singular value σ_{n+1}.

CF approximation

Use SVD construction; then drop the analytic part. (Quite tricky for $m \neq n > 0$.)

Resulting approximation has nearly circular error curve, hence is near best.
Strong asymptotic theorems about accuracy of this process.

MHG realized idea could be extended to real approx. via SVD of infinite Hankel matrix of Chebyshev coefficients. Dropping coanalytic terms even trickier here. Accuracy of method even greater.

Five key CF approximation papers

LNT, "Near-circularity of the error curve in complex Chebyshev approximation", *J. Approx. Theory* 1981. — COMPLEX POLYNOMIAL

LNT, "Rational Chebyshev approximation on the unit disk", *Numer. Math.* 1981. — COMPLEX RATIONAL

MHG + LNT, "Real polynomial Chebyshev approximation by the Carathéodory-Féjer method", *SINUM* 1982. — REAL POLYNOMIAL

LNT + MHG, "The Carathéodory-Féjer method for real rational approximation", *SINUM* 1983. — REAL RATIONAL

MHG, "Rational Carathéodory-Féjer approximation on a disk, a circle, and an interval", *J. Approx. Theory* 1984. — FOURIER & LAURENT

(+ half a dozen other papers by MHG + LNT + various coauthors)

Related work (lots!)

Carathéodory + Féjer 1911	Schur 1918
Takagi 1924 + 1925	Bernstein, Achieser, Mirakyan, 1930s
Darlington 1970	Adamjan, Arov + Krein 1968 +1971 **("AAK")**
Lam + D. Elliott 1972	Talbot 1976
Hollenhorst 1976	G. Elliott 1978
Peller + Hruščev 1982 + 1987	Glover 1984
H^{∞}-control literature	+ many more I don't know

9.28903

Cody-Meinardus-Varga 1969: approximation of e^{-x} on [0,∞) by (n,n) rationals.

"1/9 conjecture": error decreases at asymptotic rate $(1/9)^n$.

Summer 1981, Stanford. Transplant [0,∞) to [−1,1] by (1−x)/(1+x).
Rational CF up to n=18. Quadruple precision, IBM 370. Matrix size 200×200.
Late nights computing at the Stanford Linear Accelerator Center.

The ratios didn't look like 1/9. They looked more like **1/9.28903.**

I sent Martin a telegram (May 23, 1981): "9.28903?"

A exact formula was later found by Alphonse Magnus
and proved correct by Gonchar and Rakhmanov (1989).

Early Matlab

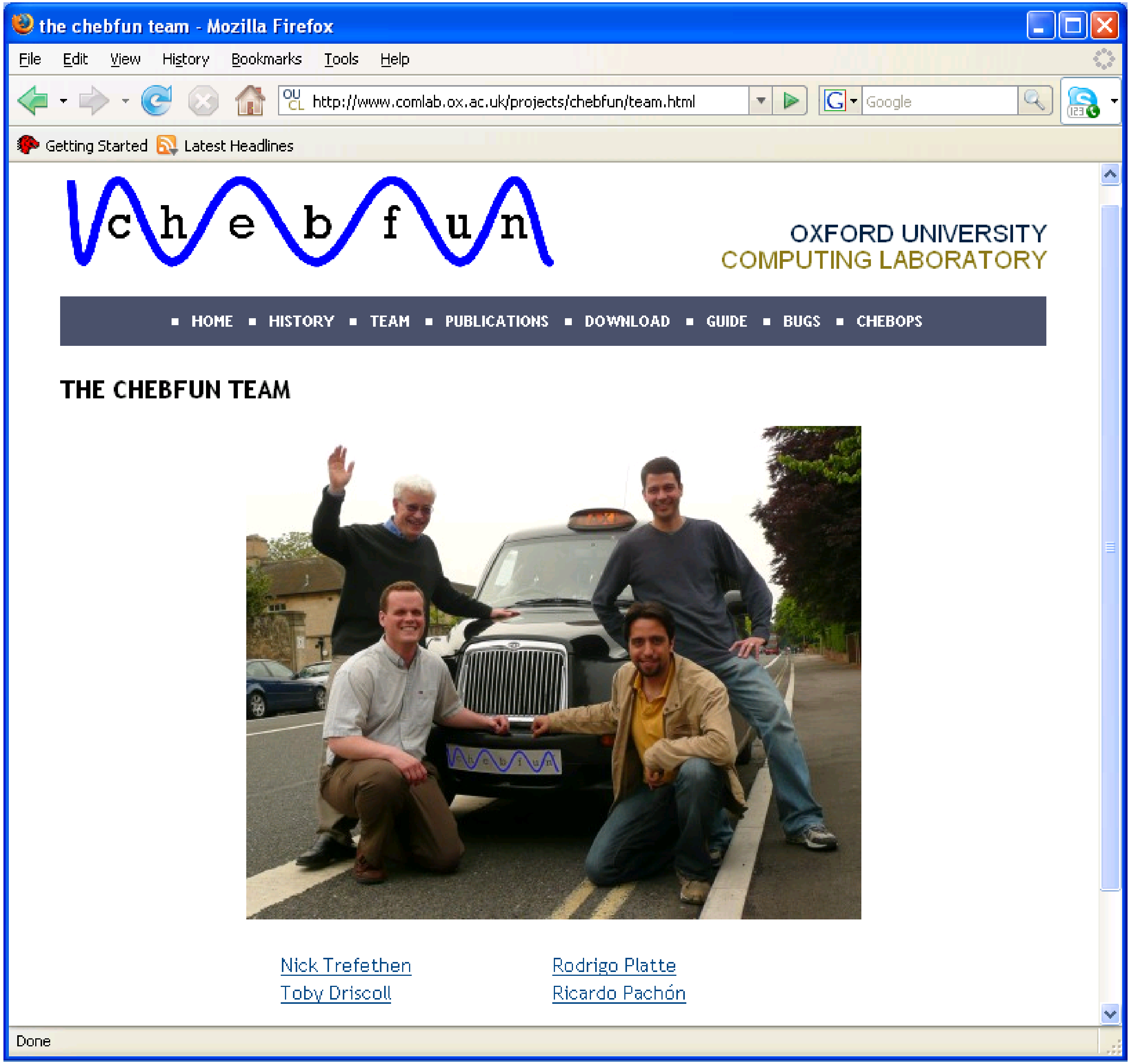
Matlab came along. With FFT and SVD, it was perfect for CF approximation.

I wrote codes CF (complex) and RCF (real) and published them in *Approximation Theory V*, 1986 (Chui, Schumaker & Ward, eds.)

Were these the first Matlab codes ever published?

By the late 1980s MHG and I had moved into other fields.

2. Now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THE CHEBFUN TEAM
Nick Trefethen
Rodrigo Platte
Toby Driscoll
Ricardo Pachón
Done

Best approximation in chebfun

(Polynomial case only so far)

R. Pachón + LNT, "Barycentric-Remez algorithms for best polynomial approximation in the chebfun system", *BIT Numer. Math.*, to appear.

```
x = chebfun('x');

f = exp(x);
pbest = remez(f,4);
plot(f-pbest)

f2 = abs(cos(3*x));
p2best = remez(f2,40);
plot(f2-p2best)
```

CF approximation in chebfun

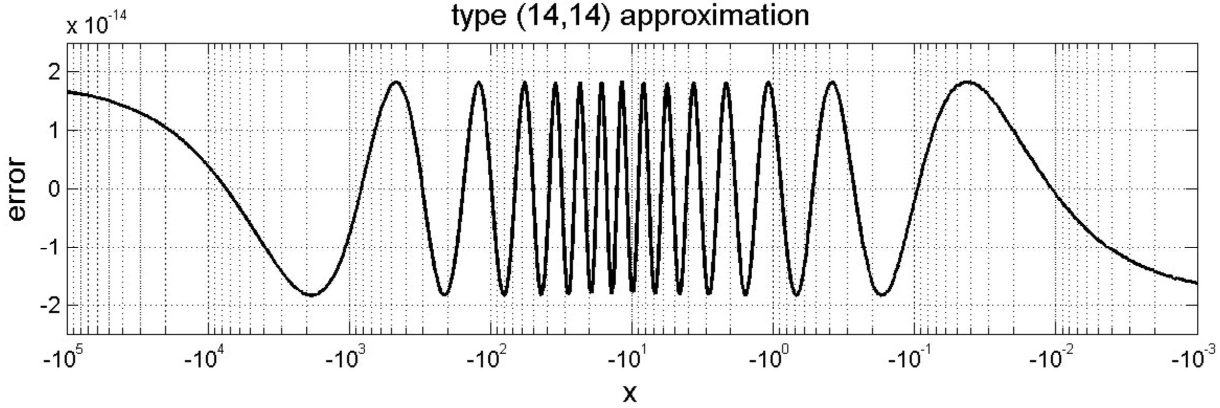
(Again polynomial case only so far. No publication as yet.)

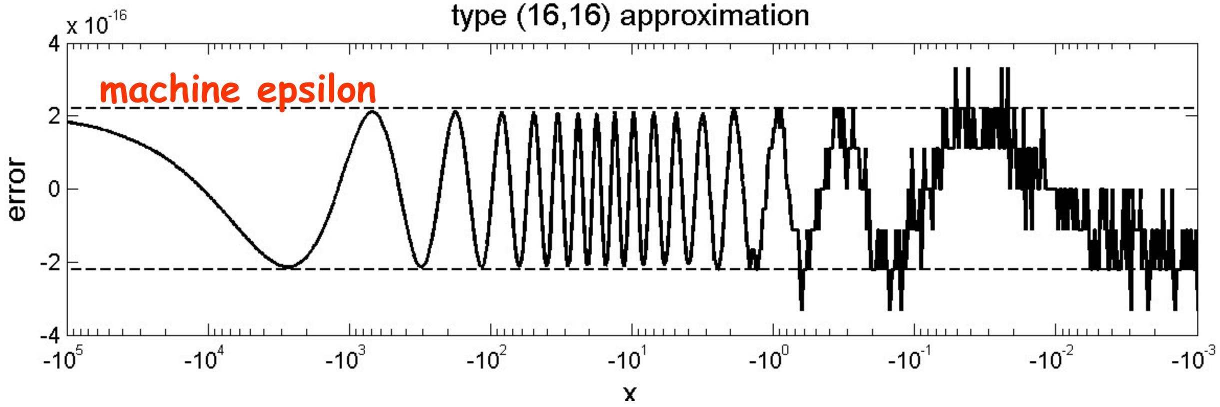
```
pcf = cf(f,4);
plot(f-pcf)
norm(pbest-pcf)

f3 = x + tanh(20*(x-1/2));
plot(f3), hold on
p3cf = cf(f3,20);
plot(p3cf,'r')
p3best = remez(f3,20);
norm(p3best-p3cf)

hold off
plot(f3-remez(f3,100))
plot(f3-cf(f3,100))
```

9.28903 — approximation of e^x on $(-\infty,0]$

Quick Matlab code: **`expx_cf`**

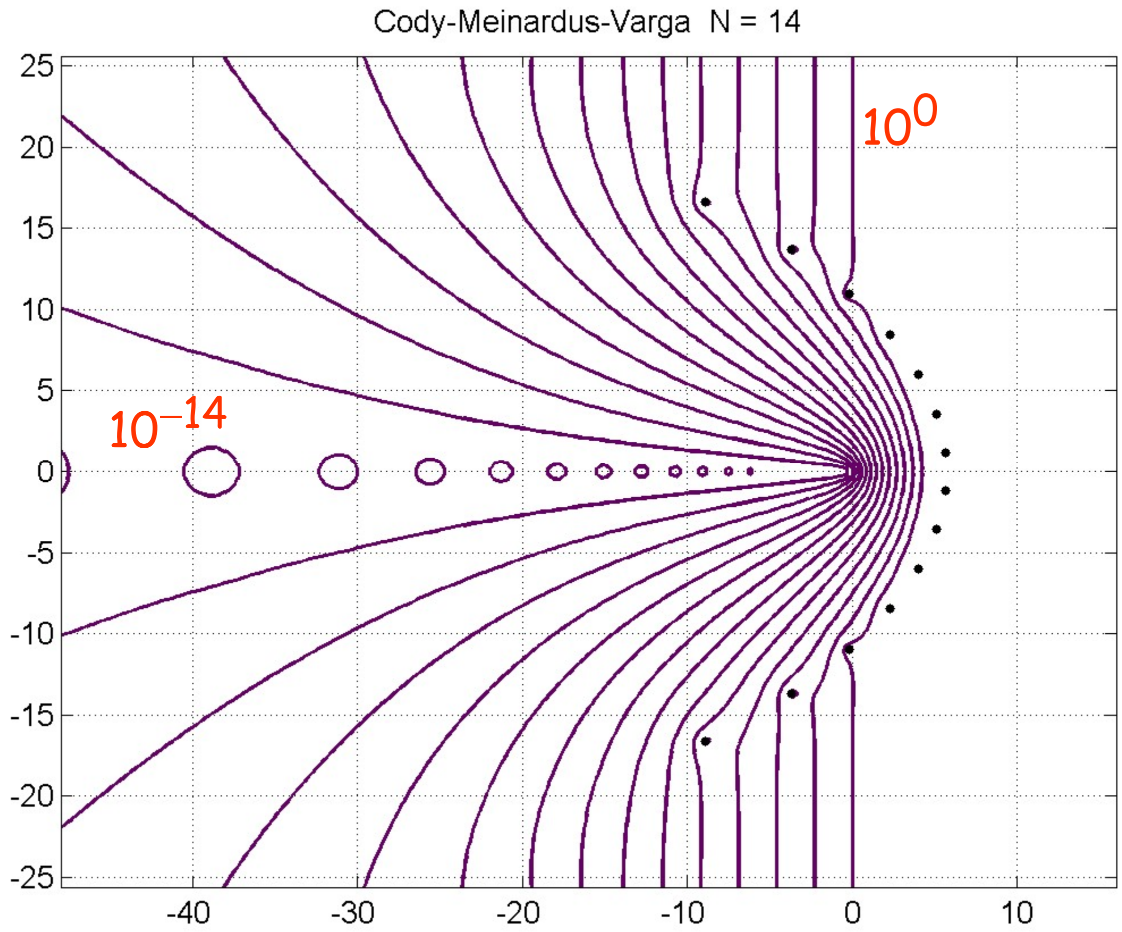
More careful calculation:



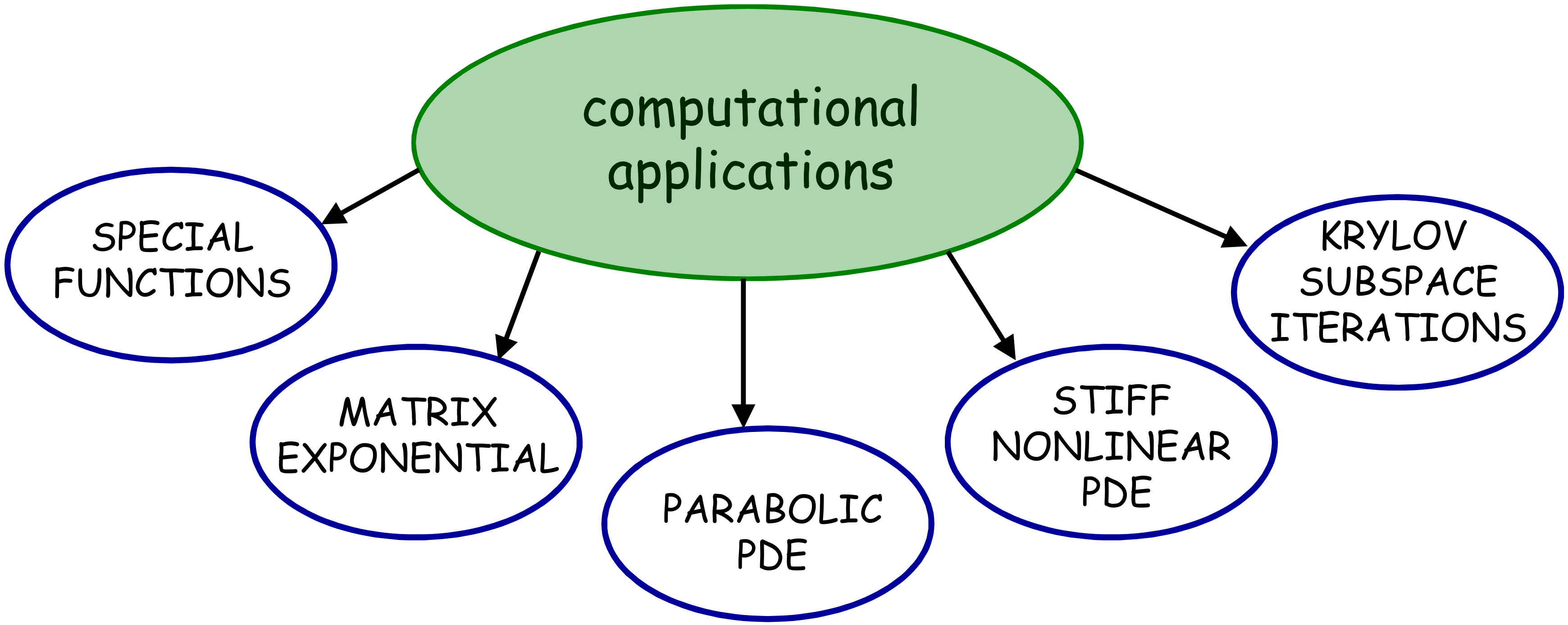


Same approximation in the complex plane

$| e^z - r(z) |$



Such approxs apply to large-scale matrix/operator problems via contour integrals.



See e.g. T. Schmelzer + LNT, "Evaluating matrix functions for exponential integrators via Carathéodory-Fejér approximation and contour integrals", *ETNA* 2007.

Conclusions

- What used to be theoretical is now good for routine computation.
- I miss quadruple precision!
- Rational approximations are especially valuable in large-scale matrix/operator applications. CF approximations are very convenient.
- Martin and I have both worked on such problems, but not together. I regret that!