Eigenmodes of planar drums and Physics Nobel Prizes

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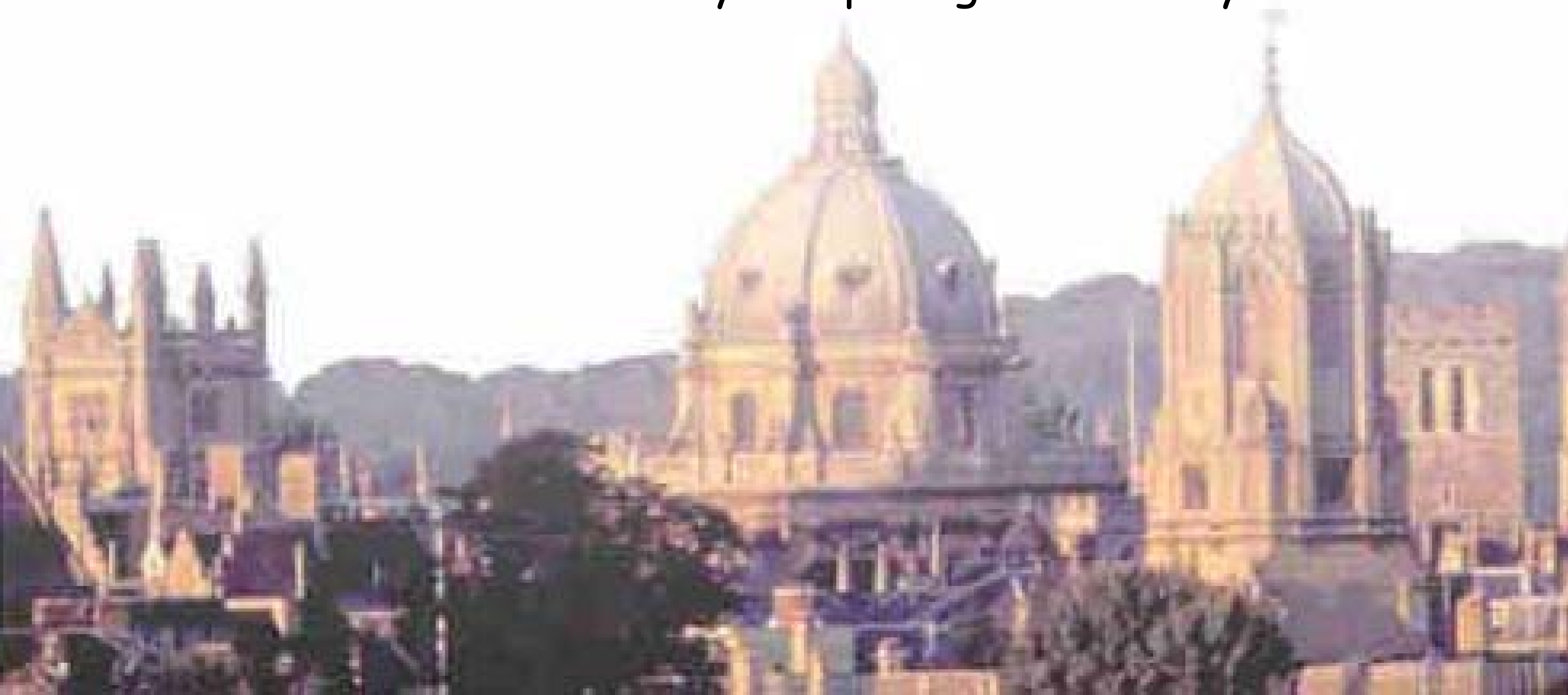
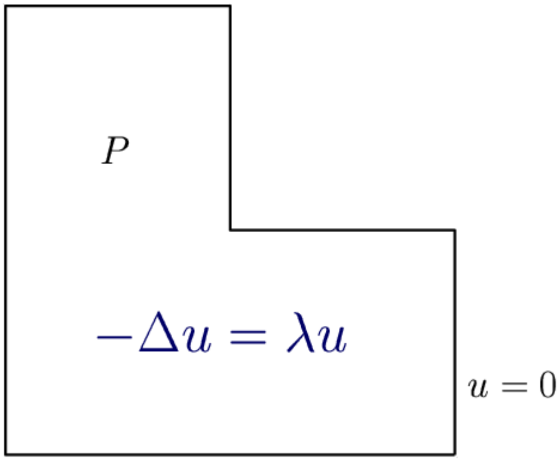


Consider an idealized "drum":
Laplace operator in 2D region
with zero boundary conditions.



Basis of the presentation: numerical methods developed with Timo Betcke, U. of Manchester

B. & T., "Reviving the method of particular solutions," *SIAM Review,* 2005

T. & B., "Computed eigenmodes of planar regions," *Contemporary Mathematics,* 2006

Contributors to this subject include

Barnett
Cohen
Heller
Lepore
Vergini
Saraceno

Banjai
Betcke
Descloux
Driscoll
Karageorghis
Tolley
Trefethen

Clebsch
Pockels
Poisson
Rayleigh
Lamé
Schwarz
Weber

Goldstone
Jaffe
Lenz
Ravenhall
Schult
Wyld

Bäcker
Boasmann
Riddel
Smilansky
Steiner
Conway
Farnham

Donnelly
Eisenstat
Fox
Henrici
Mason
Moler
Shryer

Benguria
Exner
Kuttler
Levitin
Sigillito

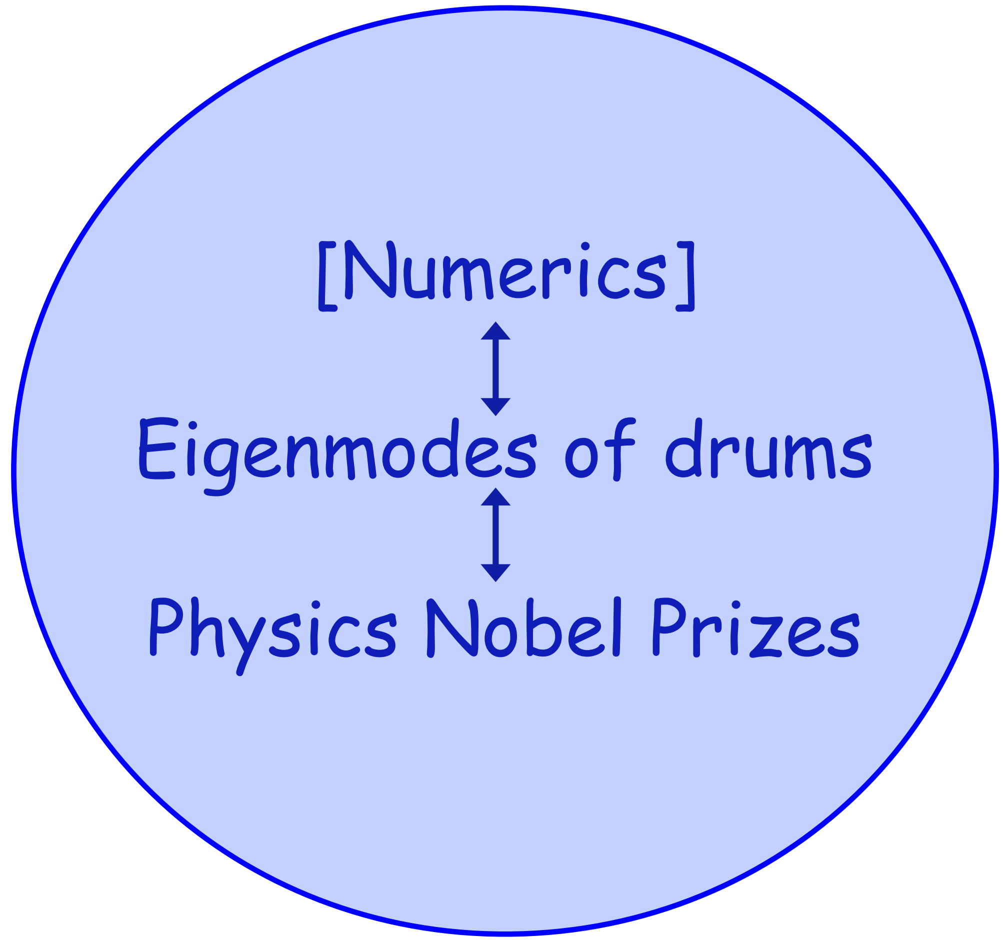
Berry
Gutzwiller
Keating
Simon
Wilkinson

and many more.

EIGHT EXAMPLES

1. L-shaped region
2. Higher eigenmodes
3. Isospectral drums
4. Line splitting
5. Localization
6. Resonance
7. Bound states
8. Eigenvalue avoidance

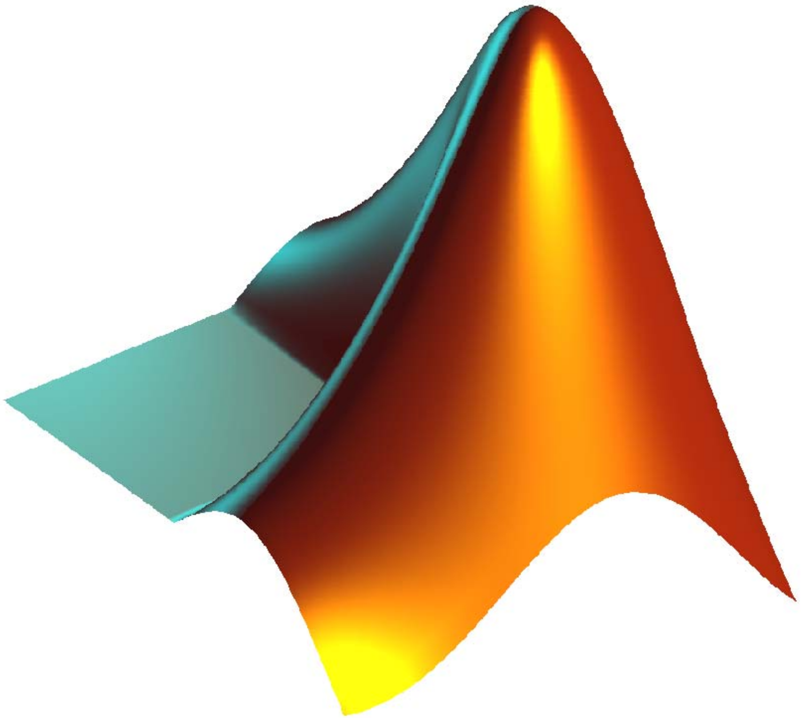
PLAN OF THE TALK



1. L-shaped region

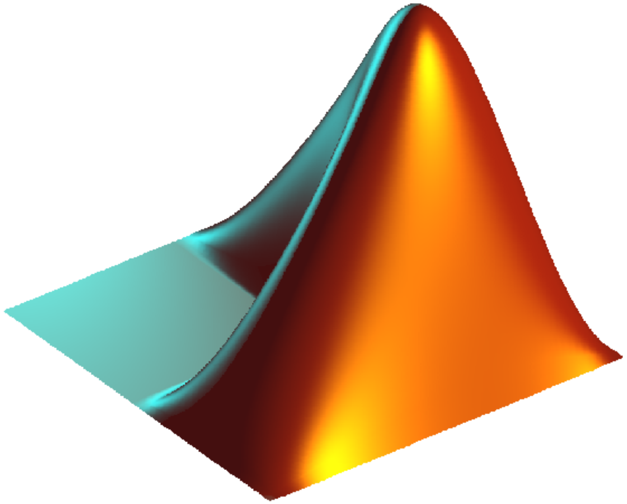
Fox-Henrici-Moler 1967

MathWorks logo

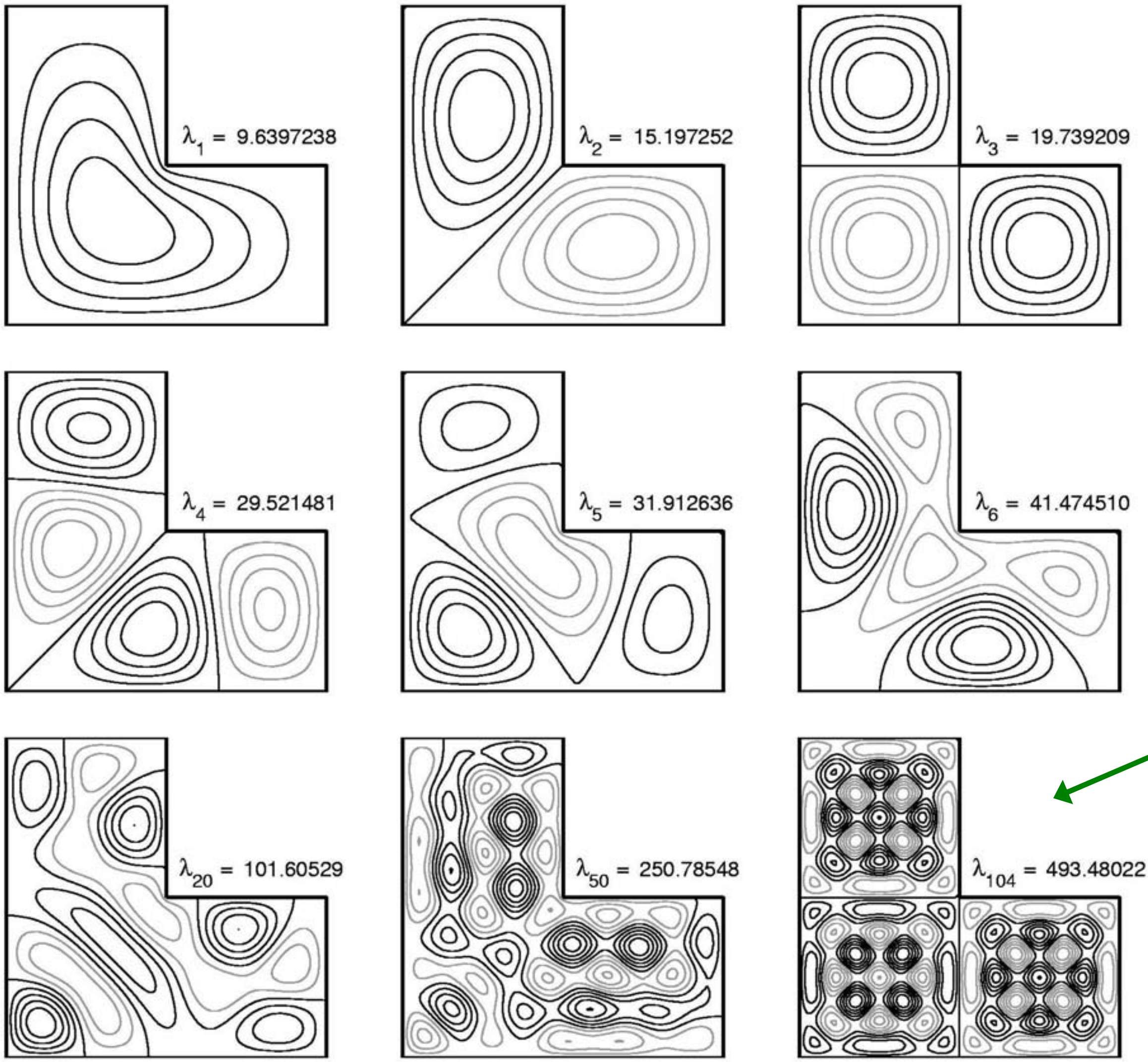


Cleve Moler, late 1970s

Mathematically correct alternative:



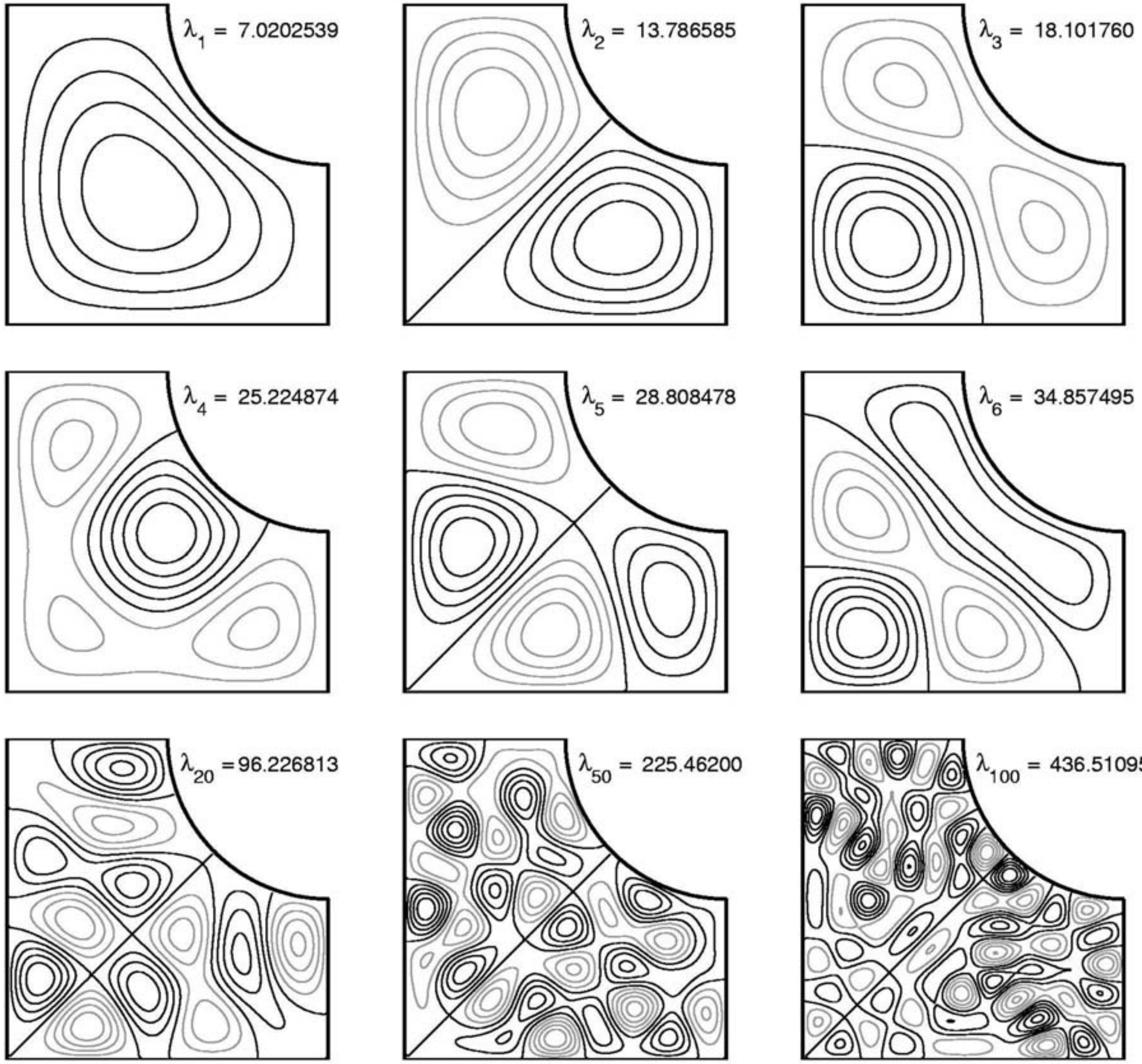
Some higher modes, to 8 digits



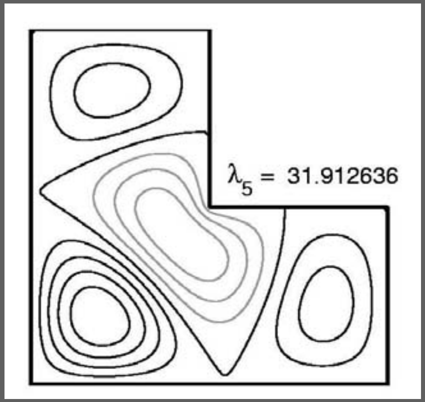
triple degeneracy:

$5^2 + 5^2 =$
$1^2 + 7^2 =$
$7^2 + 1^2$

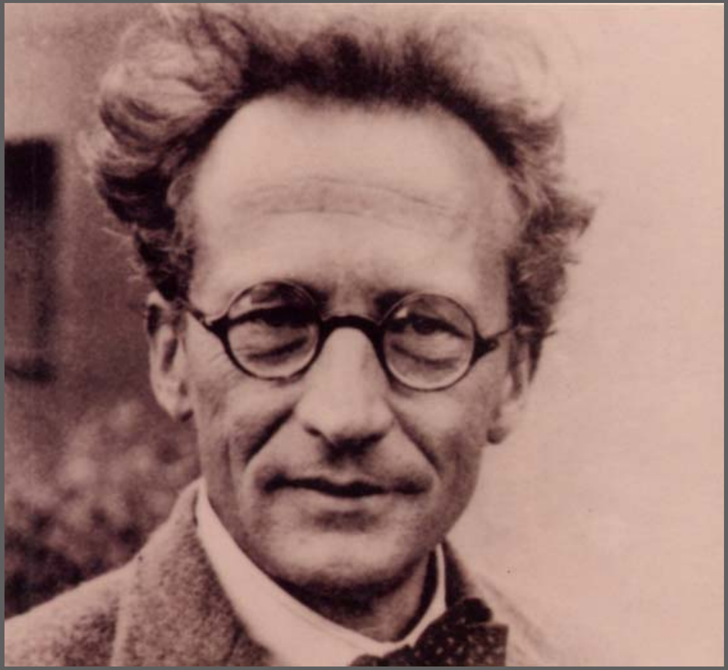
Circular L shape



no degeneracies,
so far as I know



Schrödinger 1933

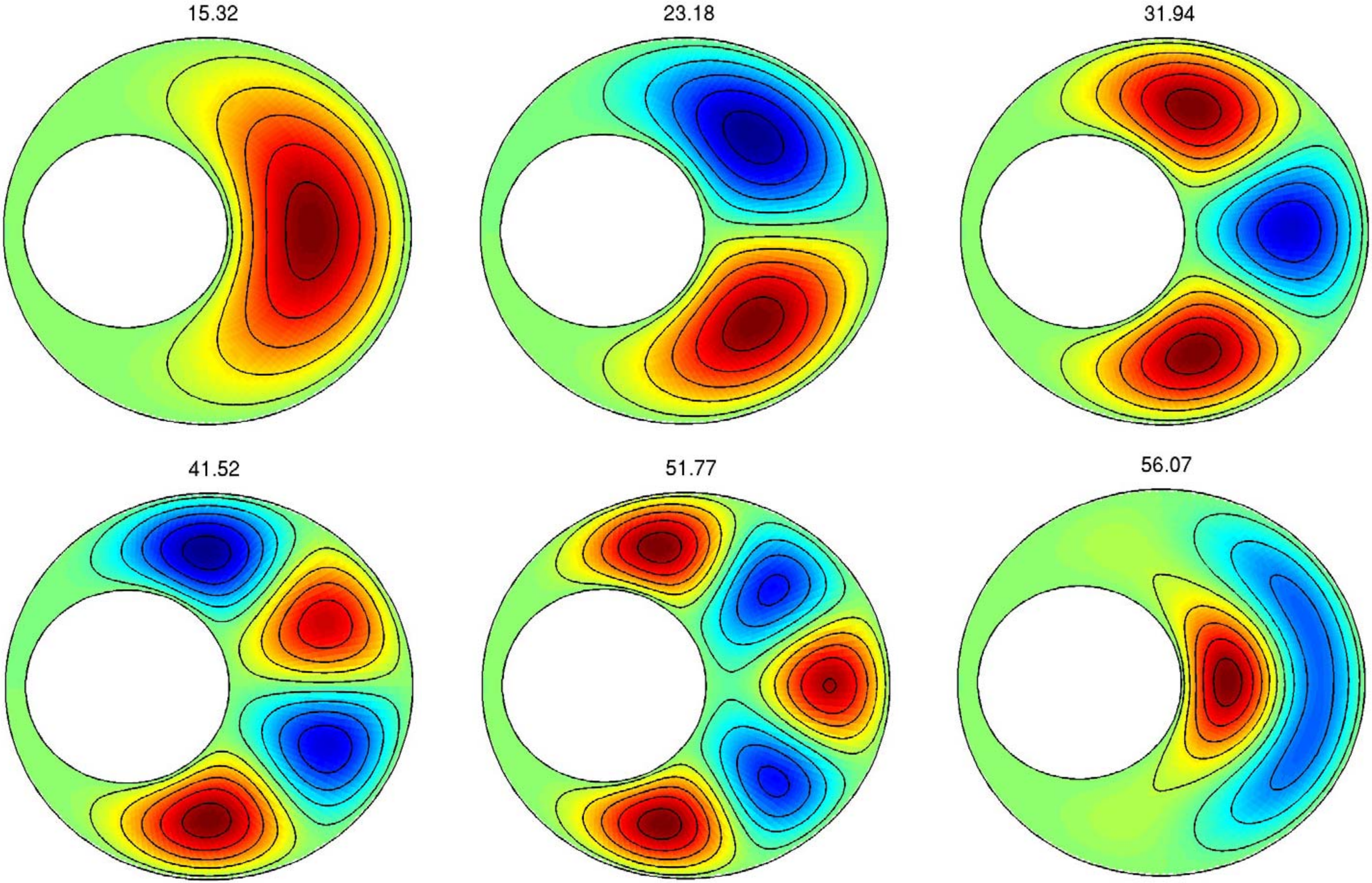


Schrödinger's equation and quantum mechanics

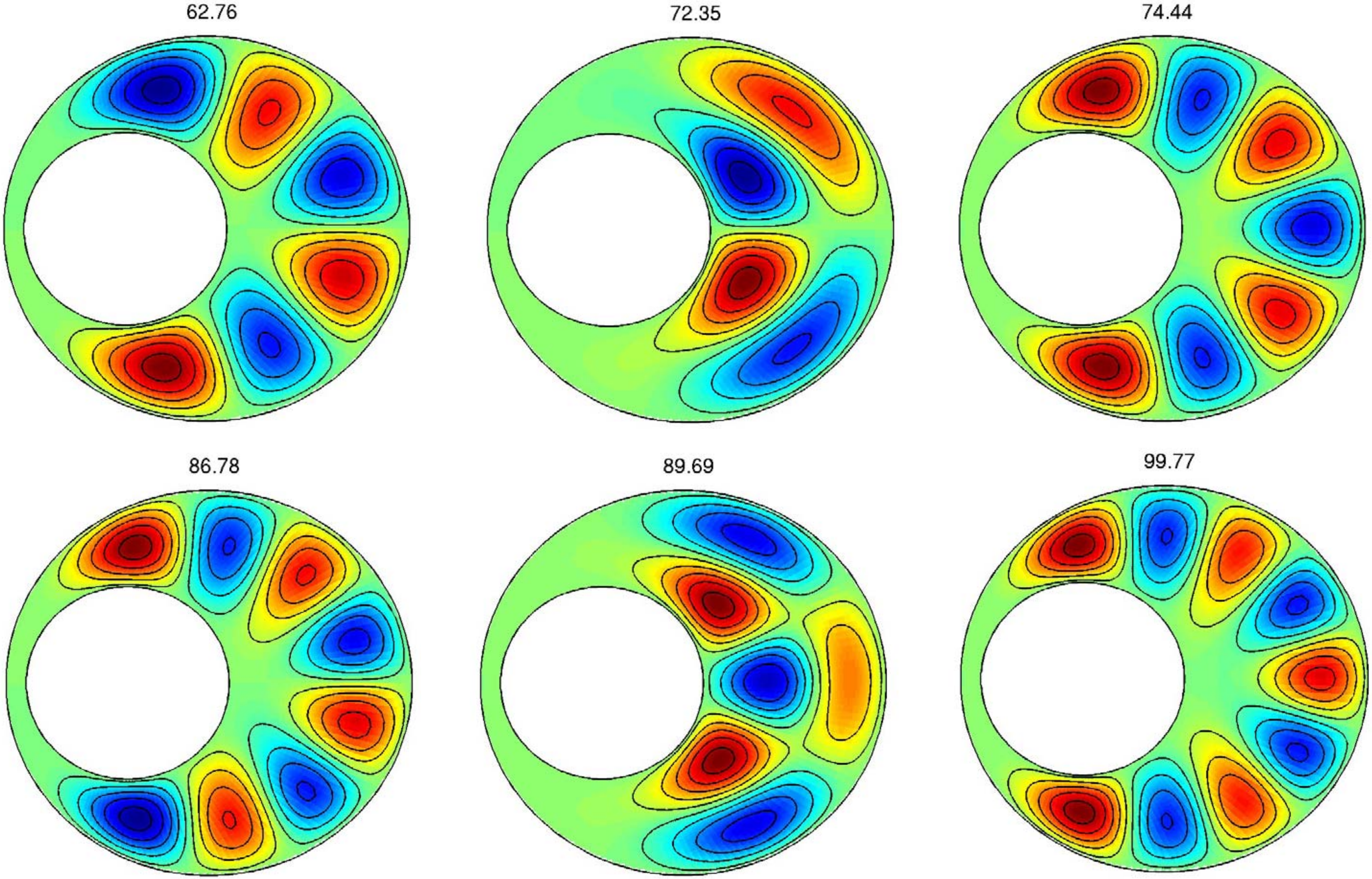
also Heisenberg 1932, Dirac 1933

2. Higher eigenmodes

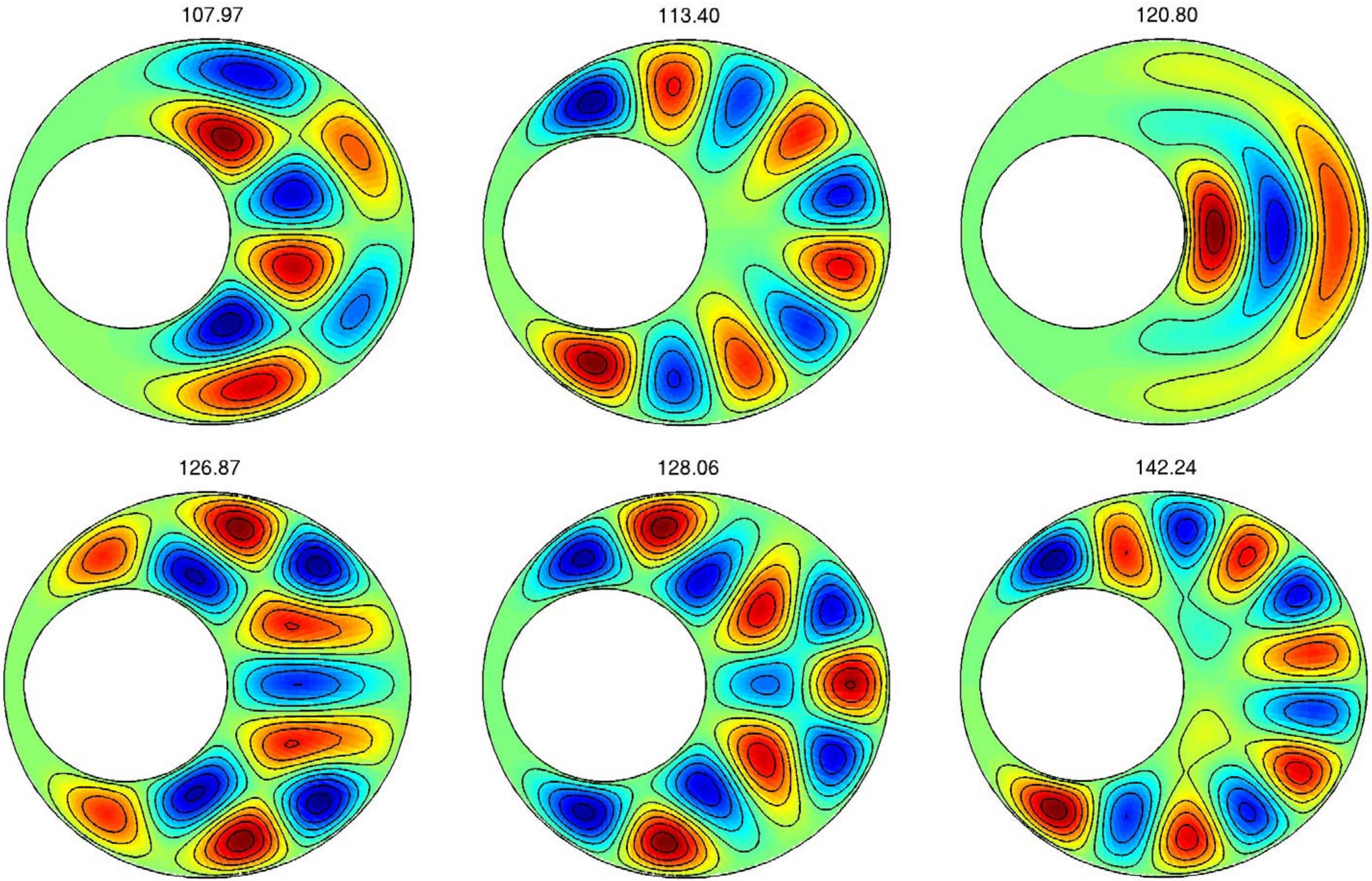
Eigenmodes 1-6



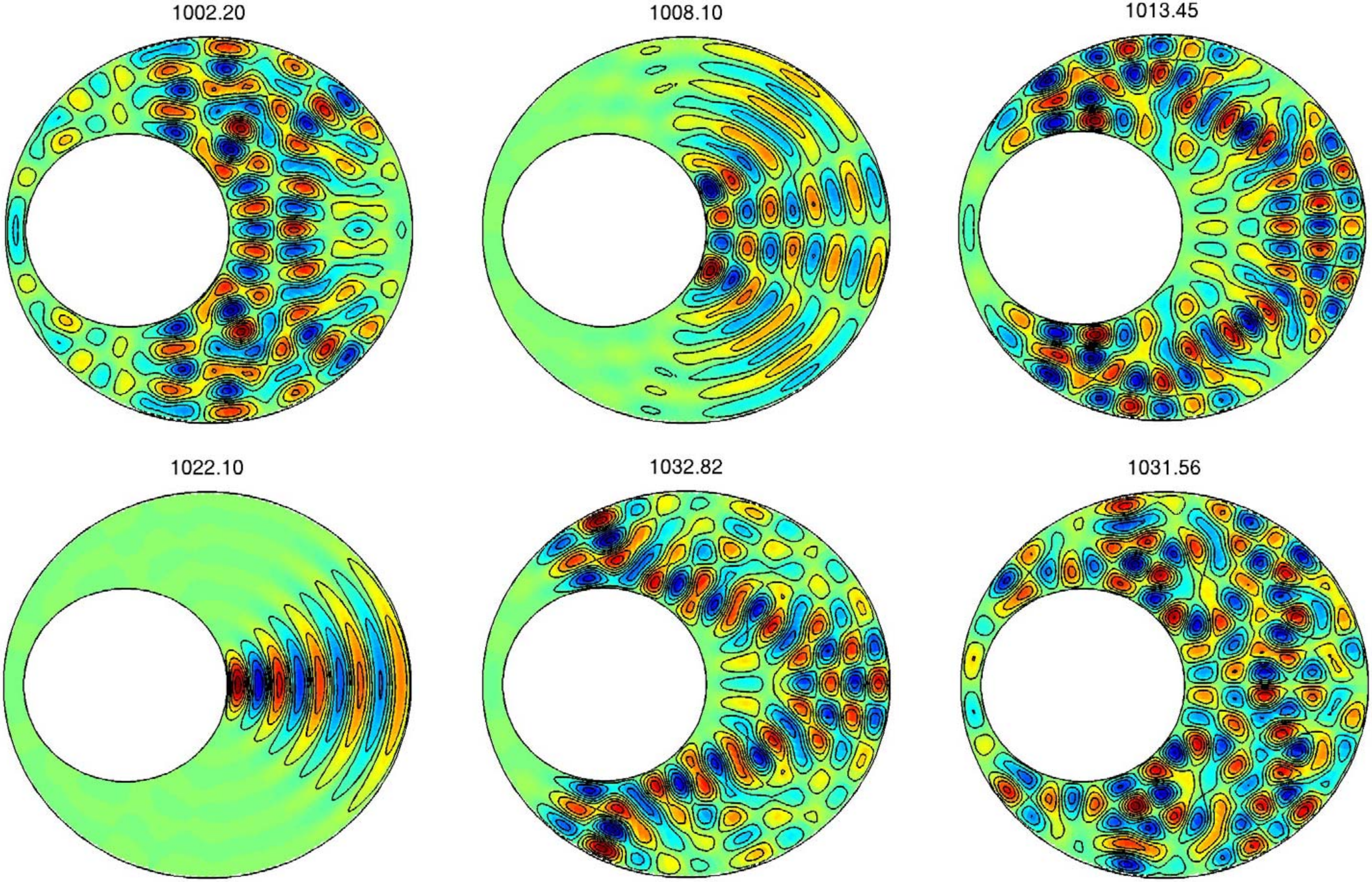
Eigenmodes 7-12

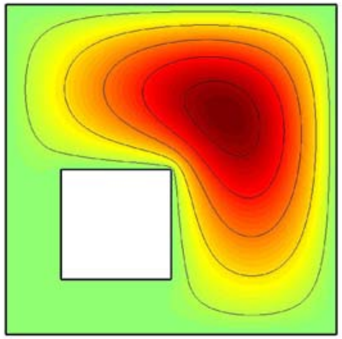


Eigenmodes 13-18

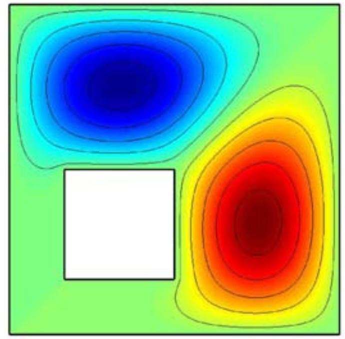


Six consecutive higher modes

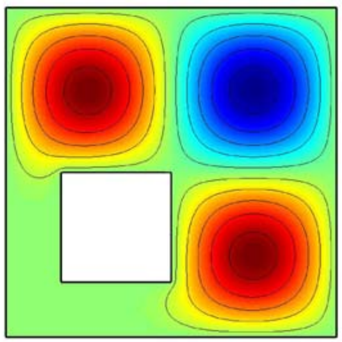




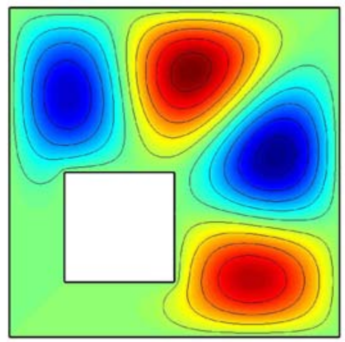
λ_1=4.2786901



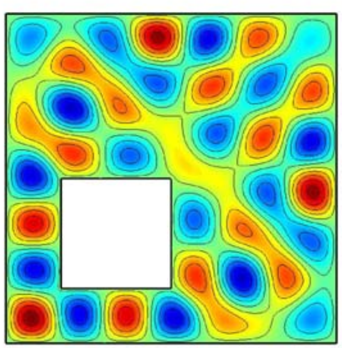
λ_2=6.718692



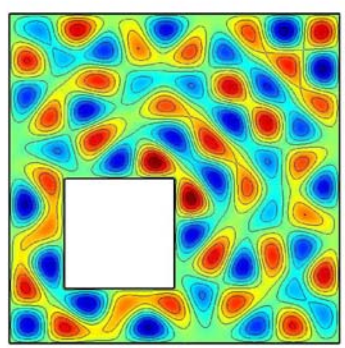
λ_3=8.713789



λ_4=13.013411



λ_{50}=94.38700

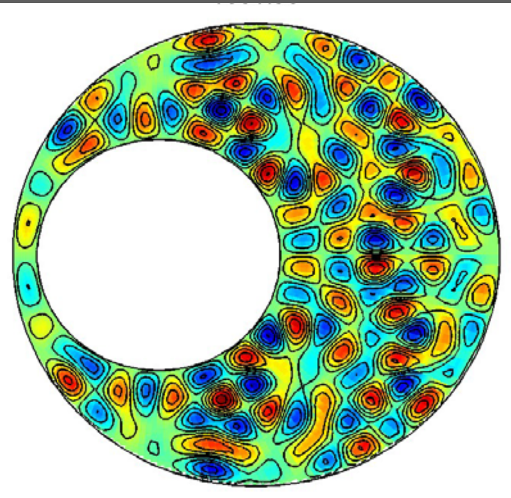


λ_{100}=181.2349

Weyl's Law

$\lambda_n \sim 4\pi n \,/\, A$

(A = area of region)



Planck 1918

blackbody radiation

also Wien 1911

3. Isospectral drums

Kac 1966

"Can one hear the shape of a drum?"

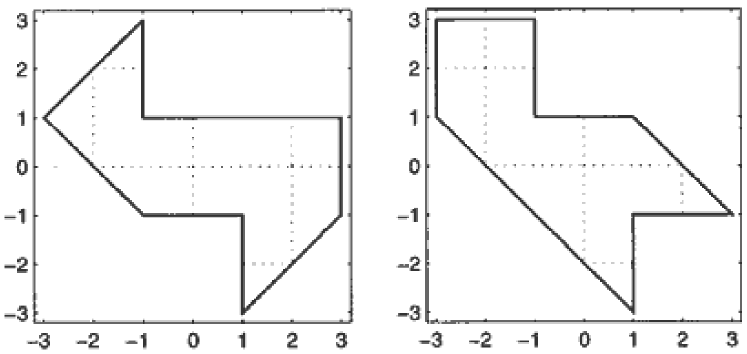
Gordon, Webb & Wolpert 1992

"Isospectral plane domains and surfaces via Riemannian orbifolds"

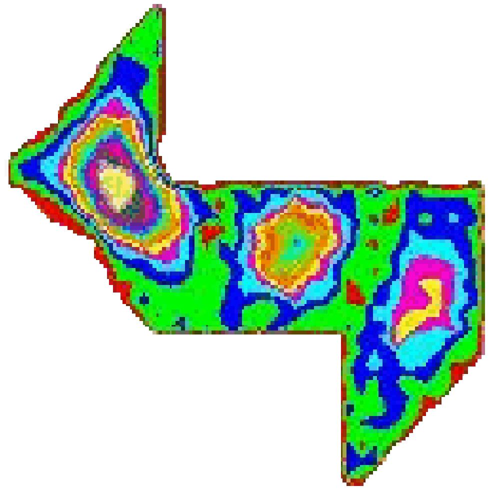
Numerical computations:

Driscoll 1997

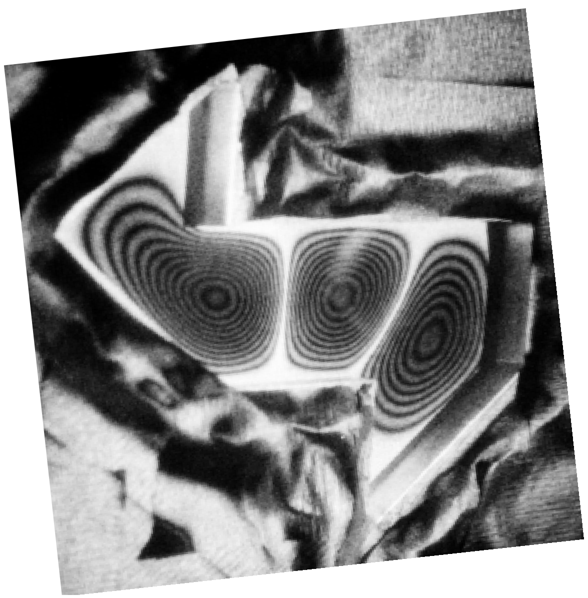
"Eigenmodes of isospectral drums"



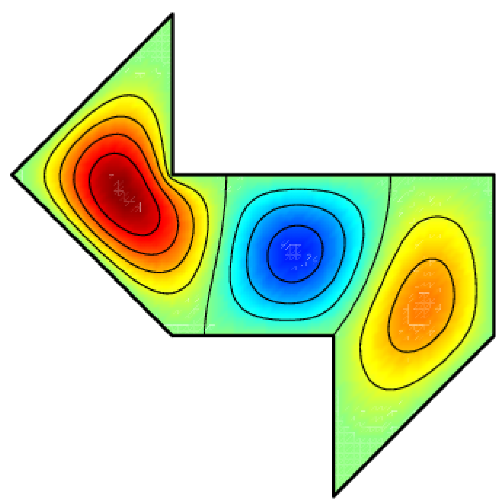
Microwave experiments
Sridhar & Kudrolli 1994

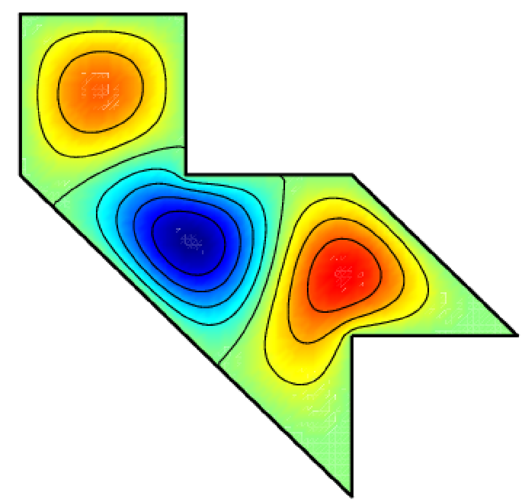


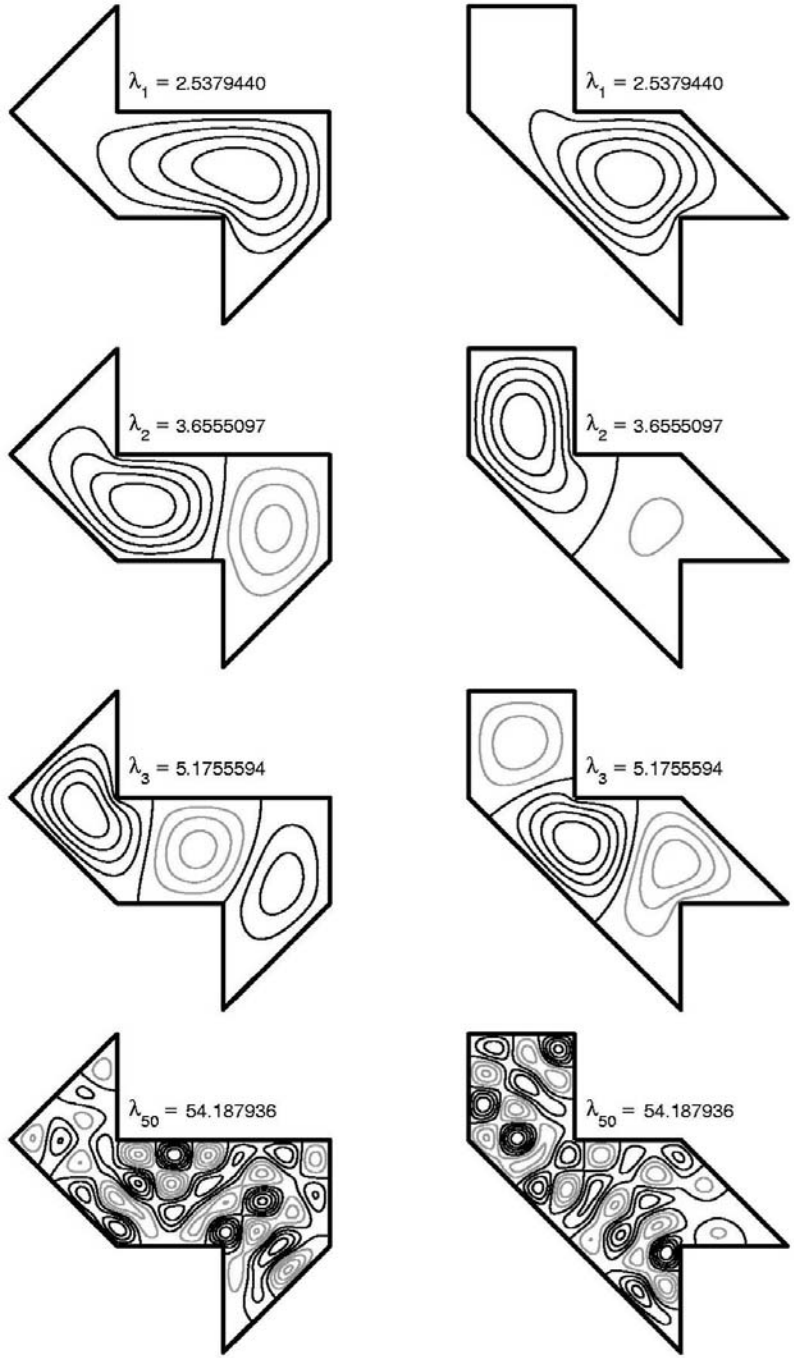
Holographic
interferometry
Mark Zarins 2004



Computations with "DTD method"
Driscoll 1997

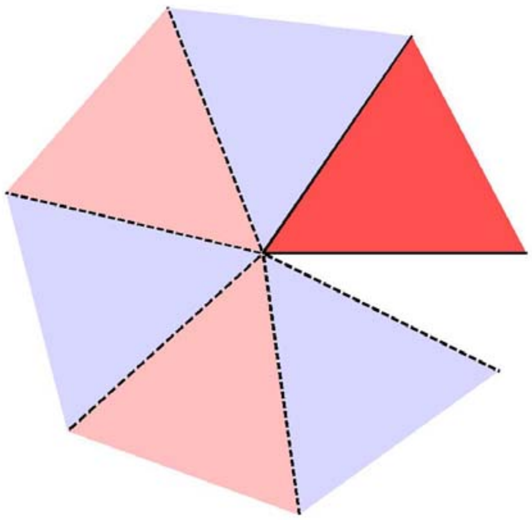






$\lambda_1 = 2.5379440$
$\lambda_1 = 2.5379440$
$\lambda_2 = 3.6555097$
$\lambda_2 = 3.6555097$
$\lambda_3 = 5.1755594$
$\lambda_3 = 5.1755594$
$\lambda_{50} = 54.187936$
$\lambda_{50} = 54.187936$

A MATHEMATICAL/NUMERICAL ASIDE: WHICH CORNERS CAUSE SINGULARITIES?

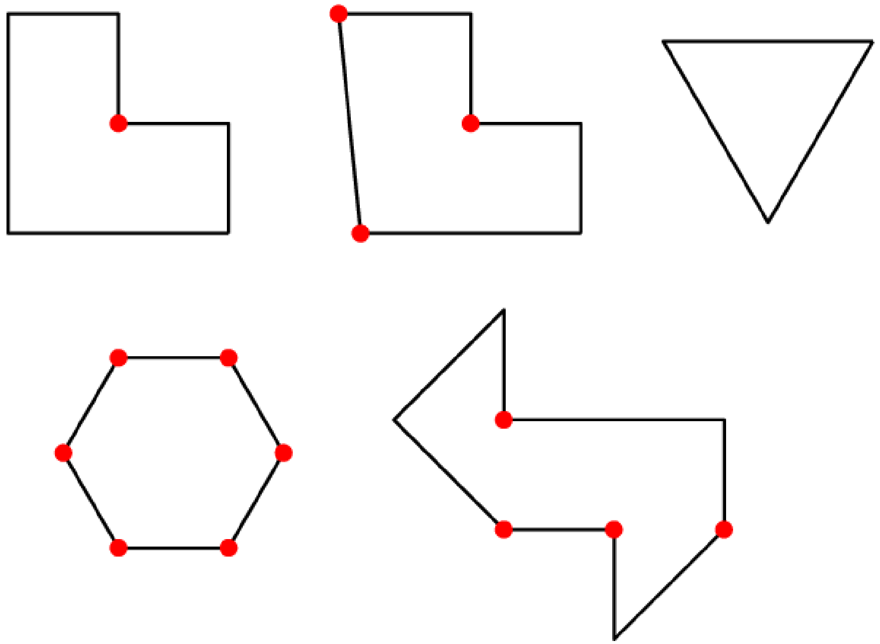


That is, at which corners are eigenfunctions not analytic?
(Effective numerical methods need to know this.)

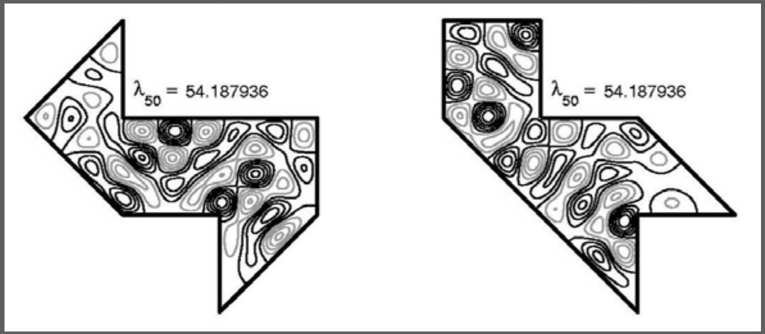
Answer: all whose angles are π/α , $\alpha \neq$ integer

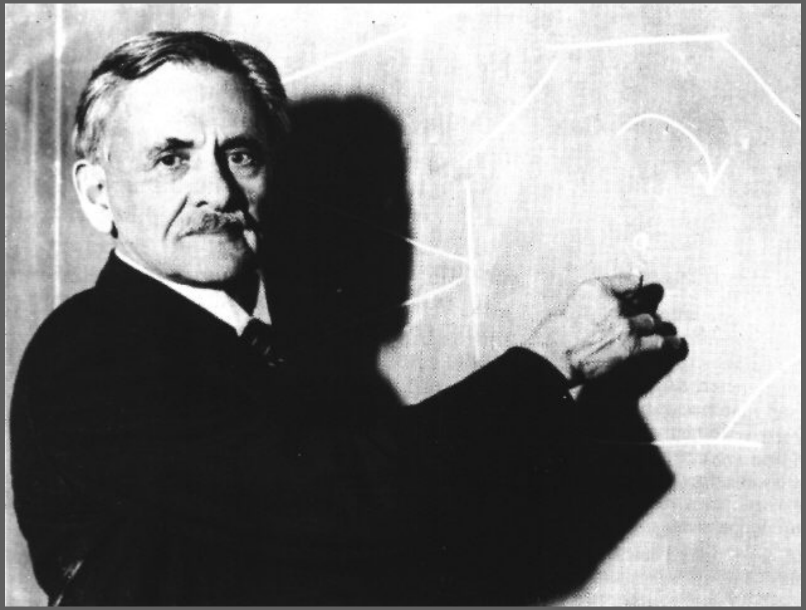
Proof: repeated analytic continuation by reflection

E.G., the corners marked in red are the singular ones:



Michelson 1907



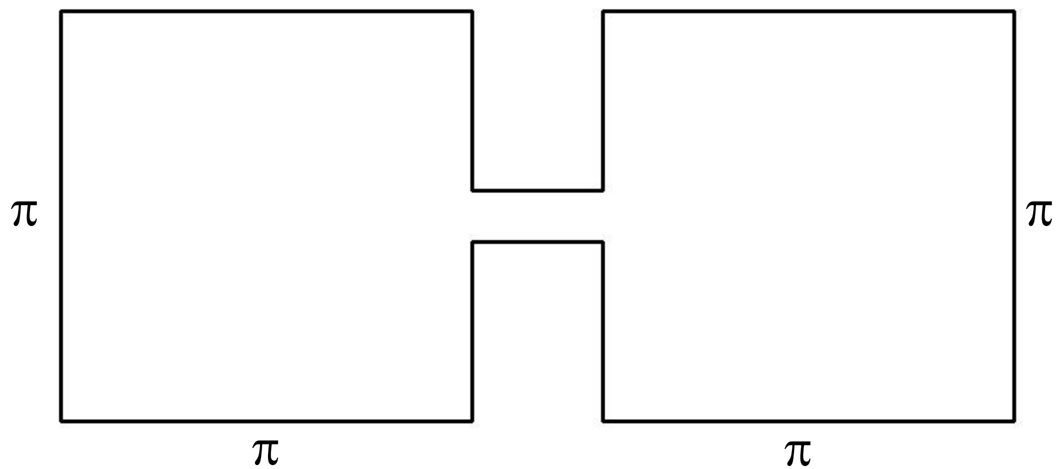


spectroscopy

also Bohr 1922, Bloembergen 1981

4. Line splitting

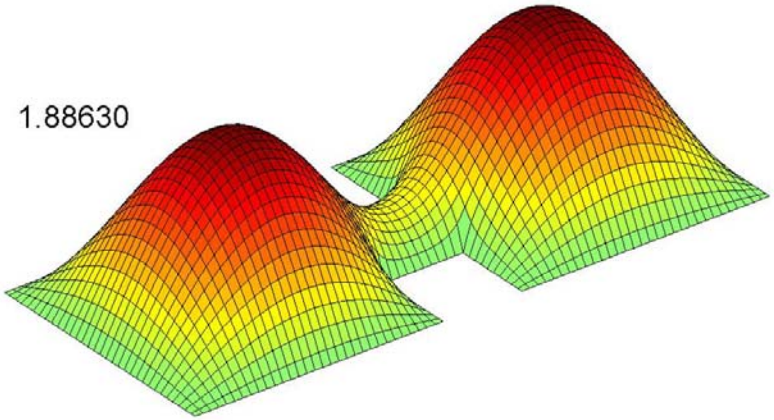
"Bongo drums"—two chambers, weakly connected.

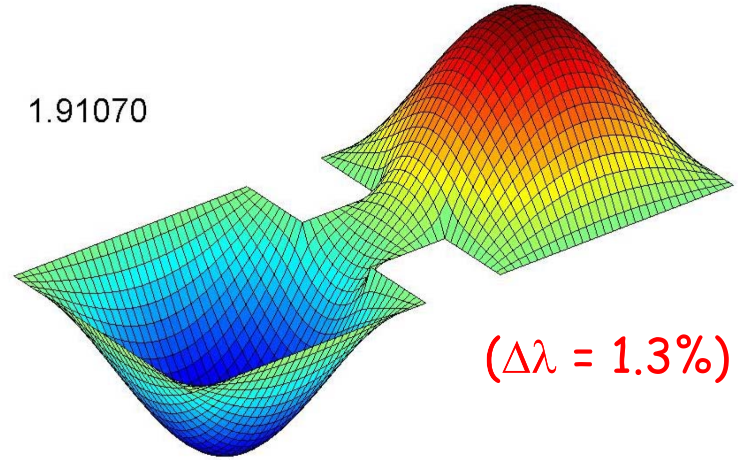


Without the coupling the eigenvalues are

$\{ j^2 + k^2 \} = \{ \quad 2, 2, \quad 5, 5, 5, 5, \quad 8, 8, \quad 10, 10, 10, 10, \ldots \}$

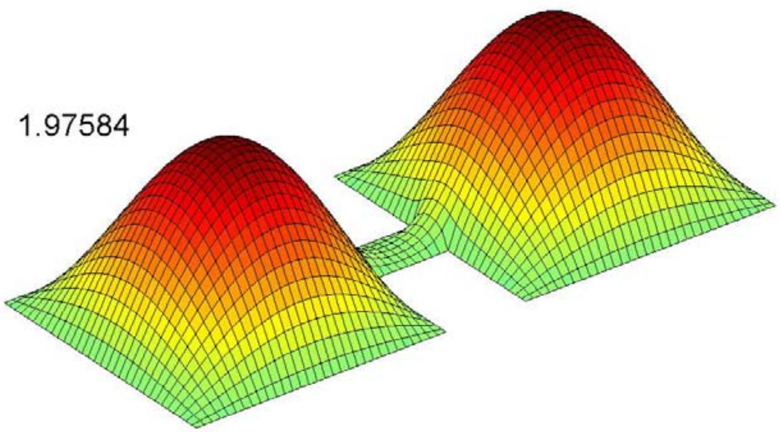
With the coupling…

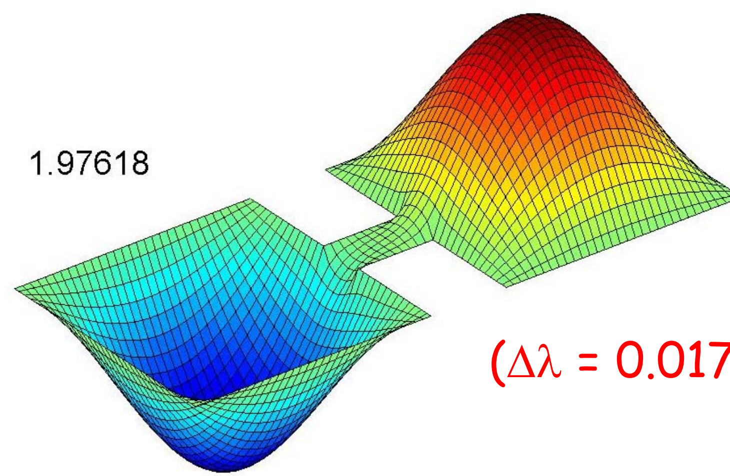




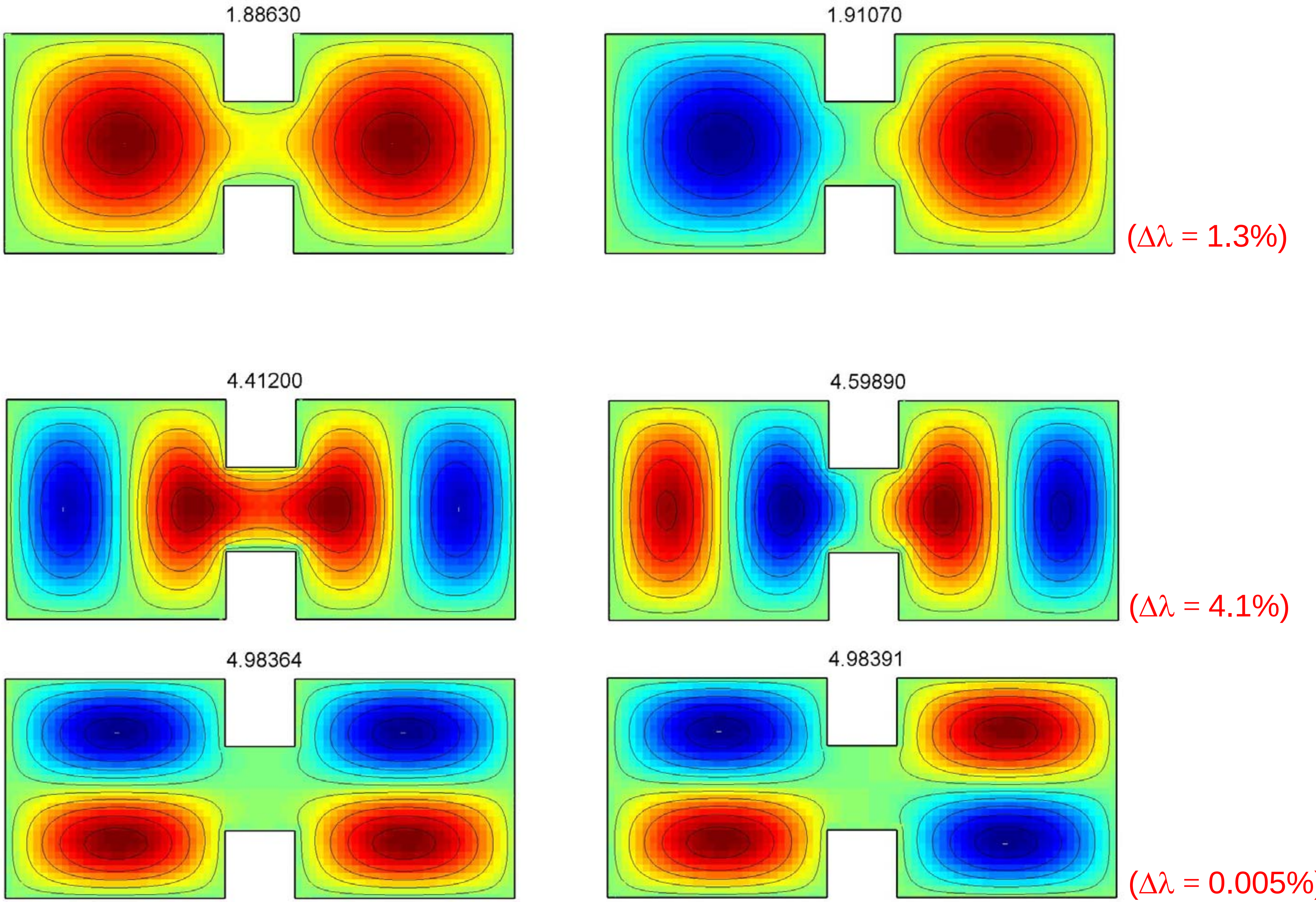
($\Delta\lambda$ = 1.3%)

Now halve the width of the connector.





($\Delta\lambda$ = 0.017%)



1.88630
1.91070
(Δλ = 1.3%)
4.41200
4.59890
(Δλ = 4.1%)
4.98364
4.98391
(Δλ = 0.005%)

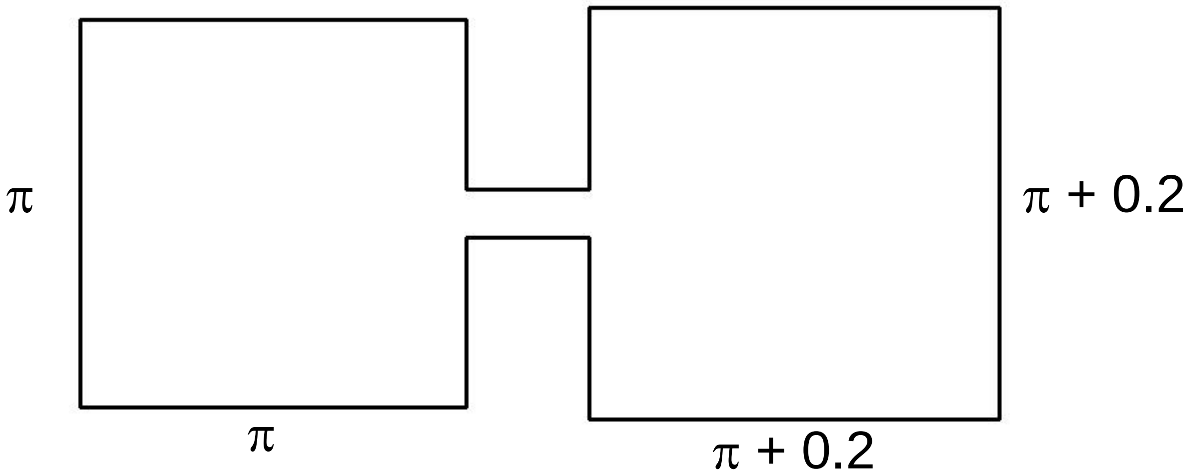
Zeeman 1902   Stark 1919

Zeeman & Stark effects:
line splitting in magnetic & electric fields

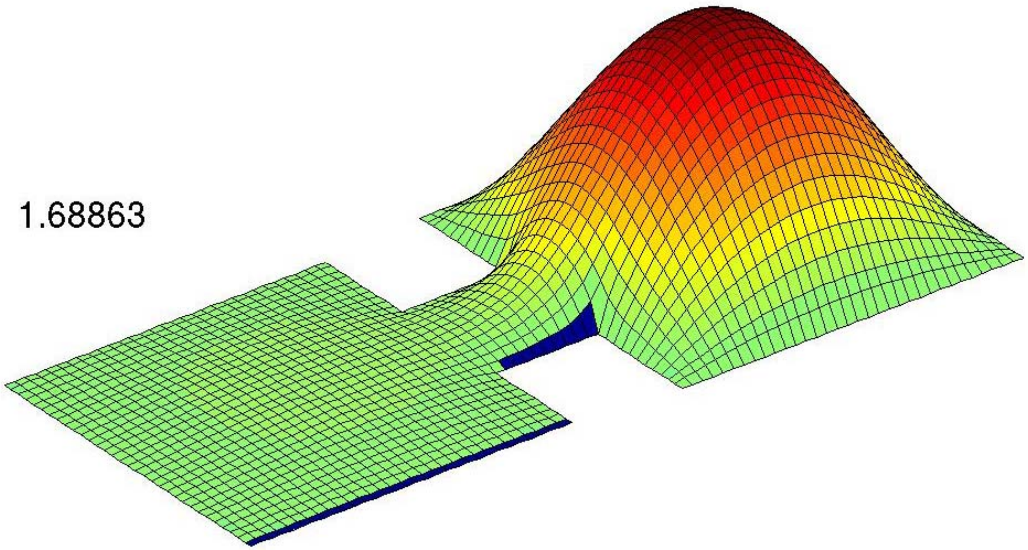
also Michelson 1907, Dirac 1933, Lamb & Kusch 1955

5. Localization

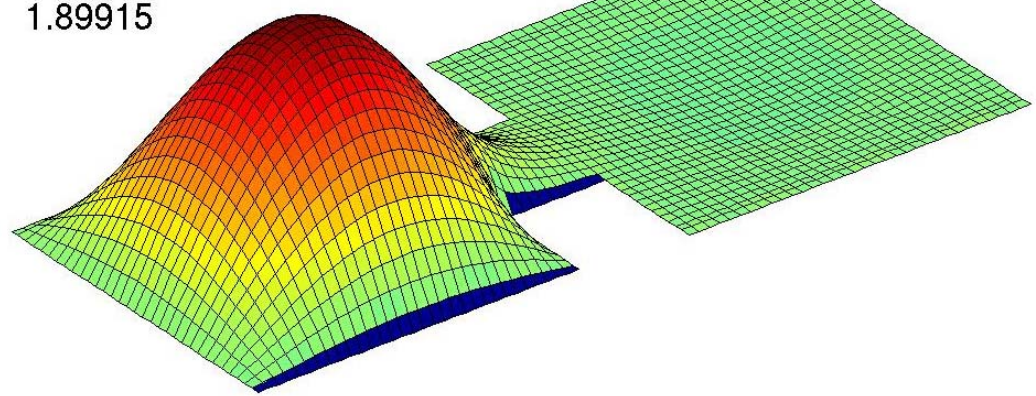
What if we make the bongo drums a little asymmetrical?

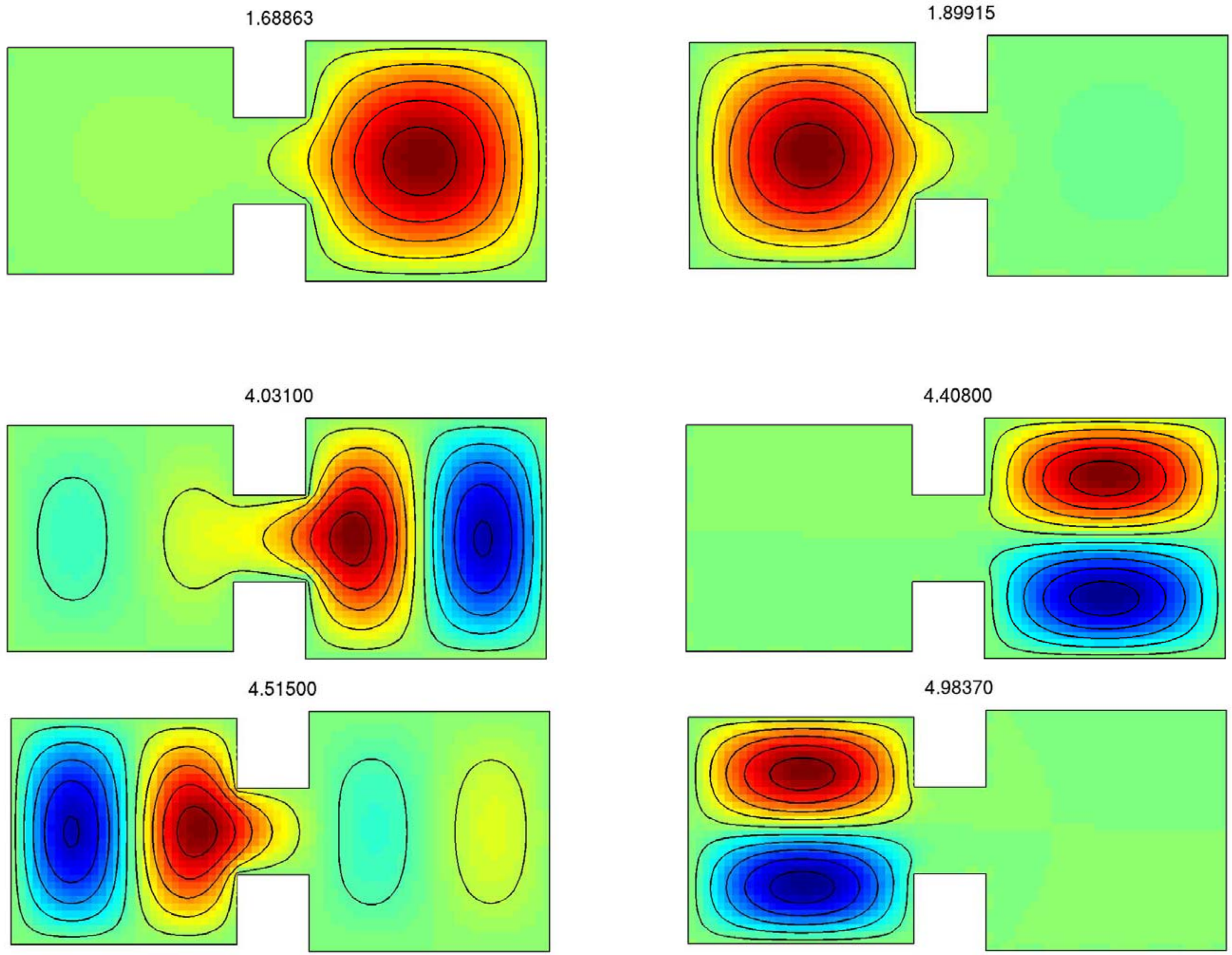


1.68863

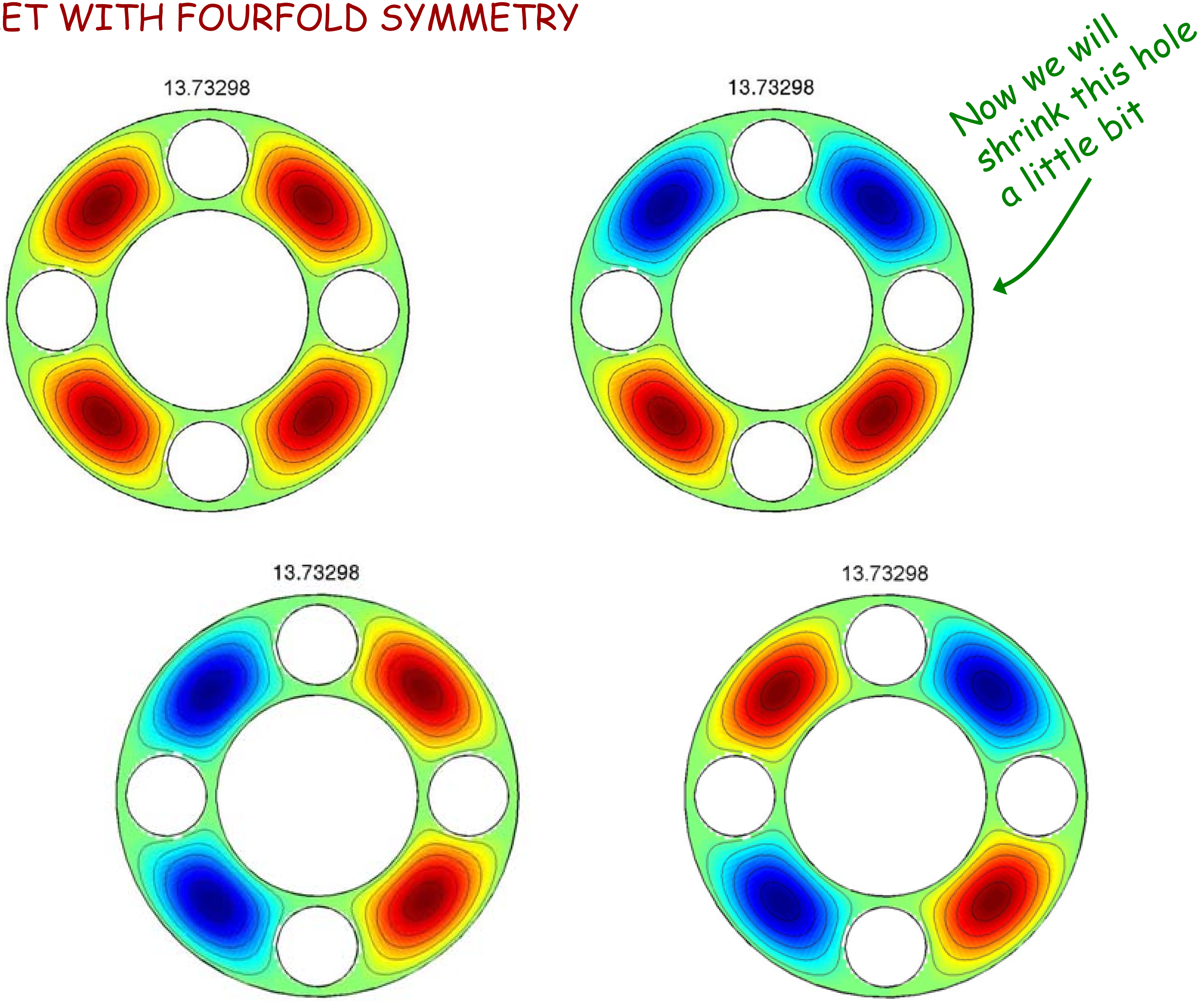


1.89915

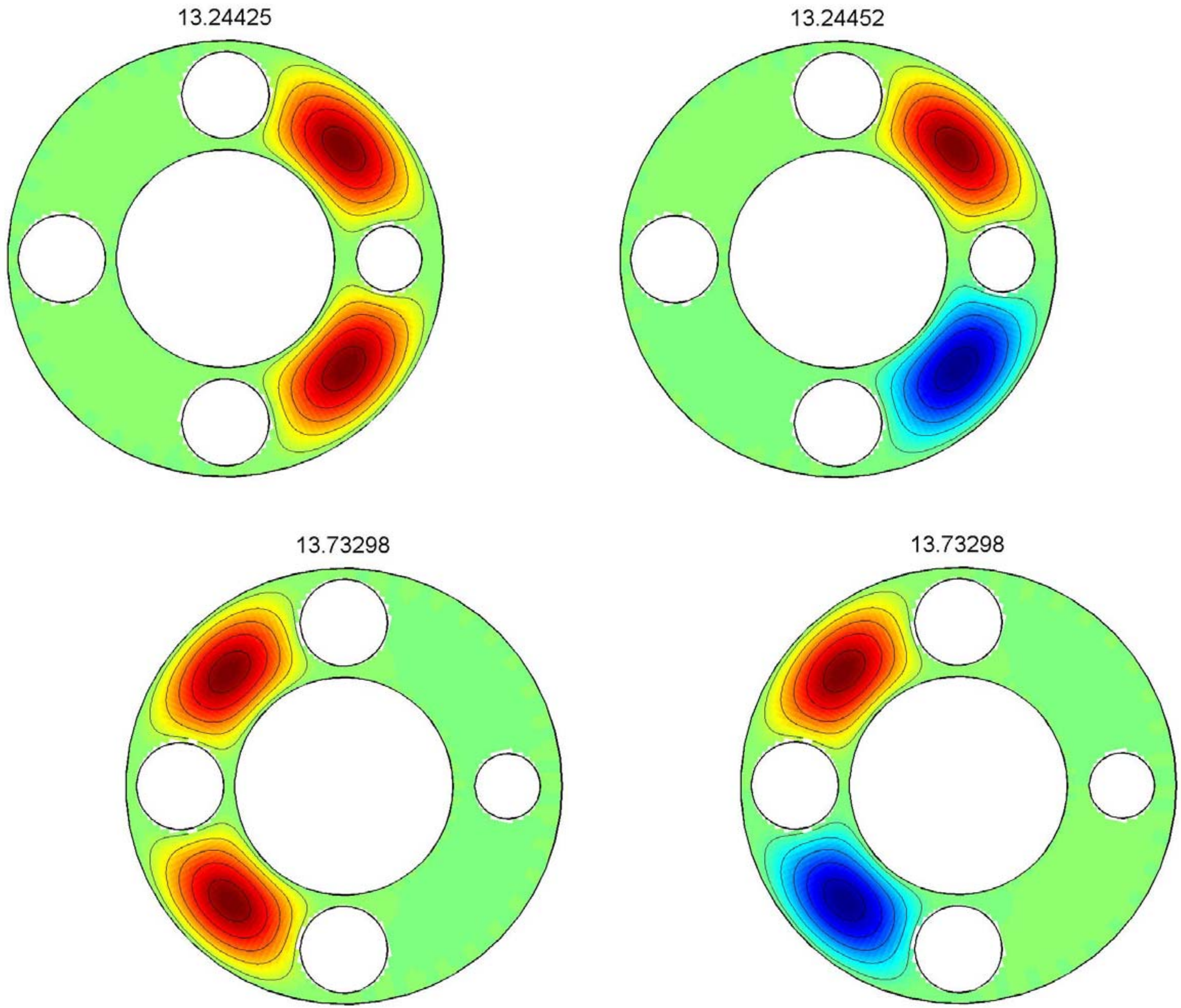



1.68863
1.89915
4.03100
4.40800
4.51500
4.98370

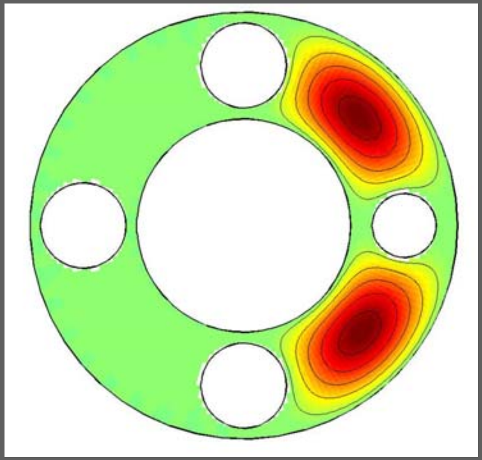
GASKET WITH FOURFOLD SYMMETRY



GASKET WITH BROKEN SYMMETRY



Anderson 1977

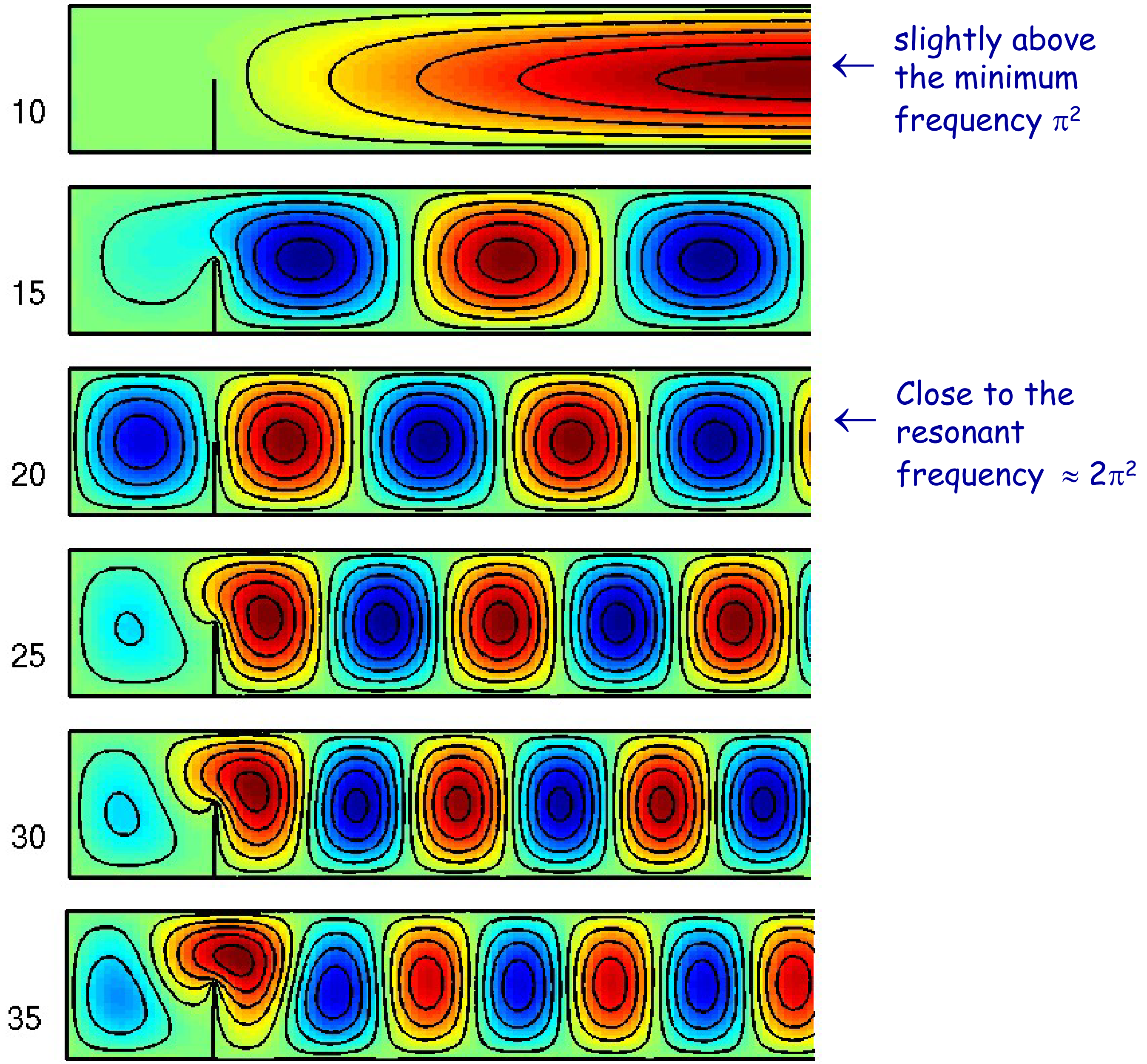


Anderson localization
and disordered materials

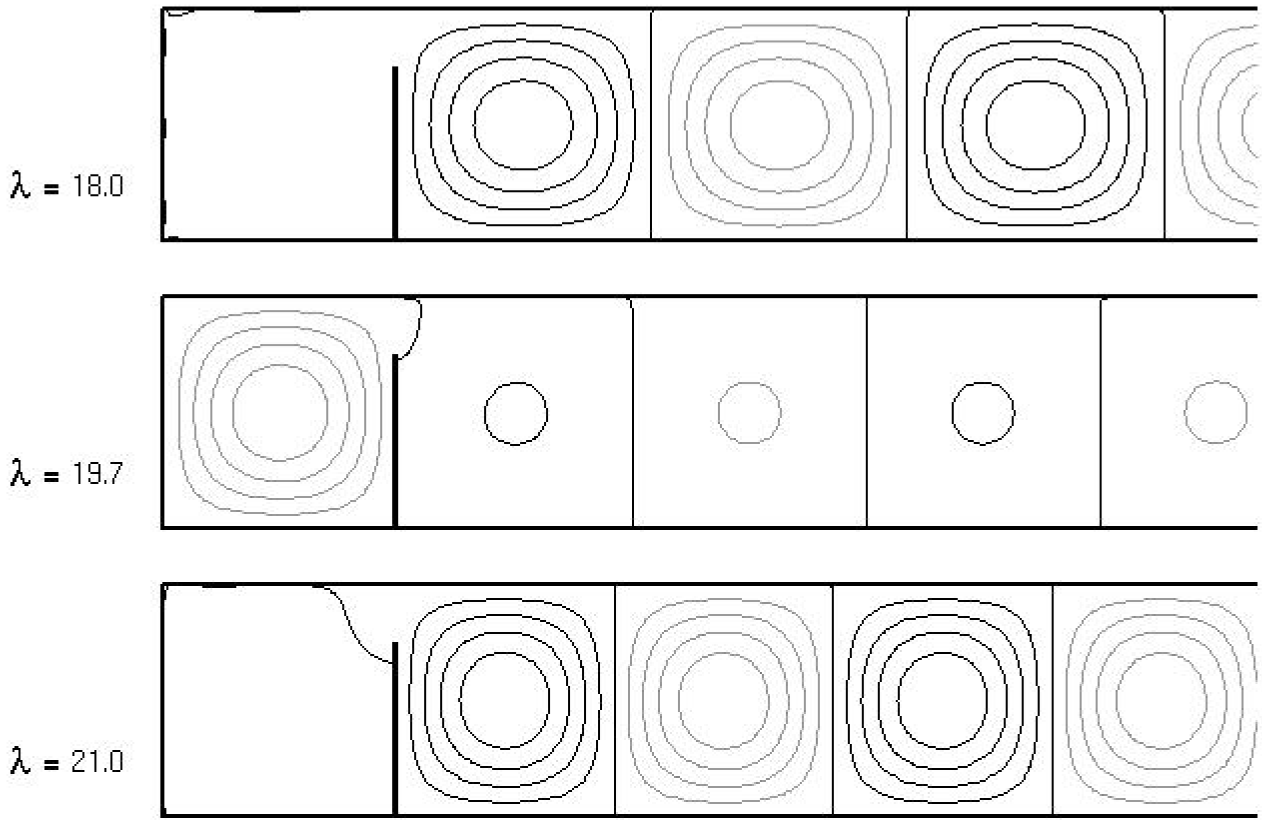
6. Resonance

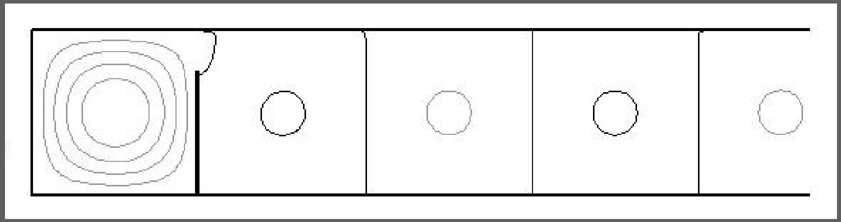
A square cavity coupled to a semi-infinite strip of width 1

The spectrum is continuous: $[\pi^2,\infty)$



Lengthening the slit strengthens the resonance:





Marconi 1909

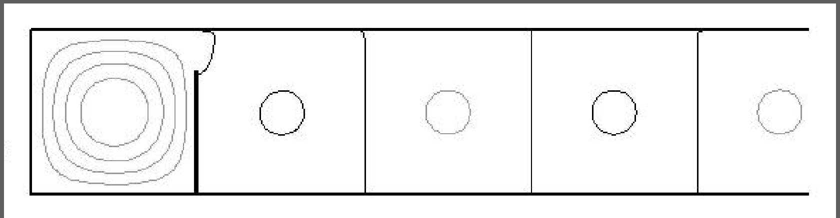
Radio

Purcell 1952

Nuclear Magnetic Resonance

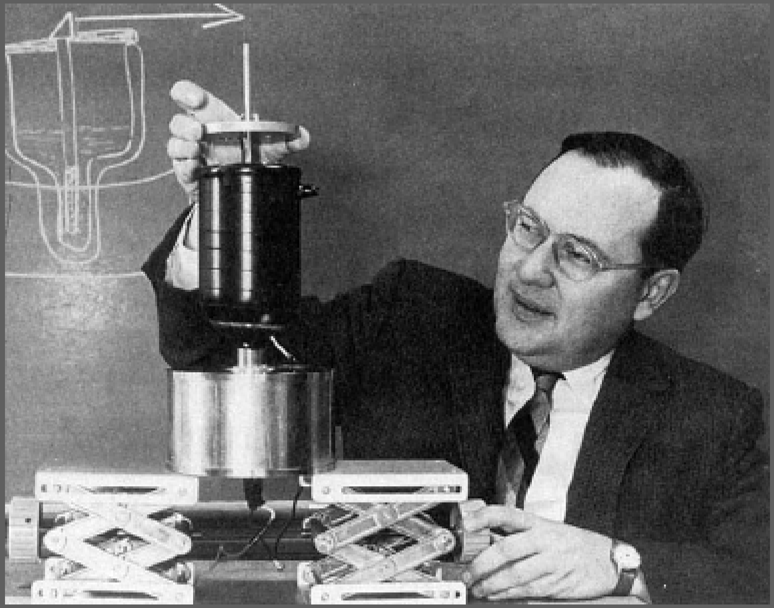
also Bloch 1952

Masers and lasers



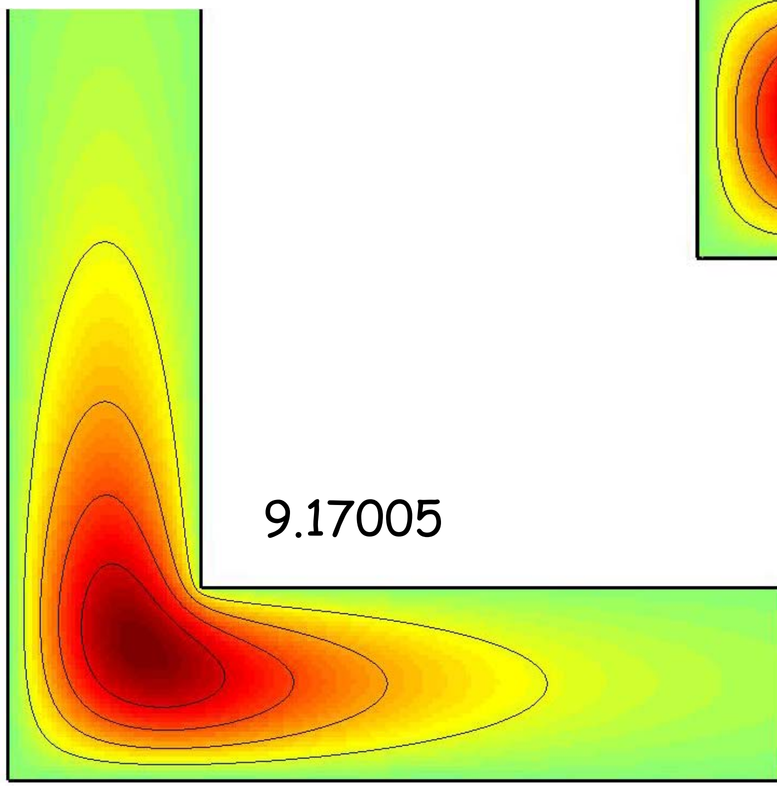
Townes 1964

Schawlow 1981

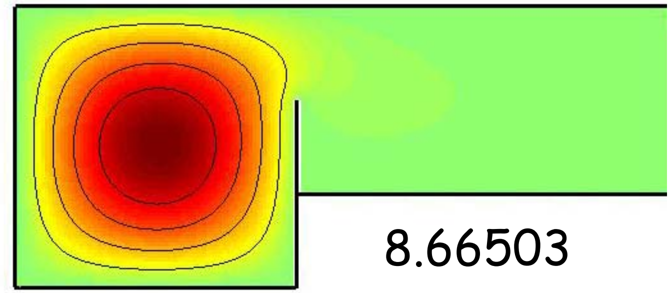


also Basov and Prokhorov 1964

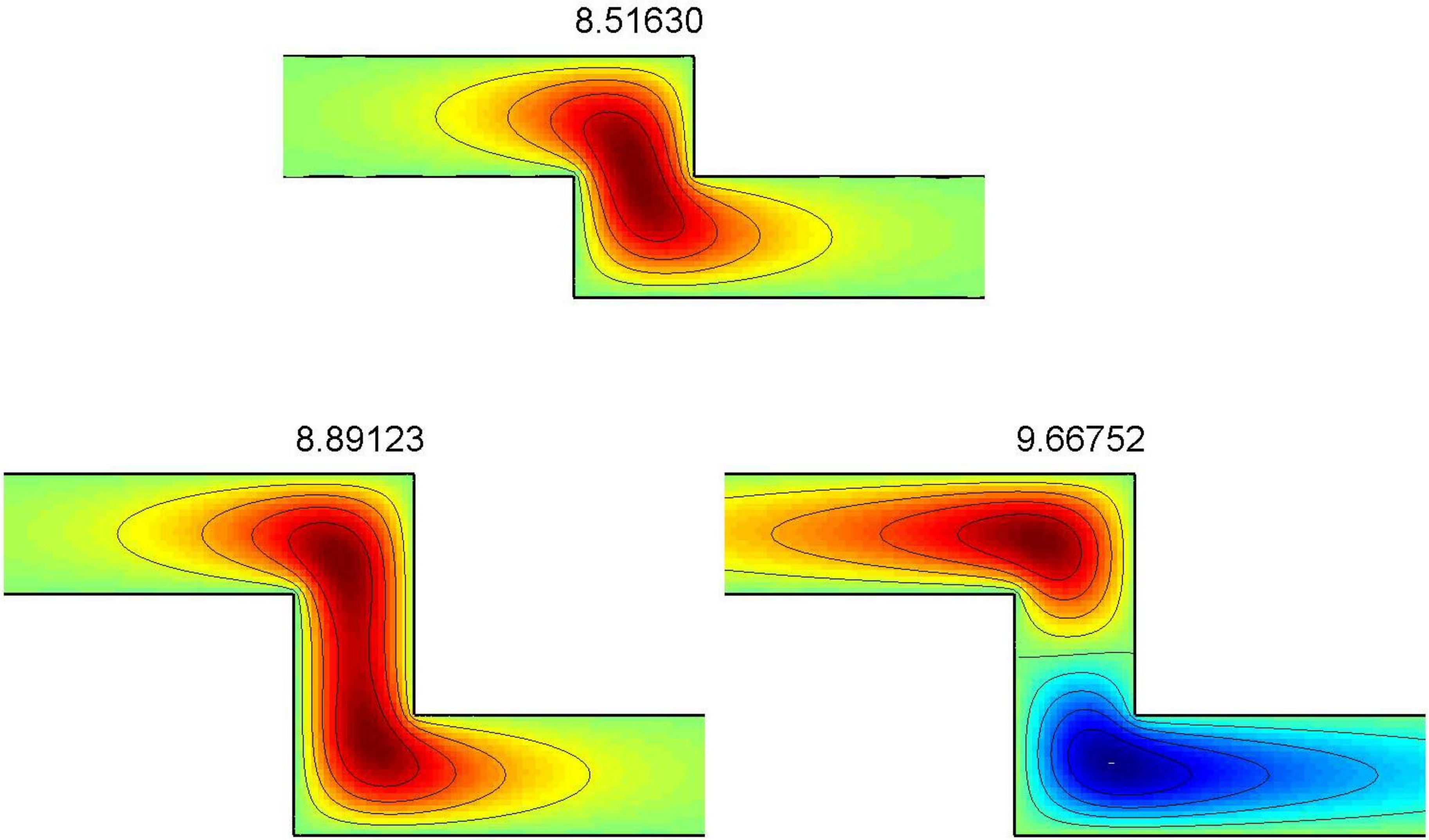
7. Bound states



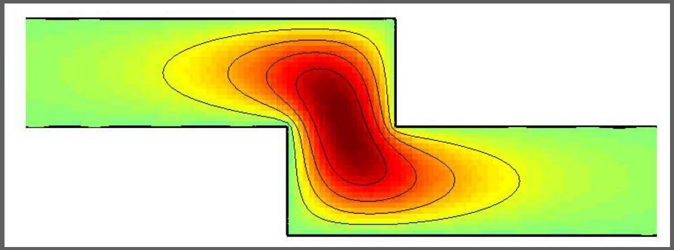
9.17005

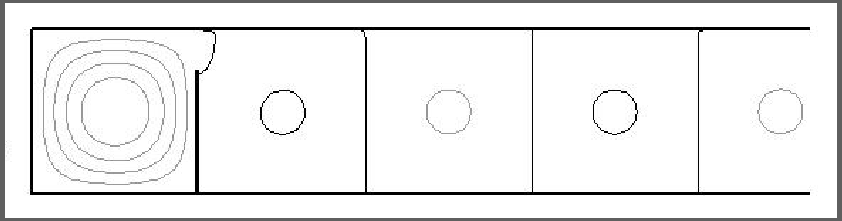


8.66503



See papers by Exner and 1999 book by Londergan, Carini & Murdock.



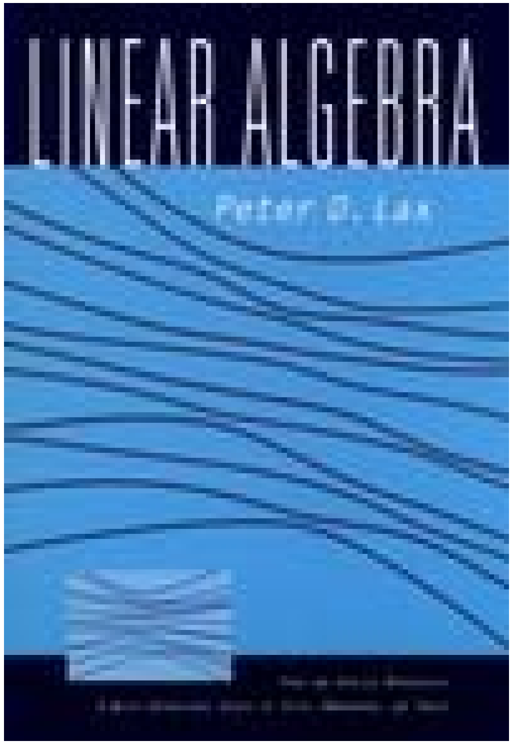


Marie Curie 1903

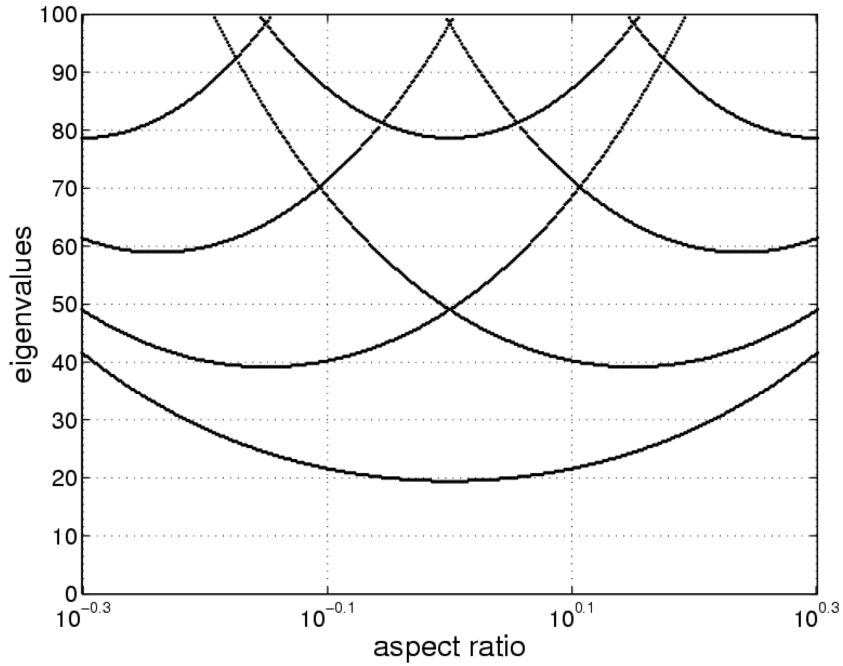
Radioactivity

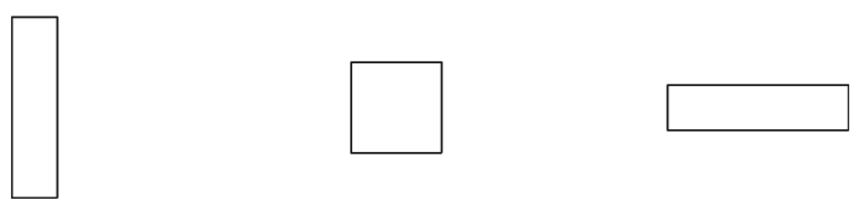
also Becquerel and Pierre Curie 1903
Rutherford 1908 [Chemistry]
Marie Curie 1911 [Chemistry]
Iréne Curie & Joliot 1935 [Chemistry]

8. Eigenvalue avoidance

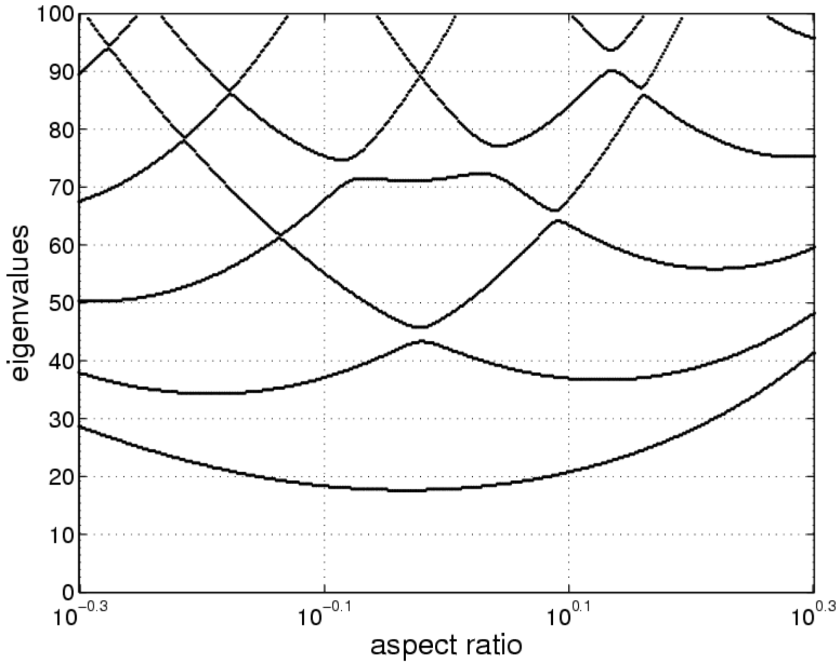


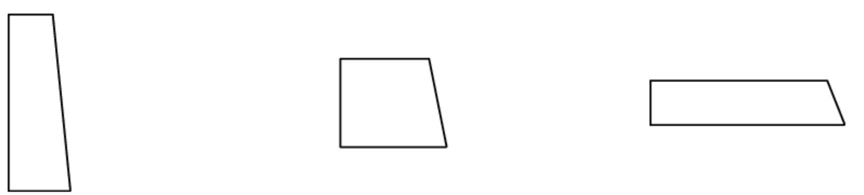
Eigenvalue degeneracy for rectangles of varying aspect ratio



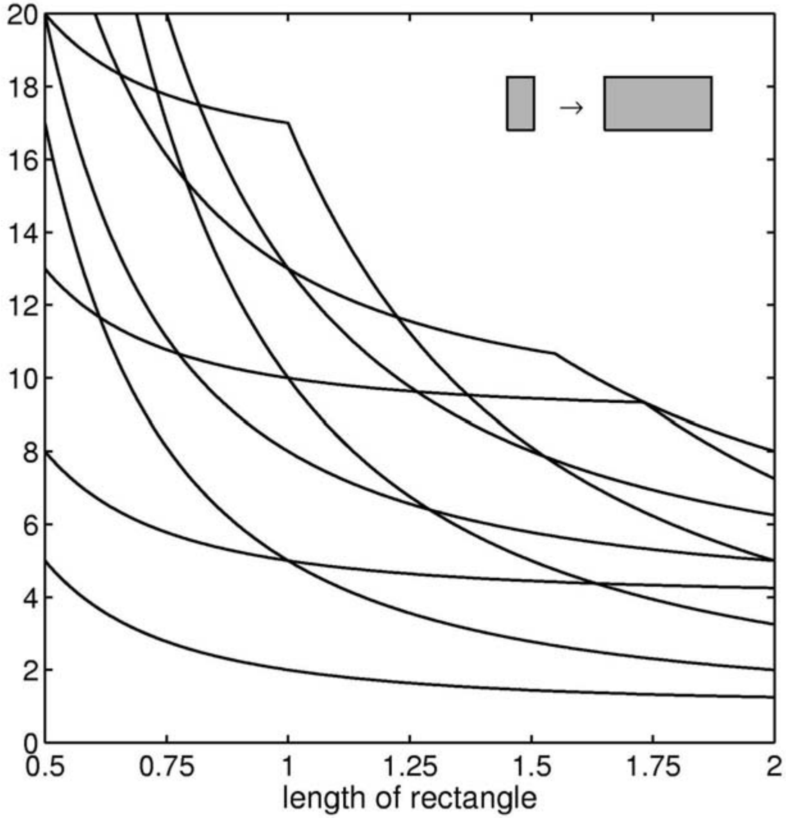


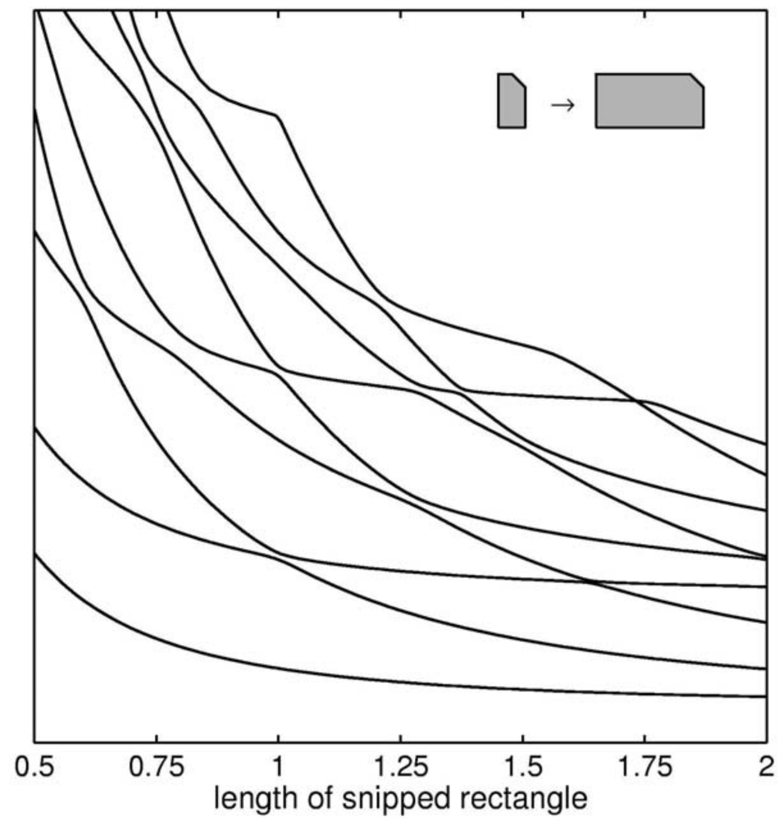
Eigenvalue avoidance for perturbed rectangles



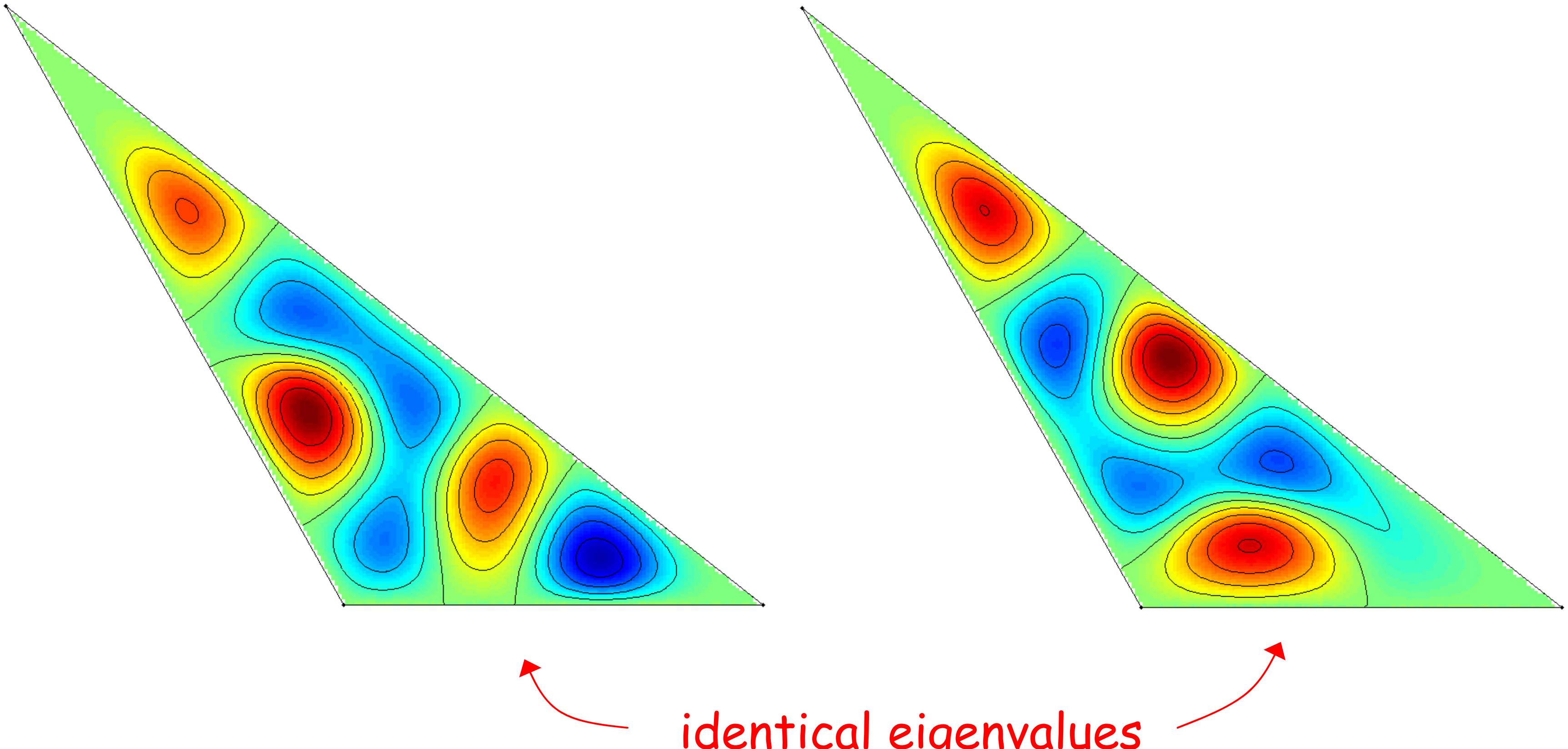


Another example of eigenvalue avoidance

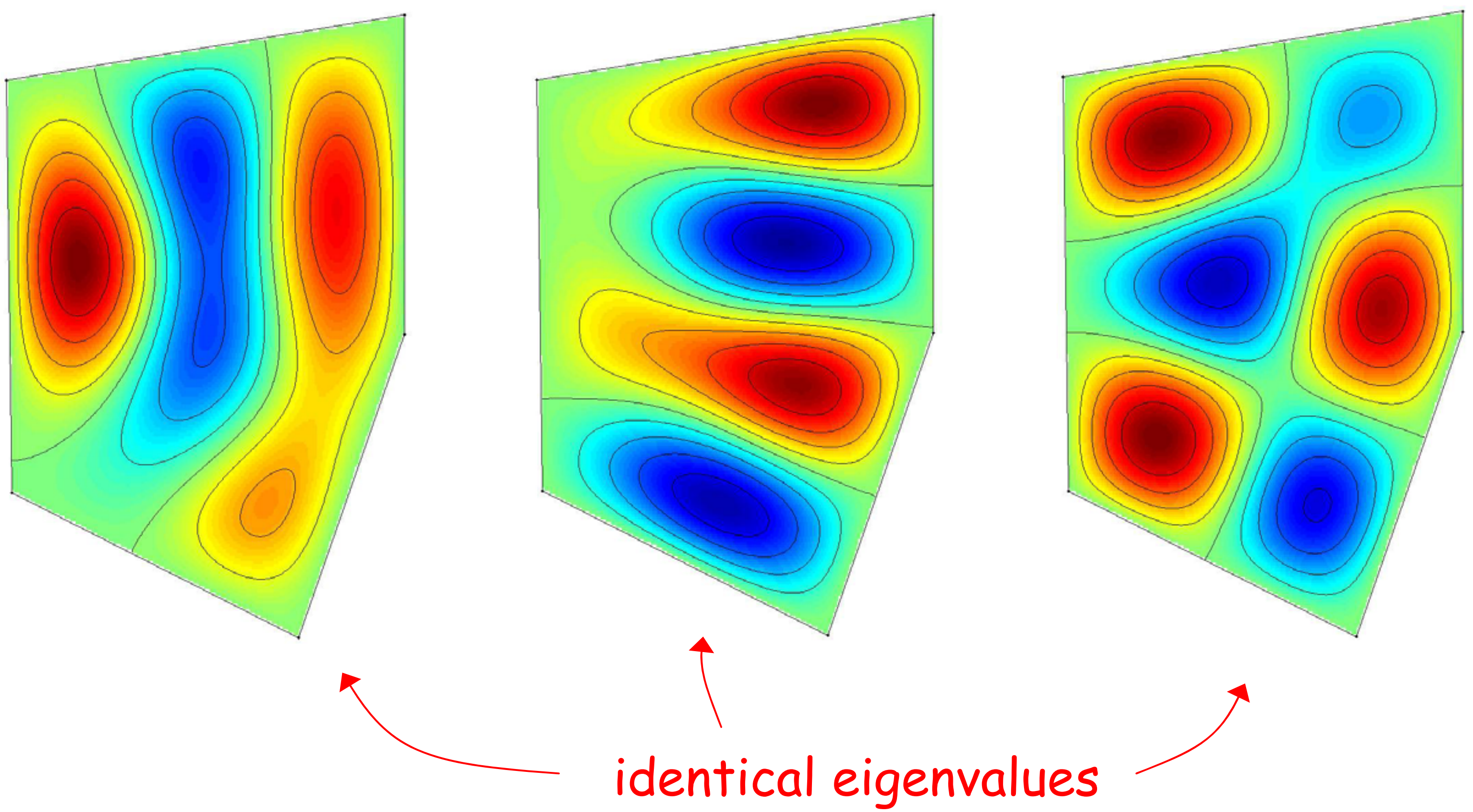




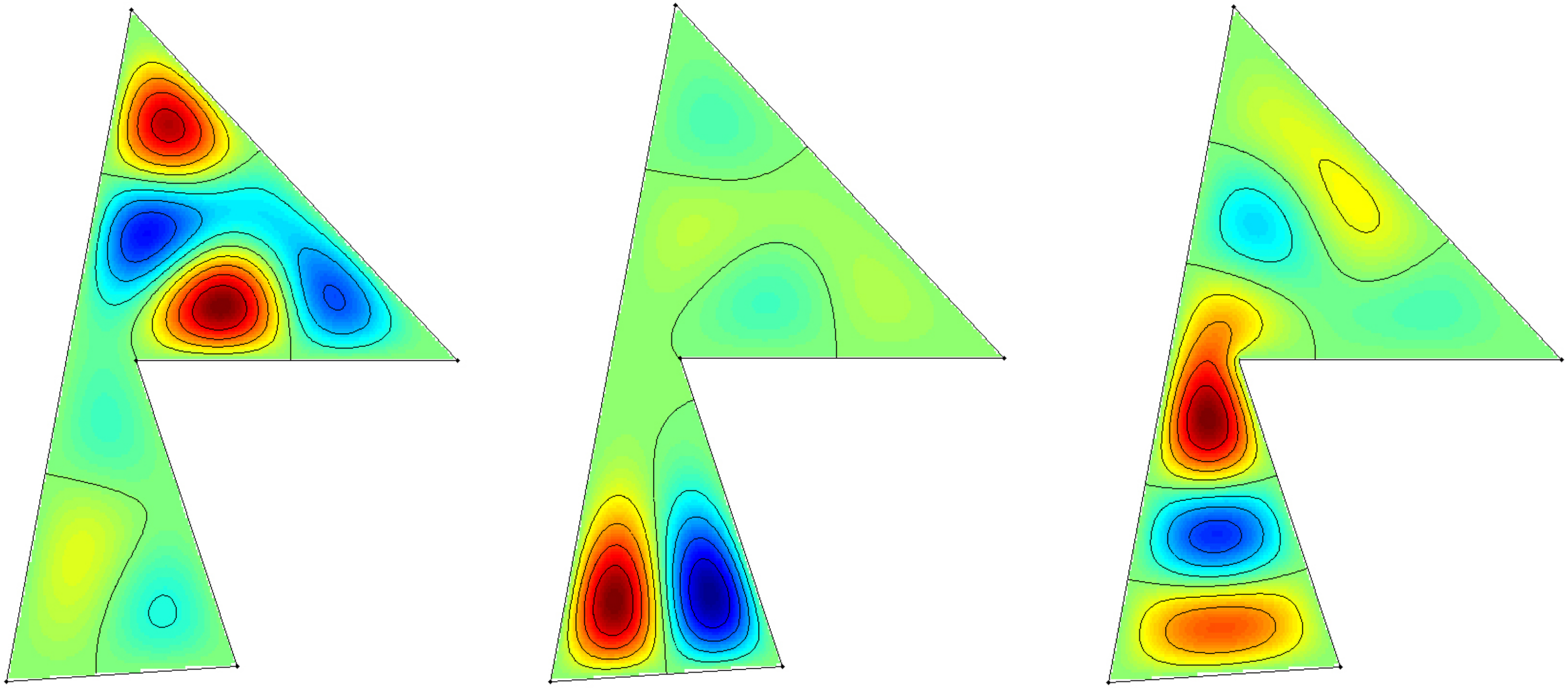
By varying 2 parameters, Berry and Wilkinson 1984
found triangular drums with "accidental" degeneracy.



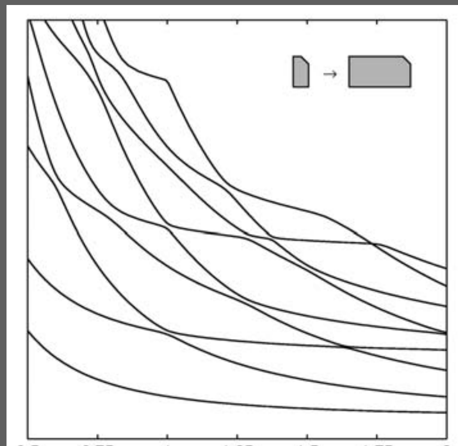
For a triple eigenvalue you need 5 parameters.
For a polygon with 5 parameters you need a pentagon.
Simon Wojczyszyn and I have found some such examples.



Here's another example of a triple eigenvalue in a pentagon.



Wigner 1963



Energy states of heavy nuclei
↔ eigenvalues of random matrices

cf. Montgomery, Odlyzko, … "The 10^{20}th zero of the Riemann zeta function and 70 million of its neighbors"

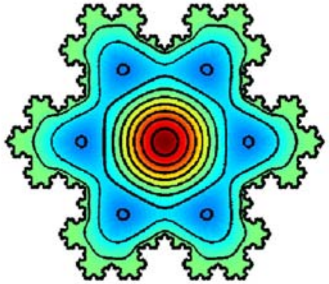
… Riemann Hypothesis?…

We've mentioned:

Zeeman 1902
Becquerel, Curie & Curie 1903
Michelson 1907
Marconi & Braun 1909
Wien 1911
Planck 1918
Stark 1919
Bohr 1922
Heisenberg 1932
Dirac 1933
Schrödinger 1933
Bloch & Purcell 1952
Lamb & Kusch 1955
Wigner 1963
Townes, Basov & Prokhorov 1964
Anderson 1977
Bloembergen & Schawlow 1981

We might have added:

Barkla 1917
Compton 1927
Raman 1930
Stern 1943
Pauli 1945
Mössbauer 1961
Landau 1962
Kastler 1966
Alfvén 1970
Bardeen, Cooper & Schrieffer 1972
Cronin & Fitch 1980
Siegbahn 1981
von Klitzing 1985
Ramsay 1989
Brockhouse 1994
Laughlin, Störmer & Tsui 1998
...



Thank you!