Notes of a Numerical Analyst

What We Learned from Galois

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I am a passionate mathematician, but with my computational perspective, I lie a standard deviation or two away from the LMS mean. I hope you find these columns stimulating, and I would be very glad to hear from you.

A portrait by his brother Alfred

Today I'd like to reflect on Évariste Galois, the fiery genius who died in 1832 at the age of 20. As is well known, the quadratic formula was discovered in antiquity and the Renaissance Italians found analogous formulae for roots of polynomials of degrees 3 and 4. It took until the 19th century before Ruffini and Abel proved that there are no such formulae for degrees $n \geq 5$.

And then came the brilliant Galois. Galois realized that the nonexistence of certain formulae was a consequence of deeper structures, of group symmetries in fields and their extensions that had not been thought about before. The eventual impact (it took a while) was enormous, as profound a paradigm shift in mathematics as you could ask for. Here is how Fernando Corbalán [1] puts it:

"It was the beginning of a true revolution: the end of algebra as understood for centuries (whose main objective was the solution of equations) and the turn to the new problem of the characterization of various structures. This was a step toward modern mathematics."

Or in the words of Michael Harris [2]:

"Galois created a new *point of view*: that what's interesting is no longer the centuries-old goal of finding a root of the equation, but rather to understand the structure of all the roots."

Heady words! It would seem that two things are true: (1) We can't compute roots of polynomials, and (2) There is no need to.

Statement (1) is false. Using standard algorithms implemented in standard software, I can calculate all the roots of a degree 1000 polynomial with random coefficients to 15 digits of accuracy on my laptop in one second. In any but the most artificial sense, roots of polynomials are as computable as π or e.

Statement (2) is false too. After Galois's ideas sank in, did the numerical values of roots of polynomials cease to matter? Of course not. What happened was, rather, that after centuries of trying to develop methods to calculate them, mainstream mathematicians lost interest in the problem. We rewrote our job description. Rather than deciding our field had doubled, we decided it had shifted.

So, for my money, Galois marks not one but two shifts in the history of mathematics. One is the birth of modern algebra. The other is the separation of pure from applied.

I could tell a story about differential equations and Poincaré. . . .

FURTHER READING

[1] Fernando Corbalán, *Galois: Revolución y Matemáticas*, 3rd ed., Nivola Libros y Ediciones, Madrid, 2010.

[2] Michael Harris, *Mathematics without Apologies: Portrait of a Problematic Vocation*, Princeton, 2015.

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