Lewy-Hörmander nonexistence and pseudospectra

From sec. 13 of T. & Embree,
Spectra and Pseudospectra,
Princeton, 2005.

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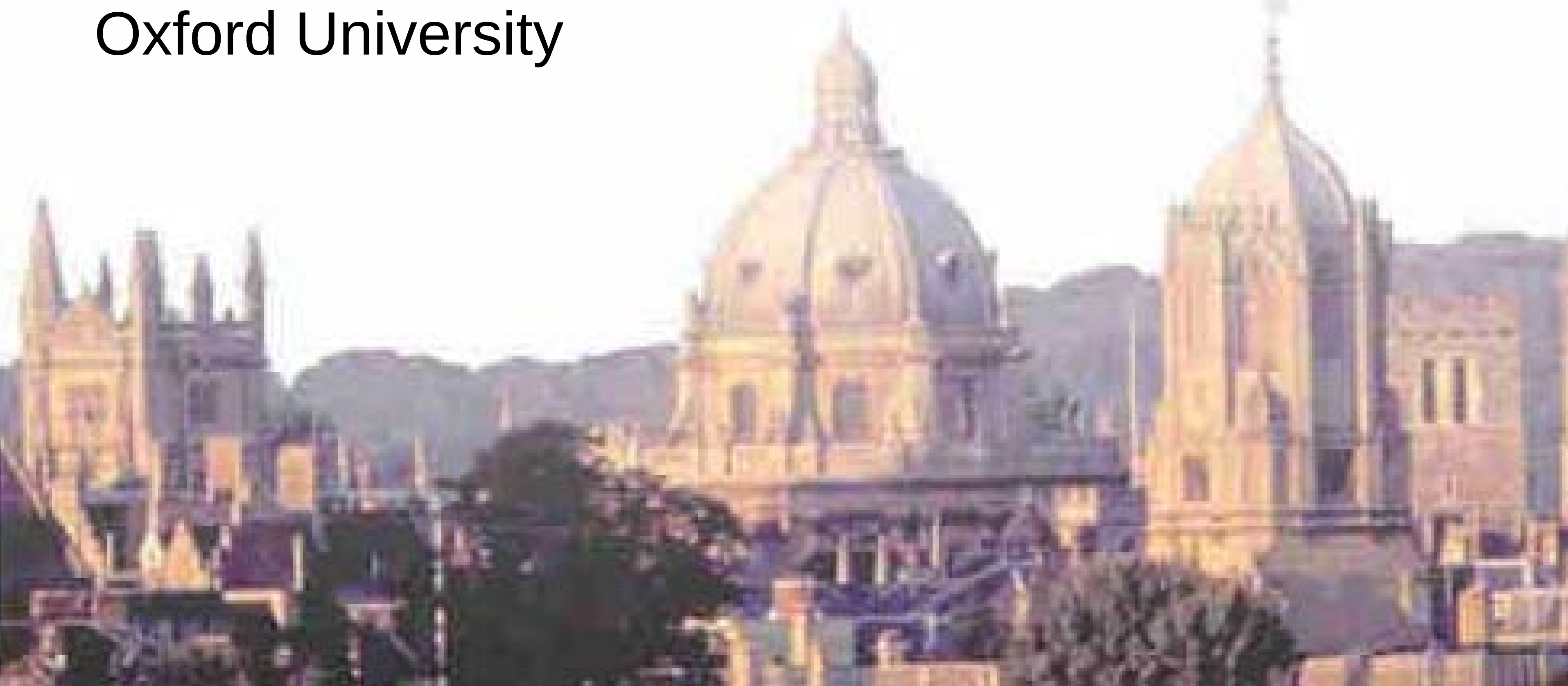

1957 Lewy shows $\exists$ C^∞ linear PDE with no solutions

1960 General theory by Hörmander

1960-2005 Major generalizations (including ΨDE) leading to characterization by "Ψ condition": Beals, Dencker, Fefferman, Garabedian, Lerner, Nirenberg, Treves, …

2001 Zworski points out connection with pseudospectra

Hörmander: Fields Medal '62
Fefferman: Fields Medal '78
Nirenberg: Crafoord Prize '82, Nat. Medal of Science '95
Dencker: Gårding Prize '03, Clay Research Prize '05

Simplest Example

Mizohata eq. : $Lu = u_x + ixu_y = f$ $(*)$

Cauchy-Kowaleski $\Rightarrow$ $(*)$ is locally solvable if f is analytic.

THEOREM. $\exists\ f \in C^\infty$ s.t. $(*)$ has no soln in any nbhd of (0,0).

Why? The idea is that nonexistence for $(*)$ is related to nonuniqueness for the adjoint equation

$$L^*v = -v_x + ixv_y = g \qquad (**)$$

This nonuniqueness stems from the existence of *wave packet pseudomodes* of L^*. Such pseudomodes exist when a certain twist or commutator or Ψ or Poisson bracket condition is satisfied.

PSEUDOSPECTRA

Equivalent Definitions

Given: matrix A, norm $\|\cdot\|$, $\varepsilon > 0$

The ε-pseudospectrum $\Lambda_\varepsilon(A)$ of A
is the set of all complex z such that

large resolvent norm → $$\|(zI - A)^{-1}\| > \varepsilon^{-1}$$

or

perturbation of spectrum → z is an eigenvalue of A+E, some $\|E\| < \varepsilon$

or

pseudo-eigenmode → $$\|Au - zu\| < \varepsilon, \text{ some } \|u\| = 1$$

or

small singular value → $$\sigma_{\min}(zI - A) < \varepsilon$$ [if $\|\cdot\| = \|\cdot\|_2$]

For operators, same defns. modulo technicalities.

(Secs. 7-13 of *Spectra and Pseudospectra* :)

	Toeplitz matrices	*Differential operators*
const. coeff.		
variable coeff.		

(Secs. 7-13 of *Spectra and Pseudospectra* :)

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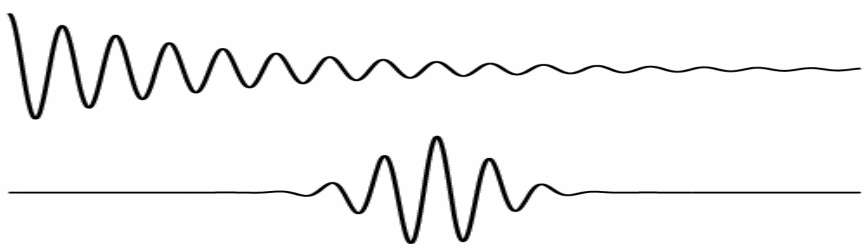
const. coeff.	boundary pseudomodes	
variable coeff.	wave packet pseudomodes	



(Secs. 7-13 of *Spectra and Pseudospectra*:)

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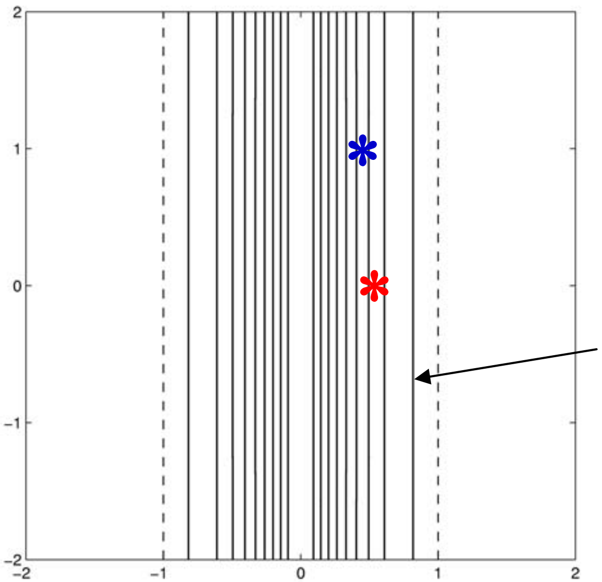
Nonhermitian quantum mechanics
"ghost" solutions of ODEs
exponential dichotomy theory
Orr-Sommerfeld eq., hydrodynamic stability

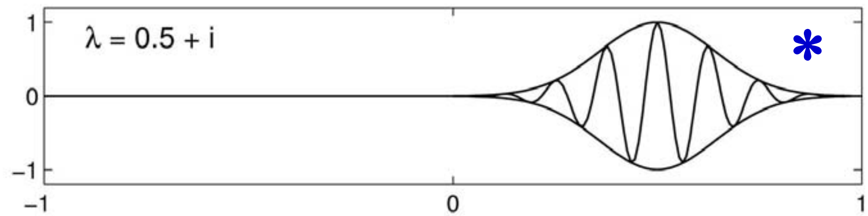
Lewy/Hörmander nonexistence

Back to adjoint-Mizohata operator: $L^* v = -v_x + ixv_y$.

For any $k > 0$, $v_k(x,y) = \exp(k(iy - x^2/2))$ * satisfies $L^* v_k = 0$.

On $x \in [-1,1]$ with zero b.c.'s, for example, it is an ε-pseudoeigenfunction for $\lambda = 0$ for an exponentially small ε .





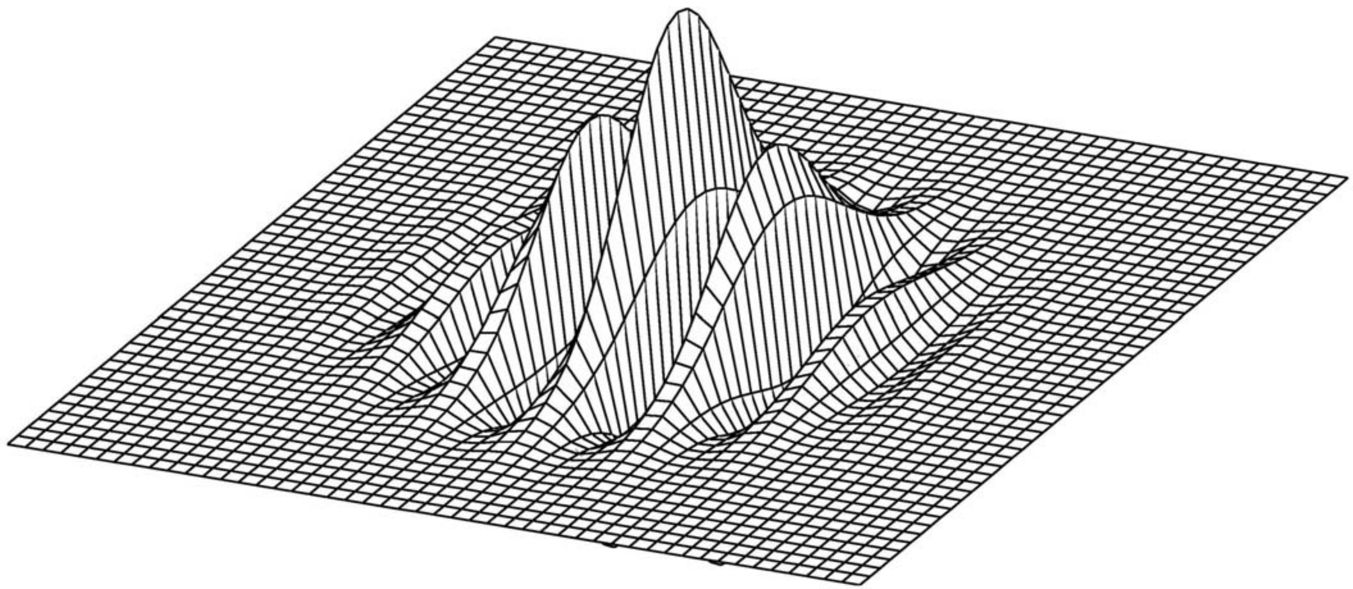
Contours
$\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, \ldots$

Many details swept under rug here!

Now imagine that $Lu = v_k$ has a soln. We get a contradiction:

$$0 = (u, L^* v_k) = (Lu, v_k) = (v_k, v_k) \neq 0 .$$

The same argument works generally for partial- or pseudodifferential operators. The key point is that the adjoint equation has a wave packet pseudosolution…



…implying nonexistence of solutions to the primal equation.

Handout.

Demonstration of EigTool (by Tom Wright) for Davies' complex harmonic Schrödinger operator

$$L\,u = -u_{xx} + i\,x^2 u$$