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What's So Special About Complex Variables?

I'd like to return to a topic I raised in my first column last year. I believe that the subject of complex variables gets too little attention in our curriculums, and is not as familiar as it should be to applied mathematicians.

Let's start with a quiz.

1. What's the difference between C^∞ and analytic?
2. How fast does the Maclaurin series of $\tanh(z)$ converge for $z = 1$?
3. What is i^i?

OK, time's up. Here are the answers, with modest editorial addenda.

FROM THE SIAM PRESIDENT

By Nick Trefethen

Question 1 is a matter of fundamental *ideas*. Both C^∞ and analytic functions have infinitely many derivatives, but an analytic function has the additional property that the Taylor series at each point converges to the function in some neighborhood. For a mathematician not to know this distinction is like a linguist not knowing the difference between French and Italian.

Question 2 relates to basic *applications* of analytic functions. $\tanh(z)$ is analytic near $z = 0$, but has poles in the complex plane at $\pm\pi i/2$. So the Maclaurin coefficients decay at the rate $(2/\pi)^n$, and that will be the rate of convergence of the series for $z = 1$. For a mathematician not to be able to work this out is like a physicist not being able to work out when a stone dropped from a height of 10 meters will hit the ground.

Question 3 touches upon *behavior* of analytic functions. Most mathematicians don't know the value of i^i off the top of their heads; I certainly don't. But what this question gets at is that although an analytic function may be single-valued locally, its values globally will depend on what paths you follow around branch points. i^i is the same as $\exp(i\log(i))$, and $\log(z)$ has a branch point at $z = 0$, so $\log(i)$ has infinitely many values, namely $\pi i/2$ plus any multiple of $2\pi i$. Therefore, i^i has infinitely many values too, an infinite set of geometrically spaced positive real numbers. For a mathematician not to know how to figure this out is like a driver not knowing how to parallel park.

But mathematics has a thousand beautiful topics that we all wish we knew better! What's so special about complex variables?

Weierstrass, Kelvin, Hardy, or von Neumann could have told you the answer. Numbers and functions are at the heart of mathematics, and you can't properly understand them if you see them only on the real line. That's why the other name for complex analysis is *function theory*. Ask Stokes, or Rayleigh, or Painlevé, who wrote that "The shortest path between two truths of the real domain often passes through the complex one." (The original French quote is given in the marvelous new book by Lax and Zalcman, *Complex Proofs of Real Theorems*.)

I'd like to propose three explanations of how complex variables have been pushed to the sidelines.

Curriculums. What has replaced function theory in mathematics education over the years? I think the answer may be linear algebra, which students spend much more time with now than they used to. Alas, I believe deeply in the importance of linear algebra, so I can't pretend that this hypothesis, if valid, points the way to a solution.

Mission creep. When I ask colleagues about complex analysis, they mention daunting topics like Riemann–Hilbert problems and Phragmén–Lindelöf theory. At least in some perceptions, it would seem that these fine advanced subjects have pushed aside basics like series, branch cuts, and contour integrals. To tell the truth, I suspect conformal mapping also gets too much attention, though I'm a card-carrying conformal mapper myself.

The oasis and the jungle. Let me finish with a couple of metaphors. Analytic functions are all the same, like Tolstoy's happy families: You can differentiate them as often as you like, their Taylor series converge absolutely and uniformly, you can manipulate the series term by term, and on and on in a symphony of tractability. Is $f(z)$ analytic in the unit disk? Then there's not much more to say about it! That disk of analyticity is an oasis of fully understood mathematics and powerful applied mathematical methods. The basic ideas had been worked out by the time Cauchy and Riemann were gone in 1866.

But now consider the boundary of that disk of analyticity, the unit circle. Is f continuous there, or differentiable, or what? Is it even defined as a function, or a distribution, or is it not defined at all? Almost anything is possible! The boundary circle is the jungle at the edge of the oasis. It's a dangerous, exciting place. And by the turn of the 20th century, that's where the research action was moving. Lebesgue and his successors exploded our notion of function, and the amazing subject of real analysis was created, right there at the edge of complex analysis. Why is real analysis a bigger specialty than complex analysis for research mathematicians? Because unhappy families are all different.

But theoretical complexity has little to do with whether a tool is *useful*. Most functions we encounter in practice are analytic or piecewise-analytic. Students should learn how to work with them right from the start. Every numerical analysis textbook should tell its readers that if the integrand is analytic, then Gauss quadrature converges geometrically. At the moment none of them do, at least none that I have seen—because "analytic" is considered too advanced a notion. Astonishing.