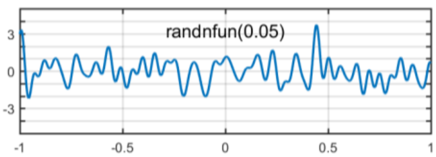
Lecture 2. Random functions and random differential equations Nick Trefethen, 12 Sept. 2017

1. 1D – randnfun

The Chebfun project is about developing continuous analogues of familiar discrete objects. What's the continuous analogue of `randn(n,1)`, i.e., a random vector?



Smooth random functions on an interval.
Desiderata: smooth, translation-invariant, $N(0,1)$ at each point.
Our choice: Fourier series with $2m+1$ random coefficients.
Pick spatial scale parameter $\lambda > 0$. On $[0, L]$,

$$f(t) = a_0 + \sqrt{2}\sum_{k=1}^{m} a_k \cos\left(\frac{2\pi k t}{L}\right) + b_k \sin\left(\frac{2\pi k t}{L}\right) \quad , \quad m = \text{round}(L/\lambda)$$

with a_k , $b_k \sim N(0,1/(2m+1))$. This is an example of a *Gaussian process*.
A priori, periodic. For nonperiodic, we chop a larger domain — "circulant embedding".

This is a finite variant of a Fourier-Wiener series (Paley, Wiener, Zygmund 1933, 1934).
See J.-P. Kahane (1926-2017), *Some Random Series of Functions*, 2nd ed., 1985.

2. 2D – randnfun2 and randnfunsphere

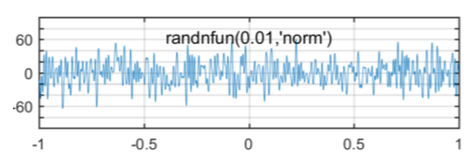
Smooth random functions on a square.
Desiderata: smooth, translation- and rotation-invariant, $N(0,1)$ at each point.
Our choice: a disk of random bivariate Fourier coefficients in wave number space.
Again, a priori periodic. For nonperiodic domains, again we chop.
This is an example of a *Gaussian random field*.

"Monochromatic" alternative (Bogomolny-Schmit, Barnett, Belyaev,...):
circle rather than disk of F. coeffs — random eigenfunctions of Laplacian.

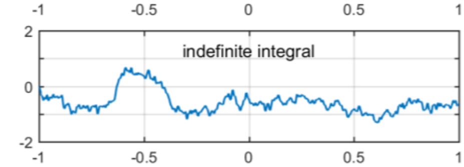
Random functions on a sphere: spherical harmonics with random coefficients.

3. The limit $\lambda \to 0$ and the white noise paradox

As $\lambda \to 0$, smooth random functions do not converge. Their integrals converge to 0 because of cancellation (amplitude $O(\lambda^{1/2})$). To get nonzero integrals as $\lambda \to 0$, we multiply by $\sim 2\lambda^{-1/2}$, giving a *normalized smooth random function*.



In Chebfun, `randnfun(lambda,'norm')`. Now we essentially have a sum of Fourier modes with coefficients from $N(0,1)$.



Indefinite integral of f: *smooth random walk*.

$$\int^t f(s)ds = a_0 t + \frac{L}{\sqrt{2}\pi}\sum_{k=1}^{m} k^{-1}\left[a_k \sin\left(\frac{2\pi k t}{L}\right) - b_k \cos\left(\frac{2\pi k t}{L}\right)\right]$$

This converges to Brownian motion as $\lambda \to 0$.

White noise paradox: amplitude and energy are ∞ in the limit $\lambda \to 0$. How to handle this?

"Physicist's (= Chebfun's) approach": avoid all technicalities by noting that in the real world, λ is always > 0. In a real problem λ is set by, e.g., a molecular scale.

"Mathematician's approach": develop the theory with $\lambda = 0$, for this is the "right" idealization. This became possible with Wiener's work of 1923. Very beautiful; and unavoidably technical. Rather than Brownian paths being the integral of noise, they become the primary objects. White noise, if you insist on working with it, becomes not a function but a distribution. Instead of adding randomness to analysis, a new field is created: *stochastic analysis*.

4. A sort of a history

Brown 1827 → Maxwell 1859, Boltzmann 1864 → Bachelier 1900, Einstein 1900, Smoluchowski 1906, Langevin 1906, Perrin 1909 → Wiener 1923 → Itô 1951, Stratonovich 1963 → Werner 2006, Smirnov 2010, Hairer 2014.

Among mathematicians, the view that noise should be conceptualized as pointwise now rules.

5. From smooth random ODEs to SDEs

In ODEs, as in analysis generally, one might imagine one could "add randomness" to the subject. Instead, technicalities of $\lambda = 0$ have required creation of a new subject: *stochastic DEs = SDEs*. On the computer, specialized numerical methods are required (Euler-Maruyama, Milstein, …).

Chebfun (= physicist's?) alternative: using smooth random functions, consider *smooth random ODEs*.
The limit $\lambda \to 0$ gives Stratonovich SDEs.
Theory stems from Wong & Zakai 1965. See Sussmann 1978 — also Lamperti 1964, Castell & Gaines 1996, Kelly & Melbourne 2016,…. (Other key references?)

Smooth random ODEs: see chap. 12 of T, Birkisson, & Driscoll, *Exploring ODEs*, available at my web site. Two other relevant books will soon appear from Springer, though neither focusses on the smooth case:
Han and Kloeden, *Random Ordinary Differential Equations and their Numerical Solution*,
Zhang and Karniadakis, *Numerical Methods for Stochastic PDEs with White Noise*.

6. Summary and an analogy

The standard in stochastic analysis is to conceive noise pointwise.
An alternative is to regard noise as smooth, and then consider the limit $\lambda \to 0$.

Smooth random ODEs:

- Provide an easy way to explore stochastic effects, e.g. in Chebfun.
- Avoid the need for a special stochastic calculus (Itô, Stratonovich).
- Avoid the need for special SDE algorithms (Euler-Maruyama, Milstein,...).
- Avoid the paradox that white noise has amplitude ∞ hence is only defined as a distribution.

Disclaimers:

- Nothing here is fundamentally new.
- I do not claim this approach is algorithmically competitive.
- Multiple dimensions may bring further subtleties.

An analogy

What if we told students they had to learn measure theory before they could talk about integrals?

```
%% Exponential growth with additive noise, y' = y + f
lam = 0.1; d = [0 5];
L = chebop(d); L.op = @(y) diff(y) - y; L.lbc = 0;
for k = 1:6
  f = randnfun(lam,d,'norm');
  y = L\f; plot(y), ylim([-10 10]), hold on, drawnow
end
hold off

%% Exponential decay with additive noise, y' = -y + f
L.op = @(y) diff(y) + y; L.lbc = 0;
for k = 1:6
  f = randnfun(lam,d,'norm');
  y = L\f; plot(y), ylim([-10 10]), hold on, drawnow
end, hold off

%% Multiplicative noise, y' = f*y
L.lbc = 1;
for k = 1:6
  f = randnfun(lam,d,'norm');
  L.op = @(t,y) diff(y) - f*y;
  y = L\0; plot(y), ylim([0 50]), hold on, drawnow
end, hold off

%% Bistable equation with additive noise, y' = y - y^3 + f
d = [0 10]; N = chebop(d);
N.lbc = 0; N.op = @(t,y) diff(y) - y + y^3;
for k = 1:20
  f = 0.1*randnfun(lam,d,'norm');
  y = N\f; plot(y), ylim([-1.5 1.5]), hold on, drawnow
end, hold off

%% Same with biased initial condition y(0) = 0.2
N.lbc = 0.1;
for k = 1:20
  f = 0.1*randnfun(lam,d,'norm');
  y = N\f; plot(y), ylim([-1.5 1.5]), hold on, drawnow
end, hold off

%% Tunneling out of a stable state, y' = -y + y^3 + ep*f
d = [0 100]; lam = 1; N = chebop(d); N.op = @(y) diff(y) - y^3 + y;
N.lbc = 0; N.maxnorm = 10; ep = 0.25;
for k = 1:6
  f = randnfun(lam,d,'norm');
  y = N\(ep*f); plot(y), ylim([-2 2]), hold on, drawnow
end, hold off

%% Reduced ep from 0.25 to 0.20
ep = 0.20;
for k = 1:6
  f = randnfun(lam,d,'norm');
  y = N\(ep*f); plot(y), ylim([-2 2]), hold on, drawnow
end, hold off
```