Lecture 4. Rational functions Nick Trefethen, 26 Sept. 2017

1. Introductory

[Ref: chapters 23, 24, 27 of *ATAP*]

P_m, R_{mn}. *Type* (m,n), *exact type* (μ,ν), and *defect* $d=\min(m-\mu, n-\nu)$ of a function $r\in R_{mn}$.

Minimax=Chebyshev=L^∞ approx on $[a,b]$: characterized by error curve equioscillating between $\geq m+n+2-d$ extrema.

Padé approx at $z=0$: characterized by error $(f-r)(z) = O(z^{m+n+1-d})$ as $z\to 0$.

Rational functions are used all the time. But they can be troublesome, and rational approximations are usually derived offline rather than online. One challenge: nonexistence/nonuniqueness of rational interpolants/approximants. Another: *Froissart doublets*, i.e., poles with very small residues, or equivalently, nearly-coincident pole/zero pairs. These arise both in exact analysis and also, even more, when there are rounding errors.

Though rational functions are used widely, they have never become a core topic of numerical analysis.

2. Three famous problems (illustrating how rational functions may far outperform polynomials)

[Ref: chapters 25, 28 of *ATAP*]

1. Approximation of $|x|$ on $[-1,1]$. Donald Newman, 1964. Singularity. Root-exponential convergence.

2. Approximation of e^x *on* $(-\infty,0]$. Cody-Meinardus-Varga, 1968. Infinite domain. Exponential convergence.

3. Aitken/eta/epsilon acceleration of convergence. Going beyond the disk of convergence of a Taylor series.

3. What is a rational function? (the familiar definition has held us back)

[Ref: Nakatsukasa, Sète, and T, "The AAA algorithm for rational approximation," *SISC*, submitted.]

Misleading definition: a *polynomial* is a linear combination $p(z) = \sum_{j=0}^{n} a_j z^j$

(misleading because bases of monomials z^j are exponentially ill-conditioned on domains other than a disk).

Another misleading definition: a *rational function* is a quotient $r(z) = p(z)/q(z)$

(misleading because p and q often vary exponentially over a domain even though r does not).

Instead of a quotient of polynomials, it is good to regard r as a quotient n/d of *partial fractions.* For type (m,m), let $z_0,\dots,z_m$ be any set of distinct *support points.* We then consider the *barycentric representation*

$$r(z) = \frac{n(z)}{d(z)} = \sum_{j=0}^{m} \frac{w_j f_j}{z-z_j} \Big/ \sum_{j=0}^{m} \frac{w_j}{z-z_j}.$$

With arbitrary f_j and $w_j\neq 0$, this spans the subset of R_{mm} with no poles at $\{z_j\}$.

The freedom to choose $\{z_j\}$ enables well-conditioned representations of rational functions otherwise inaccessible.

4. The AAA algorithm (barycentric representation with adaptive selection of support points)

[Ref: same as above; also subsequent manuscript by Filip, Nakatsukasa, T, and Beckermann on minimax approx]

Taking $m = 0,1,\dots,$ choose support points z_j one after another.

Next support point: point z_j where error $|f_j - r(z_j)|$ is largest.

Barycentric weights $\{w_j\}$ at each step: chosen to minimize linearized least-squares error $\|fd-n\|$.

For example, approxs to $|x|$ can now be computed in IEEE arithmetic for which Varga, et al. required 200 digits.

Does this open the door to wider use of rational functions, including in online calculations?

Not discussed (to appear in next lecture): rational functions and quadrature formulas.