21. Beam equation — *waves in an elastic rod*

When you ping a ruler projecting a length L over the edge of a table, the frequency is inversely proportional to the *square* of L, so you only have to shorten the ruler to $L/\sqrt{2}$ for the note to rise by an octave. The prongs of a tuning fork vibrate more musically but in a similar way. The simplest PDE modelling these transverse vibrations of a bending beam was derived by Daniel Bernoulli in a letter to Euler in 1735:

$$\boxed{u_{tt} = -u_{xxxx}.} \tag{1}$$

This equation involves the fourth x-derivative of u, instead of the second derivative that occurs in the wave equation ($\rightarrow$ *ref*), and in order for $u = \exp(\mathrm{i}(kx + \omega t))$ to be a solution, the frequency ω and wavenumber k must obey the dispersion relation $\omega^2 = k^4$, i.e., $\omega = \pm k^2$. Consequently, bending waves have the property that the shorter is the wavelength, the faster are both the phase velocity $c = -\omega/k = \pm k$ and the group velocity $c_g = -\mathrm{d}\omega/\mathrm{d}k = \pm 2k$.

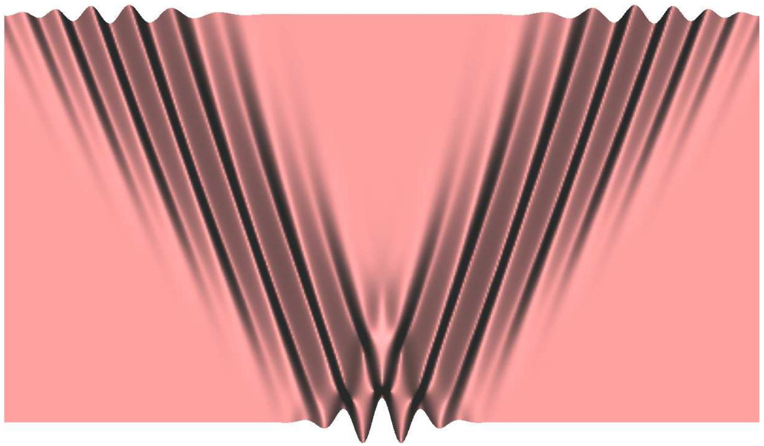

Fig. 1: Wave propagation with $c = \pm 4$, $c_g = \pm 8$

Figure 1 shows the behaviour of an infinite beam with initial data

$$u(x,0) = \cos(4x)\exp(-x^2/4), \qquad u_t(x,0) = 0.$$

Two groups of waves are formed, one travelling left and one right. Since each group has an average wavenumber $k \approx \pm 4$, the average phase velocity is $c \approx \pm 4$ and the average group velocity is $c_g \approx \pm 8$. Each group travels as a whole at velocity ± 8. Within that overall motion, two other features can be seen. First, the wavecrests travel at the slower phase velocity $c \approx \pm 4$ and so are continually falling behind: new wavecrests appear at the leading edge of the group, move more slowly than the group, and disappear when they are left behind by the trailing edge. Second, the group of waves gradually disperses, with shorter wavelengths appearing near the leading edge and longer wavelengths near the rear.

The beam equation can be factored into leftgoing and rightgoing free-space Schrödinger equations,

$$\partial_t^2 + \partial_x^4 \;=\; (\partial_t - i\partial_x^2)(\partial_t + i\partial_x^2). \tag{2}$$

It follows that if u obeys a Schrödinger equation, then it also obeys (1), and so do $\mathrm{Re}\,(u)$ and $\mathrm{Im}\,(u)$. Conversely, if u satisfies (1), then $v = u_t \pm \mathrm{i}u_{xx}$ satisfies the Schrödinger equation.

The response of a *finite* beam, $0 < x < 1$, depends on the boundary conditions. For the case corresponding to pinging a ruler we have

$$u(0,t) = u_x(0,t) = 0, \qquad u_{xx}(1,t) = u_{xxx}(1,t) = 0,$$

and the response when we displace the free end to $u(1,0) = -1$ and then release it from equilibrium ($u(x,0) = (x^3 - 3x^2)/2$, $u_t(x,0) = 0$) is shown in Figs. 2 and 3. The motion is a linear combination of the normal modes of the beam, and is predominantly at the lowest frequency, which accounts for the nearly-periodic behaviour. However, the higher modes have frequencies that are not rational multiples of the fundamental (or of each other), in contrast to the modes of a string. The actual behaviour is therefore not exactly periodic, as is evident in Fig. 3.

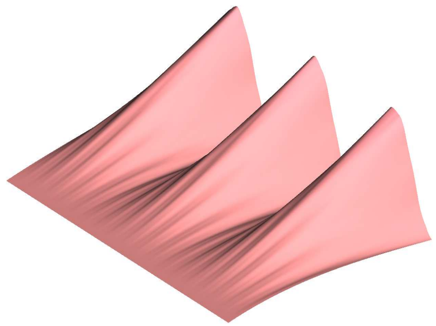

Fig. 2: $u(x,t)$ for a pinged ruler

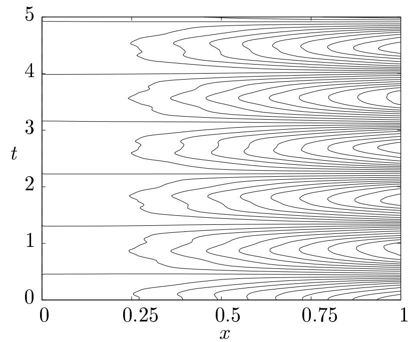

Fig. 3: Same data shown by contours

Including more detailed models of the physics of a bending beam results in extra terms on the right-hand side of (1). The finite thickness of the beam results in certain *thick-bending* terms, the first two of which are $+au_{xxtt} - bu_{tttt}$, due to rotatory inertia and shear deformation. Frictional effects introduce terms such as $-cu_t$. An applied tension or compression introduces a term $+du_{xx}$ or $-du_{xx}$, and the second of these leads to the Euler buckling instability if d exceeds a critical value. Nonlinear effects can also arise, and for a nonuniform beam, the basic equation becomes $\rho(x)u_{tt} = -(B(x)u_{xx})_{xx}$, which can again be the subject of further modifications.

References

Lord Rayleigh, *The theory of sound*, Dover, 1945.

W. Weaver, S. P. Timoshenko, and D. H. Young, *Vibration problems in engineering*, Wiley, 1990.

