The *biharmonic equation* is the "square of the Laplace equation",

$$\boxed{\Delta^2 u = 0,} \qquad (1)$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplacian operator. Like the Laplace equation, the biharmonic equation is elliptic, but, being of order four rather than two, it requires two boundary conditions rather than one to define a unique solution. In 2D, it is the equation satisfied, to a good approximation, by a small transverse deflection of a thin flat elastic plate.

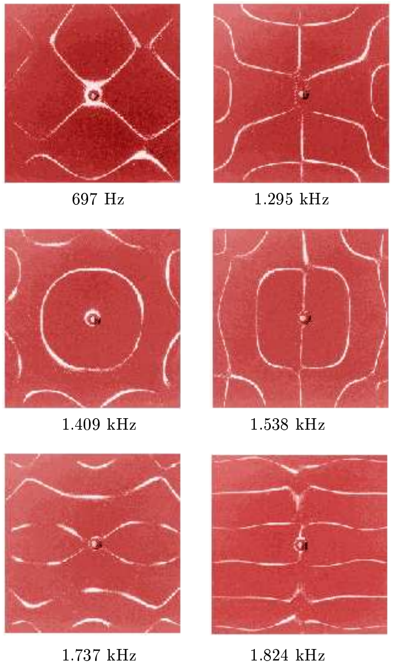

Fig. 1: Chladni figures for a square plate

The eigenvalues and eigenfunctions of the biharmonic operator Δ^2, with suitable homogeneous boundary conditions, give the modes of transverse vibration of such a plate. The physicist and astronomer Ernst Chladni (1756–1827) carried out a famous series of experiments using particles of sand to locate the nodal curves of a plate clamped at its centre and excited in various modes. The resulting patterns are known as *Chladni figures* and some results from an experiment of this kind are shown in Figure 1. One can try the experiment oneself at some science museums.

The biharmonic equation has two independent fundamental solutions (spherically symmetric and singular at the origin), one of which is the fundamental solution of the Laplace equation. These are $\log r$ and $r^2 \log r$ in 2D, r^{-1} and r in 3D, r^{-2} and $\log r$ in 4D, and r^{2-d} and r^{4-d} in dimensions $d \geq 5$. In the 2D case, just as any harmonic function in the (x, y)-plane is the real part of an analytic function $f(z)$, where $z = x + \mathrm{i}y$, so any biharmonic function is the real part of a function of the form $f(z) + \overline{z}\, g(z)$, where f and g are analytic and $\overline{z} = x - \mathrm{i}y$ is the complex conjugate of z. For instance, $r^2 \log r = \Re(\overline{z}\, g(z))$ where $g(z) = z \log z$. The functions f and g (which are not uniquely determined) are the *Goursat functions* of the problem.

In two dimensions, the function $u(x, y)$ minimising the value of the integral

$$I(u) = \iint_\Omega \left\{ (u_{xx})^2 + 2\,(u_{xy})^2 + (u_{yy})^2 \right\} \mathrm{d}x\ \mathrm{d}y$$

over a given domain Ω, subject to suitable conditions on the boundary, can be shown to satisfy the biharmonic equation on Ω. This is an analogue for the biharmonic equation of the Dirichlet integral for the Laplace equation ($\rightarrow$ *ref*). In particular, if Ω is the whole of $\mathbb{R}$, then this integral is minimised, subject to u taking given values $u(X_j, Y_j) = U_j$ at a finite number of points (at least 3 of them), when u is a so-called *thin-plate spline*—an interpolating function of the simple form

$$u(x, y) = a + b_1 x + b_2 y + \sum_j c_j r_j^2\ \log r_j,$$

where $r_j^2 = (x - X_j)^2 + (y - Y_j)^2$, whose later coefficients c_j satisfy the conditions $\sum_j c_j = \sum_j c_j X_j = \sum_j c_j Y_j = 0$ to ensure that $I(u)$ is finite. The thin-plate spline is a standard device for constructing a smooth function through data given at points arbitrarily distributed in the plane. The technique can be extended to 3D by adding in a term $b_3 z$ and a condition $\sum_j c_j Z_j = 0$ and replacing the 2D fundamental solution $r^2 \log r$ by the corresponding 3D solution r. Extension to more than three dimensions is possible, but one then needs to replace the biharmonic equation by a *polyharmonic equation* $\Delta^r u = 0$ with $r > 2$. Generalisations of these data fitting methods based on functions other than solutions of the biharmonic equation are the business of the field of *radial basis functions*.

The biharmonic equation also arises in the theory of steady Stokes (i.e., speed ≈ 0) flow of viscous fluids, where it is the equation satisfied by the stream function. For example, it has been shown that in the vicinity of a right-angle corner, the eigenfunctions of the biharmonic operator oscillate infinitely often in sign, with each successive region of oscillation being 16.56743 times smaller in size and 36267.55 times smaller in amplitude than the last. This result can be interpreted as the statement that a plate distorted in a certain way will in principle bend back and forth infinitely often near a corner, or that a fluid motion at Reynolds number 0 will in principle exhibit an infinite sequence of counter-rotating "*Moffatt vortices*".

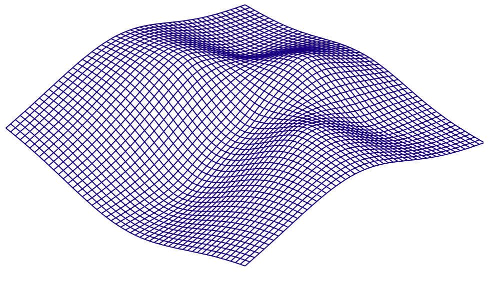

Fig. 2: A thin-plate spline, interpolating function values given at 9 scattered points

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