51. Complex Ginzburg–Landau equation — *a laboratory for turbulence*

The cubic complex Ginzburg–Landau equation

$$\boxed{u_t = (1 + \mathrm{i}\nu)u_{xx} + u - (1 + \mathrm{i}\mu)u|u|^2, \qquad u \in \mathbb{C}} \tag{1}$$

was derived by Newell and Whitehead in 1969 as an amplitude modulation equation for modelling the onset of instability in fluid convection problems. In these problems, at some critical parameter value, a spatially homogeneous steady state loses stability to oscillations whose wavelength and frequency can be understood in terms of a linearised equation. Newell and Whitehead found that when nonlinear effects are included, these oscillations are modulated over long time and space scales by a quantity u satisfying (1). To use an AM radio analogy, u is the *music* superimposed on the *carrier frequency* of the original PDE.

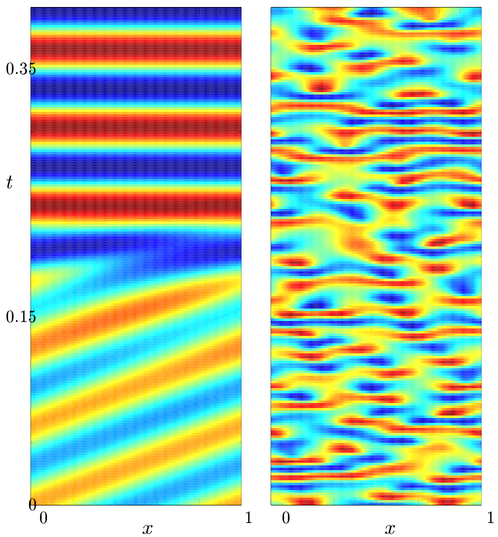

Fig. 1: Rotating waves for $\mu = \nu = 2$, $R = 50$; turbulence for $-\mu = \nu = 2$, $R = 500$ (Reu shown)

In fact, (1) arises almost generically in stability analyses, especially in fluid dynamics. For example, Stewartson and Stuart in 1971 discovered it in the context of plane Poiseuille flow. The derivation of (1) can be understood rigorously in terms of an infinite-dimensional *centre manifold* theory.

With μ, $\nu = 0$, (1) is the Allen–Cahn equation ($\to$ *ref*) except with a complex dependent variable. In this case, with the addition of extra terms to model the effect of the magnetic field, it is used as a model for superconductivity ($\to$ *ref*). This complex Allen–Cahn equation is a *gradient system* which evolves according to an energy minimisation principle, so that in the absence of external forcing, the dynamics as $t \to \infty$ are steady. However, with μ or ν nonzero, (1) is not a gradient system and its long-term behaviour can be more exotic—e.g. periodic or chaotic.

For example, consider (1) on $[-1, 1]$ with periodic boundary conditions, with the linear term u replaced by Ru ($R \in \mathbb{R}$) for full generality. Then there exist explicit *rotating wave* solutions

$$u_k = c\,\mathrm{e}^{\mathrm{i}(2k\pi x - \omega t)}, \qquad |c| = \sqrt{R - 4k^2\pi^2}, \qquad \omega = \mu R + 4k^2\pi^2(\nu - \mu), \tag{2}$$

where $\arg c$ is arbitrary. The linear stability of these solutions may be analysed by writing $u(x,t) = u_k(x,t)(1 + h(x,t))$ for small h. This leads to a coupled pair of complex, autonomous linear ODEs for h and $\overline{h}$. Doering *et al.* showed that for $1 + \mu\nu < 0$, all rotating waves are unstable, and when R is sufficiently large, numerical simulations indicate chaotic behaviour (Fig. 1).

Aside from specific applications, the complex Ginzburg–Landau equation has attracted attention as a simple PDE with chaotic solutions. In particular, it has become a test bed for new ideas in statistics of turbulence, control of chaos, and rigorous PDE bounds. We might say it is a kind of *laboratory for turbulence*. True fluid turbulence, of course, requires three space dimensions.

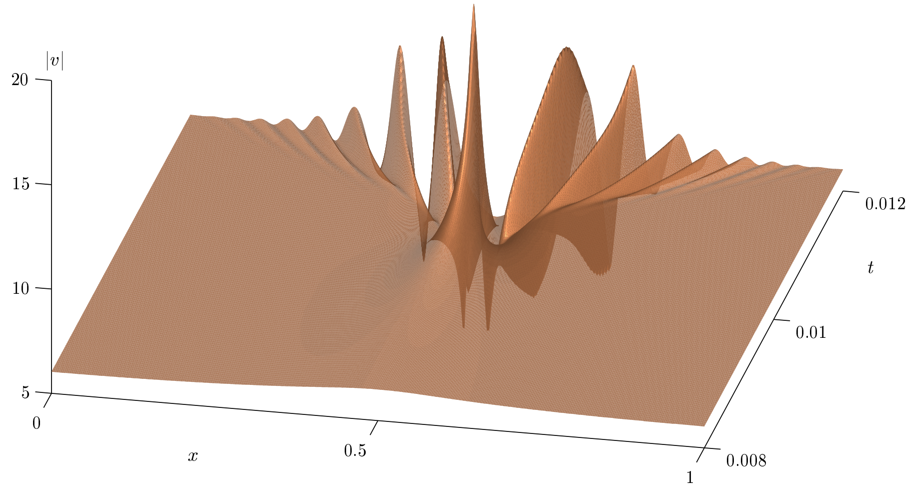

Fig. 2: Burst and collapse for a quintic complex Ginzburg–Landau equation

For $-\mu$, $\nu > 0$, with u and $|u|^2$ replaced by Ru and $|u|^{2q}$, (1) may be rescaled and written

$$v_\tau = (\mathrm{i} + \nu^{-1})v_{xx} + R\,\nu^{-1}v + (\mathrm{i} + \mu^{-1})v|v|^{2q}. \tag{3}$$

As $-\mu$, $\nu \to +\infty$, (3) becomes the focusing nonlinear Schrödinger equation ($\to$ *ref*), whose solutions blow up in finite time for $qd \geq 2$ (in d space dimensions). On the other hand, it can be proved that (3) has regular solutions for all time for $qd \leq 2$. For $qd = 2$, (3) hae *burst solutions* which follow the blow-up of the nonlinear Schrödinger equation for a time, until the small $1/\nu$ dissipation causes their collapse. Figure 2 shows a case with $d = 1$, $q = 2$, $-\mu = \nu = 25$.

References

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