31. Lewy equation — *a smooth PDE with no solution*

$$\boxed{2\mathrm{i}(x+\mathrm{i}y)u_t = u_x + \mathrm{i}u_y} \tag{1}$$

Once it is said, it is clear to everybody;
but Lewy was the first to say it!
K.O. Friedrichs

It was long believed that in the absence of boundary conditions any reasonable-looking partial differential equation should have many solutions. The surprise came in 1957.

Consider the kth-order linear partial differential equation

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x), \tag{2}$$

where f and the coefficients a_α are complex-valued functions defined and continuous on $\mathbb{R}^n$ together with all their partial derivatives of all orders (i.e., C^∞ functions), and pose the following question:

Given $x_0 \in \mathbb{R}^n$, can we find a function u that satisfies (2) *in some neighbourhood of x_0?*

If f and the coefficients a_α are analytic functions and $a_\alpha(x_0) \ne 0$ for some α with $|\alpha| = k$, the famous Cauchy–Kovalevskaya theorem furnishes an affirmative answer. But what if f and the coefficients a_α are "merely" C^∞ functions? In 1957, Hans Lewy gave a surprisingly simple counterexample which showed that in this case the set of solutions may be empty.

For the linear partial differential operator P defined on $\mathbb{R}^3$ by

$$P = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} - 2\mathrm{i}(x+\mathrm{i}y)\frac{\partial}{\partial t},$$

Lewy proved that if there is a function u, continuously differentiable in (x, y, t) and such that $Pu = f$ in some neighbourhood of the origin, where f is a continuous function depending only on t, then f is analytic at $t = 0$. It follows that if f is not analytic at $t = 0$, then $Pu = f$ has no continuously differentiable solution.

A simple example of a C^∞ function that is not analytic at 0 is

$$f(t) = \begin{cases} \mathrm{e}^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases} \tag{3}$$

In fact, Lewy modified his argument to show the existence of C^∞ functions f for which the equation $Pu = f$ has no solution near any point even in the sense of distributions. Worse still, the collection of all such functions f is a set of second (Baire) category in the Schwartz space of rapidly decreasing functions. This means that the set of C^∞ functions for which $Pu = f$ has no solution near any point is very much larger than the set of f for which a solution to this equation exists.

A few years after the publication of Lewy's example, Hörmander discovered a general necessary condition for local solvability which clarified the causes of non-existence and laid the foundations of modern local existence theory for linear partial differential equations. Over the last forty years many other linear partial differential equations have been discovered which exhibit similar behaviour. The following simple example, attributed by Egorov to V. V. Grushin, is based on an idea of Garabedian. Consider

$$u_x + \mathrm{i}x u_y = g(x, y), \tag{4}$$

where g is a nonnegative C^∞ function with compact support in $\mathbb{R}^2$ and $g(x, y) = g(-x, y)$. Suppose that for $x \ge 0$, g is equal to zero outside the set $\bigcup_{j=1}^\infty D_j$, where $\{D_j\}$ is a sequence of mutually disjoint disks that lie in the half-plane $x > 0$ and converge as $j \to \infty$ to the origin $x = y = 0$. Again, this equation has no solution in any neighbourhood of the origin, not even a solution in the sense of distributions.

Fig. 1: Solutions to (1)

As both of the examples presented so far concern differential operators with complex coefficients, the reader may wonder whether linear partial differential equations with *real* coefficients must always have solutions. The answer to this question is, again, negative. By iterating Lewy's operator P to eliminate the complex coefficients, Trèves discovered in 1962 that the real fourth-order linear partial differential equation $P\bar{P}P\bar{P}v = f$ has no solution if f is the function defined in (3); for otherwise, $u = \bar{P}P\bar{P}v$ would satisfy Lewy's equation $Pu = f$, contradicting the fact that this has no solution.

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