46. Perona–Malik equation — *image sharpening by anisotropic diffusion*

The heat equation $u_t = \Delta u$ is the canonical smoothing process, diffusing data at a constant rate in all directions ($\rightarrow$ *ref*). When applied to an image $u(x,0)$ in the case $x = (x_1, x_2)$ of two space dimensions, the result is a blurred version $u(x,t)$ for $t > 0$. Noise may be eliminated, but at the cost of making edges fuzzy.

Suppose we want a modified heat equation that will attenuate noise while maintaining or even enhancing edges. In 1987 and 1990, Pietro Perona and Jitendra Malik proposed that this could be accomplished by replacing the constant diffusion coefficient of $u_t = \Delta u$ by a nonlinear function of u that decreases to 0 as the gradient of u gets steeper. The *Perona–Malik equation* is

$$u_t = \nabla \cdot \left(g\left(|\nabla u|\right) \nabla u \right), \qquad (1)$$

where $g(s)$ is a smooth nonincreasing function with $g(0) = 1$, $g(s) \geq 0$ and $g(\infty) = 0$.

Figure 1 illustrates how this works in the one-dimensional case $u_t = (g(|u_x|)u_x)_x$ with $g(s) = (1+s^2)^{-1}$. The initial data is a noisy signal. Notice how (1) maintains the overall features while diminishing the noise.

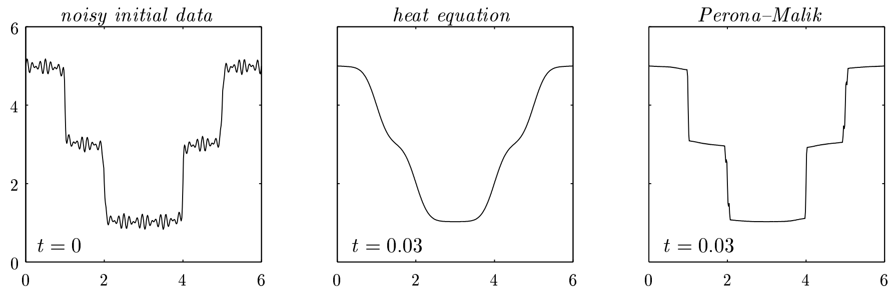

Fig. 1: Edge enhancement in 1D

Equation (1) enhances edges because the diffusion process is conditional: near points x where $|\nabla u(x,t)|$ is large, the diffusion is weak, so that edges may be kept; where $|\nabla u(x,t)|$ is small, the smoothing is strong. But in fact, the diffusion is not merely variable, it is *anisotropic*. To see this we can expand the derivative to get

$$u_t = (g u_x)_x = g u_{xx} + g_x u_x,$$

where $g = g(|u_x|)$, or in several dimensions,

$$u_t = g \Delta u + \nabla g \cdot \nabla u.$$

The physics can now be seen as locally constant diffusion assisted by direction-dependent drift. Approaching an edge from the left, say, $|u_x|$ increases and $g = g(|u_x|)$ decreases, so the drift term has the same sign as $u_t = -u_x$, which means that it moves material rightward: into the edge. Leaving the edge from the right, $|u_x|$ decreases and g increases, so the drift term is like $u_t = u_x$ and moves material leftward: into the edge once more. We have *anisotropic diffusion*, and the signs are such as to enhance edges. Yet the conservation form of (1) ensures that there is none of the ill-posedness of the backward heat equation ($\rightarrow$ *ref*).

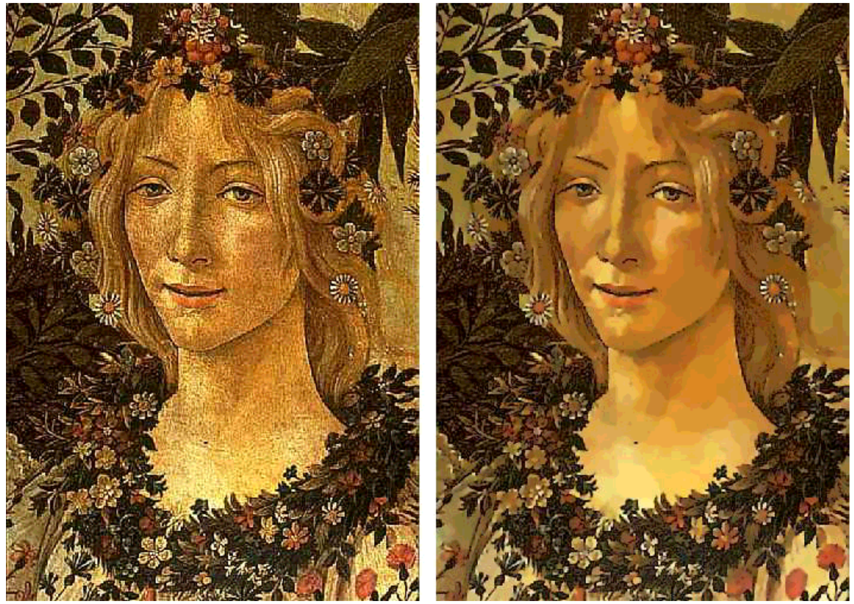

Fig. 2: Botticelli's *Primavera* before and after slowed anisotropic diffusion

The Perona–Malik equation in the pure form we have described has some difficulties in practice. For example, there is a scale-dependence that means that if an image is severely distorted with noise, the oscillations of $|\nabla u|$ are large and the noisy edges are kept. Since the original Perona–Malik papers, there has been lively research to develop related equations that do even better. One such model, *slowed* anisotropic diffusion, is illustrated in Figure 2 by images due to Kacur and Mikula.

References

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