35. Porous medium equation — *nonlinear diffusion with sharp interfaces*

The *porous medium equation* is like the heat equation (→ *ref*), except that the linear diffusion term Δu is replaced by $\Delta(u^m)$ for some $m > 1$:

$$\boxed{u_t = \Delta(u^m).} \qquad (1)$$

Equivalently we may rewrite (1) as a heat equation with a nonlinear diffusion constant,

$$u_t = \nabla \cdot (m u^{m-1} \nabla u). \qquad (2)$$

From either (1) or (2) it is clear that the diffusion 'turns off' as $|u| \to 0$. This is analogous to the situation with compacton equations (→ *ref*) which are like the KdV equation (→ *ref*) except that the linear dispersion term u_{xxx} is replaced by a nonlinear term $(u^m)_{xxx}$ with $m > 1$. Just as with compactons, the effect of the nonlinearity in (1) is that there are solutions with compact support.

Where might such an equation arise, in which the diffusion constant varies in proportion to some power of the diffused quantity? Some applications are in biology, notably in models of animal and insect dispersal, and some are in plasma physics. Another, as indicated by the name, is in the study of the flow of a gas in a porous medium. We can derive (1) in a simplified manner as follows. Ignoring certain constants, the flow is governed by the three equations

$$\begin{aligned} \rho_t &= -\nabla \cdot (\rho v) && \text{(conservation of mass)},\\ v &= -\nabla p && \text{(Darcy's Law (→ } ref\text{))},\\ \rho &= p^{\gamma} && \text{(equation of state)} \end{aligned}$$

where ρ is the density, p is the pressure, v is the velocity, and γ is the (constant) ratio of specific heats. Eliminating v and p gives

$$\rho_t = \nabla \cdot (\,\rho\, \nabla(\rho^{\gamma^{-1}})),$$

and since $\rho\,\nabla(\rho^{\gamma^{-1}}) = \gamma^{-1}\rho^{\gamma^{-1}}\nabla\rho$, whereas $\nabla(\rho^{1+\gamma^{-1}}) = (1+\gamma^{-1})\rho^{\gamma^{-1}}\nabla\rho$, we can rewrite this as

$$\rho_t = \frac{\gamma^{-1}}{1+\gamma^{-1}}\Delta(\rho^{1+\gamma^{-1}}) = \frac{1}{1+\gamma}\Delta(\rho^{1+\gamma^{-1}}).$$

Setting $u = \rho$ and rescaling t by $1/(1+\gamma)$ gives (1).

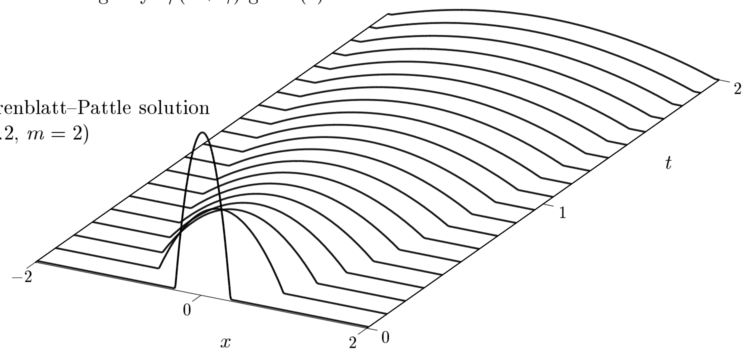

Fig. 1: 1D Barenblatt–Pattle solution ($\Gamma = 0.2$, $m = 2$)

Barenblatt and Pattle independently found an explicit formula for the solution of (1) beginning from a delta function of integral Γ at the origin:

$$u(|\mathbf{x}|, t) \;=\; \max\left\{0,\; t^{-\alpha}\left[\Gamma - \frac{\alpha(m-1)}{2dm}\frac{|\mathbf{x}|^2}{t^{2\alpha/d}}\right]^{1/(m-1)}\right\}, \qquad (3)$$

where d is the number of space dimensions and $\alpha = (m-1+2/d)^{-1}$. This solution is radially symmetric and has compact support. Figures 1 and 2 give an idea of a typical 'spreading drop' in one and two dimensions, respectively.

It is evident from (3) and from the figures that the solution to (1) may contain an interface where the gradient is discontinuous. A precise mathematical treatment thus involves the notion of *weak solutions*. We find that at the inner edge of the interface, the gradient is infinite if $m > 2$, finite if $m = 2$, and zero (but with a nonzero derivative) if $m < 2$. The question also arises whether these solutions are necessarily unique. The answer is yes, and this can be shown with the aid of a *comparison principle* of the kind used for many parabolic PDEs, which establishes that if for two solutions, the boundary and initial data of one are greater than for the other, then this solution is dominant for all t.

Like any diffusion equation, the porous medium equation conserves mass: $\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = \text{constant}$ in the absence of flows across boundaries.

More complicated initial conditions than in our figures may give rise to more complicated behaviour. For example, an interface may remain exactly stationary for an initial 'waiting time'. Eventually, however, it must move outward so as to enlarge a region where u is positive.

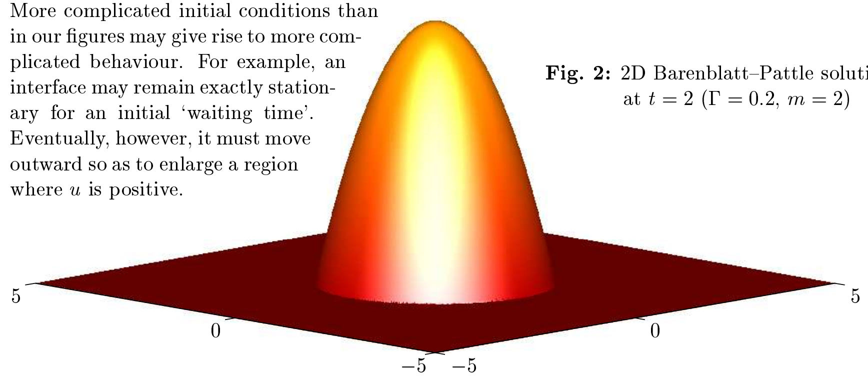

Fig. 2: 2D Barenblatt–Pattle solution at $t = 2$ ($\Gamma = 0.2$, $m = 2$)

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