Polynomials and multivariate polynomials

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1. Univariate polynomials of degree d (see T, *Approximation Theory and Approximation Practice*)

a. Taylor series. Locally, everything's a polynomial (Newton). Convergence in a disk. The basis of complex variables.
b. Interpolation. Always possible in $d+1$ distinct points. Stable algorithm: barycentric formula.
c. Quadrature. Almost always based on integrating polynomial interpolants (Newton-Cotes, Gauss, Clenshaw-Curtis,...).
d. Approximation theory. Continuous functions are approximable (Weierstrass), exponential convergence if analytic (Runge).
e. Rootfinding. Algorithms increasingly based on matrix eigenvalue problems. PURE MATHS: GALOIS THEORY, ALGEBRA.
f. Optimization. Usually $d \leq 2$ à la Newton's method, for $d > 2$ does not improve complexity. Exception: Chebfun global opt.
g. Finite difference formulas for ODE IVPs and BVPs. Derived from polynomials (Adams, R-K, difference calculus,...).
h. Piecewise polynomials. Splines. The big advantage is exponential decay of errors away from singularities.
i. Spectral methods for ODE BVPs. Based on global polynomials.
j. Other high-order polynomials. Rarely used until Chebfun, which uses them for everything.

Univariate polynomials are the universal starting point for numerical methods in 1D.

2. Multivariate polynomials of degree d in s dimensions

a. Multivariate Taylor series. Again locally, everything's a polynomial. Basis of several comp. vars. — a subject rarely used.
b. Interpolation. Only possible in unisolvent point sets. The number of points is $\binom{d+s}{s}$.
c. Cubature. Integrating polynomials is problematic, so other ideas are used more (tensor products, Monte Carlo,...).
d. Approximation theory. Polynomials have similar power in principle as in 1D, but little has been done in this area.
e. Rootfinding. Good numerical algorithms have been slow to develop. PURE MATHS: ALGEBRAIC GEOMETRY.
f. Optimization. Usually $d \leq 2$. Developments with $d > 2$: Steihaug, Parrilo/Nesterov/Lasserre, Cartis/Gould/Toint,....
g. Finite difference formulas for PDE IVPs and BVPs. Usually based on univariate pieces — e.g. $\Delta u = u_{xx} + u_{yy} + y_{zz}$.
h. Piecewise multivariate polynomials. Finite elements. The big advantage is geometric flexibility. $d \leq 2$ usually, $\lesssim 6$ sometimes.
i. Spectral methods for PDE BVPs. Based mainly on univariate ideas (tensor products).
j. Other high-order multivariate polynomials. Rarely used.

Multivariate polynomials have a limited role in numerical computation, probably because they are hard to work with and in most cases we can get away with univariate tools instead. On the other hand they are at the very heart of algebraic geometry.

3. Univariate nonuniformity across an interval (see chap. 22 of *ATAP*)

The issue is different behavior for $x \approx \pm 1$ than for $|x| \ll 1$ by all sorts of measures. The simplest starting point is the change of variables $x = \cos\theta$, relating Fourier (uniform wrt $\theta \in [-\pi, \pi]$) with Chebyshev (nonuniform wrt $x \in [-1,1]$).

a. Series. Monomials make exponentially ill-conditioned bases. Need orthogonal polys instead, which oscillate faster near ± 1.
b. Interpolation. Equispaced interpolants have exp. large Lebesgue consts. (Runge phenom.). Need clustered points instead.
c. Quadrature. Equispaced nodes (Newton-Cotes) diverge as $n \to \infty$. Need clustered nodes (Clenshaw-Curtis, Gauss,...).
d. Approximation theory. Estimates confirm greater resolution near ± 1. For analytic case, this follows from Bernstein ellipses.
i. Spectral methods for ODEs. Need clustered grids. The enhanced resolution near boundaries is useful for boundary layers.

The $\pi/2$ factor. The nonuniformity of polynomials carries an asymptotic cost of a factor of $\pi/2$. (Gauss quadrature is not optimal!) This can be recovered via conformal maps ("ellipse → sausage") — see Kosloff-Tal-Ezer 1984 for spectral methods and Hale-T 2008 for quadrature. Another approach is prolate spheroidal wave functions (Slepian & Pollak 1961).

What is the degree of x^n? From an approx point of view it is $O(\sqrt{n})$! See Newman-Rivlin 1976, Lanczos's *Applied Analysis*, and the Szasz-Müntz theorem. A degree n orthogonal poly like T_n, however, has degree n in every sense (leading coeff 2^n).

4. Multivariate nonisotropy around a square/cube/hypercube (see T, *SIAM Review* 2017 and *Proc AMS* 2017)

The *degree* (= *total degree*) of a multivariate polynomial is defined by the 1-norms of its exponents.
This makes sense for calculus as $h \to 0$ and it is also the right definition for isotropy (e.g., approximation in a spherical ball).

However, a square/cube/hypercube is not isotropic, and polynomials of a given total degree underresolve along diagonals.
The right balance is to approximate by polynomials of a given *Euclidean degree*, defined by the 2-norms of the exponents.

When s is large, the volume of the s-hypercube is $\approx s^{s/2}$ times that of the inscribed ball. This is a lot of anisotropy!
(cf. concentration of measure).

There is also *maximal degree*, defined by the ∞-norms of the exponents. This is essentially a univariate idea, relevant to tensor products. Easy, but expensive ($s!$ more parameters than with total degree). Overresolves along diagonals.