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REAL POLYNOMIAL CHEBYSHEV APPROXIMATION BY THE CARATHÉODORY–FEJÉR METHOD*

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Abstract. A new method is presented for near-best approximation of a real function F on $[-\tau, \tau]$ by a polynomial of degree m. The method is derived by transplanting the given problem to the unit disk, then applying the Carathéodory–Fejér theorem. The resulting near-best approximation is constructed from the principal eigenvalue and eigenvector of a Hankel matrix of Chebyshev coefficients of F.

It is well known that as $\tau \to 0$, the mth partial sum of the Chebyshev series of F agrees with the best approximation to a relative error $O(\tau)$. In contrast, our approximation is shown to differ from best by at most $O(\tau^{2m+3})$. A similar result is given for approximation on $[-1, 1]$ as $m \to \infty$. Such high-order agreement is of both practical and theoretical importance. In particular, it establishes a real analogue of the phenomenon that on the complex unit disk best approximation error curves tend to closely approximate circles.

Several numerical examples are presented.

Introduction. Two main ideas are combined in this paper. The first is that by means of the Joukowski map $x = \frac{1}{2}(w + w^{-1})$, the real Chebyshev approximation problem on the unit interval $[-1, 1]$ can be related to a complex Chebyshev approximation problem on the unit disk $|w| < 1$. Under this transplantation near-best approximations for one problem often correspond to near-best approximations for the other, and so an approximation method for one problem can be carried over to the other. The second is that exceedingly good near-best approximations on the unit disk can be computed by an application of the Carathéodory–Fejér theorem, which involves the principal singular value and singular vector of a Hankel matrix of Taylor series coefficients. Transplanting this technique to $[-1, 1]$ leads to a powerful real near-best approximation method that is based upon the principal eigenvalue and eigenvector of a Hankel matrix of Chebyshev series coefficients of the function to be approximated.

Both of these ideas take advantage of the phenomenon that in approximation on the unit disk best approximation error curves tend to closely approximate perfect circles. The complex "CF method" was developed by Trefethen in response to this phenomenon, partly to explain and partly to exploit it [18], [19]. The transplantation technique was proposed in connection with Padé approximation by Frankel, Gragg and Johnson [5], [7], and has been extended by Gutknecht [8]. Combining the two leads to the conclusion that a real best approximation error curve not only equioscillates, but also approximates the real part of a multiply winding Blaschke product in the complex plane. In fact, the great majority of analytically known examples of Chebyshev approximations are either complex examples with perfectly circular error curves (finite Blaschke products), or transplantations of these to a real interval. Such examples will be discussed in a unified way by Gutknecht in a forthcoming paper.

Both the Joukowski transplantation and the CF theory extend to rational approximation; for the latter see [9] and [19]. However, as the rational theory is more complicated and as some of its results are more limited, this paper is confined to the polynomial case. Section 1 presents the CF method. Section 2 establishes two a

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posteriori theorems to the effect that if the error curve of some approximation nearly equioscillates, then the approximation is close to best. Section 3 applies these theorems and asymptotic results from [18] concerning approximation on small disks to the problems of approximation on $[-\tau, \tau]$ as $\tau \to 0$ (Theorem 3.4) and on $[-1, 1]$ as $m \to \infty$ (Theorem 3.5). Section 4 presents a few numerical examples.

1. Description of the method. Let the unit interval, disk and circle be denoted $I = [-1, 1]$, $D = \{z : |z| < 1\}$ and $\partial D = \{z : |z| = 1\}$, and let $I(\tau) = [-\tau, \tau]$. Let $\|\cdot\|_{I(\tau)}$, $\|\cdot\|_{\partial D}$, etc. be the corresponding supremum norms. In the sequel x will generally denote a real and w a complex variable; lower case letters will be used for functions of w and upper case for functions of x.

We begin with a real function $F(x)$ that is continuous on $I(\tau)$. Let P_m^* denote the unique best approximation to F of degree at most m with respect to $\|\cdot\|_{I(\tau)}$. For any finite $l \geqq 0$, F possesses a partial Chebyshev expansion of the form

$$F(x) = F_l(x) + R_l(x) = 5 \sum_{k=0}^{l}{}' a_k T_k\left(\frac{x}{\tau}\right) + R_l(x), \tag{1.1}$$

where each coefficient a_k is defined by an inner product (cf. [2, p. 117])

$$a_k = \frac{2}{\pi} \int_{-\tau}^{\tau} F(x) T_k\left(\frac{x}{\tau}\right) \frac{dx}{\sqrt{\tau^2 - x^2}}.$$

Our fundamental transplantation is the map

$$x = x(w) = \frac{1}{2}\tau(w + w^{-1}), \tag{1.2}$$

which for $x \in I(\tau)$, $w \in \partial D$ leads to the well-known formula

$$T_k\left(\frac{x}{\tau}\right) = \frac{1}{2}(w^k + w^{-k}). \tag{1.3}$$

In particular,

$$F_M(x) - F_m(x) = \frac{1}{2} \sum_{k=m+1}^{M} a_k(w^k + w^{-k}) = \frac{1}{2}[f(w) + f(w^{-1})], \tag{1.4}$$

where

$$f(w) \equiv \sum_{k=m+1}^{M} a_k w^k. \tag{1.5}$$

Here is the version of the Carathéodory–Fejér theorem that we shall make use of:

THEOREM 1.1. *f has a unique extension q of the form*

$$q(w) = \sum_{k=-\infty}^{M} b_k w^k,$$

with $b_k = a_k$ for $m + 1 \leqq k \leqq M$, that is analytic outside ∂D and bounded there except near the pole at ∞, and has minimal norm $\|q\|_{\partial D}$ among all such analytic extensions. q is given by the Blaschke product

$$q(w) = \lambda w^M \frac{u_1 + \cdots + u_{M-m} w^{M-m-1}}{u_{M-m} + \cdots + u_1 w^{M-m-1}}, \tag{1.6}$$

where λ is the largest eigenvalue in absolute value of the real symmetric Hankel matrix

$$A \equiv \begin{bmatrix} a_{m+1} & a_{m+2} & \cdots & a_M \\ a_{m+2} & & \cdot & \\ \vdots & \cdot & & 0 \\ a_M & & & \end{bmatrix} \tag{1.7}$$

and $u = (u_1, \cdots, u_{M-m})^T$ is a corresponding real eigenvector.

Proof. Due to Carathéodory and Fejér [1] and Schur [15]. See also [6], [18] and [19]. □

From (1.6) it follows that $\|q\|_{\partial D} = |\lambda|$. Takagi [17, p. 17] showed that there is always an eigenvector u with $u_1 \neq 0$ so the coefficients $\{b_k\}$ can be computed recursively by

$$b_k := \frac{-1}{u_1}(b_{k+1}u_2 + \cdots + b_{k+M-m-1}u_{M-m}), \qquad k = m, m-1, \cdots. \tag{1.8}$$

The CF *method* for near-best approximation on ∂D [18] consists of taking $-\sum_{k=0}^{m} b_k w^k$ as an approximation to $f(w)$. Such a truncation of q can be transplanted by (1.2) to provide a corresponding near-best approximation on $I(\tau)$. Here even more accuracy can be achieved than in the complex case, however, for q can be truncated at the term $k = -m$ rather than $k = 0$. We define the *real* CF *approximation* $P_{m,M}^{cf}$ of F on $I(\tau)$ by

$$P_{m,M}^{cf}(x) = F_m(x) - \sum_{k=-m}^{m} b_k T_{|k|}\left(\frac{x}{\tau}\right). \tag{1.9}$$

$P_{m,M}^{cf}$ is related to

$$p(w) \equiv -\sum_{k=-m}^{m} b_k w^k \tag{1.10}$$

by

$$P_{m,M}^{cf}(x) - F_m(x) = \frac{1}{2}[p(w) + p(w^{-1})]. \tag{1.11}$$

In particular, for $x \in I(\tau)$,

$$P_{m,M}^{cf}(x) - F_m(x) = \text{Re}\, p(w), \tag{1.12}$$

and from (1.4),

$$F_M(x) - F_m(x) = \text{Re}\, f(w). \tag{1.13}$$

From these equations it is apparent that if $f - p$ is nearly circular on ∂D with winding number at least $m+1$, then $F_M - P_{m,M}^{cf}$ will nearly equioscillate on $I(\tau)$ at a set of at least $m+2$ points.

It follows from Theorem 1.1 that in complex Chebyshev approximation on the unit disk $|\lambda|$ provides a lower bound for the optimal error:

$$|\lambda| \leq \|f - p_m^*\|_{\partial D}.$$

One might speculate that the same would hold for real approximation on an interval, but this is not true. A counterexample is degree-0 approximation on $[-1, 1]$ to

$$F(x) = 4x - 4x^3 = T_1(x) - T_3(x) = \text{Re}\,(z - z^3).$$

In this problem

$$|\lambda_1| = \frac{1+\sqrt{5}}{2} \approx 1.618,$$

while the optimal error is

$$\|F - P_0^*\|_I = \|F - 0\|_I = \frac{8}{3\sqrt{3}} \approx 1.540.$$

2. Two a posteriori estimates for near-best approximation. If $F - P_m$ nearly equioscillates on a set of $m+2$ points in $I(\tau)$ then, by the de la Vallée Poussin inclusion theorem, P_m approximates F with nearly minimal error. We wish to conclude that $\|P_m - P_m^*\|_{I(\tau)}$ is small. Such a conclusion can be derived from either a strong unicity theorem or a theorem on Lipschitz continuity of the best approximation operator [2, pp. 80–82]. In either case, we need here a constant that is uniform for all intervals $I(\tau)$ in some range $\tau \in (0, \tau_0]$, and we need to know how it depends on m.

The existence of a uniform strong unicity constant or a uniform Lipschitz constant (the former implies the latter) is guaranteed under suitable assumptions, cf. [4], [10], [13]. Essentially, one needs to have that for each τ there exists an alternant with well separated points. Here, we make a more specific assumption, namely that the alternation points are close to the Chebyshev abscissae.

THEOREM 2.1. *Let P_m be an arbitrary approximation to F of degree at most m on $I(\tau)$. Suppose there exist $m+2$ points $\tau \geqq x_0 > x_1 > \cdots > x_{m+1} \geqq -\tau$ in $I(\tau)$ on which $F - P_m$ alternates in sign with*

$$\|F - P_m\|_{I(\tau)} - \min_j |(F - P_m)(x_j)| \leqq \varepsilon \tag{2.1}$$

for some ε, and suppose that $x_j = \tau \cos \phi_j$ with

$$\left|\phi_j - \frac{j\pi}{m+1}\right| \leqq \frac{1}{2\chi(m+1)^2} \tag{2.2}$$

for each j and some $\chi > 1$. Then

$$\|P_m - P_m^*\|_{I(\tau)} \leqq \kappa\varepsilon \tag{2.3}$$

where

$$\kappa = (2m+1)\frac{\chi}{\chi - 1}.$$

Proof. It is easy to construct a continuous function $\hat{F}$ with $\|\hat{F} - F\|_{I(\tau)} = \varepsilon/2$ such that $\hat{F} - P_m$ exactly equioscillates between $\pm\|\hat{F} - P_m\|_{I(\tau)}$ on the point set $\{x_j\}$. P_m is then the best approximation to $\hat{F}$ on $I(\tau)$. To prove the theorem, it will suffice to show that $1/\kappa$ is a lower bound (uniform in τ) for the strong unicity constant of $\hat{F}$ and P_m. This will imply that 2κ is a corresponding Lipschitz constant [2, p. 82], i.e., that if $\tilde{F}$ is any continuous function on $I(\tau)$ and $\tilde{P}_m^*$ its best approximation, then

$$\|\tilde{P}_m^* - P_m\|_{I(\tau)} \leqq 2\kappa\|\tilde{F} - \hat{F}\|_{I(\tau)}.$$

Choosing $\tilde{F} = F$ will yield (2.3).

To compute κ we use a construction presented by Cline [3], which in essence appears also in an earlier paper of Maehly and Witzgall [11]. To begin with, one has

$$\frac{1}{\kappa}=\min_{Q}\max_{j}\operatorname{sign}\{(\hat{F}-P_m)(x_j)\}Q(x_j), \tag{2.4}$$

where Q is any polynomial of degree m and norm 1; cf. [2], [3]. But, as shown by Maehly and Witzgall and by Cline, one may restrict Q in (2.4) to range over the $m+2$ polynomials $Q_l/\|Q_l\|_{I(\tau)}$, where Q_l interpolates $\operatorname{sign}(\hat{F}-P_m)$ on $\{x_j\}_{j\neq l}$:

$$(-1)^j\sigma Q_l(x_j)=1 \quad \forall j\neq l. \tag{2.5}$$

Moreover, $\kappa=\max\{\|Q_l\|_{I(\tau)}\}_{l=0}^{m+1}$.

Now $Q_l(\tau\cos\phi)$ is a trigonometric polynomial of degree m in ϕ, which implies by Bernstein's inequality

$$\left|\frac{dQ_l(\tau\cos\phi)}{d\phi}\right|\leqq m\|Q_l\|_{I(\tau)};$$

see, e.g., [14, p. 103]. Setting $x_j^T=\tau\cos(j\pi/(m+1))$, we get from (2.2) and (2.5)

$$|Q_l(x_j^T)|\leqq 1+m\|Q_l\|_{I(\tau)}\frac{1}{2\chi(m+1)^2}. \tag{2.6}$$

Cline showed, in an example [3, § 4], that $(-1)^j\sigma Q(x_j^T)=1(\forall j\neq l)$ implies $\|Q\|_{I(\tau)}\leqq 2m+1$. Together with (2.6) this implies

$$\|Q_l\|_{I(\tau)}\leqq(2m+1)\left[1+\frac{m\|Q_l\|_{I(\tau)}}{2\chi(m+1)^2}\right],$$

and therefore

$$\|Q_l\|_{I(\tau)}\leqq(2m+1)\left(1-\frac{1}{\chi}\right)^{-1}. \qquad \square$$

Remark. Condition (2.2) may be replaced by

$$|x_j-x_j^T|\leqq\frac{\tau}{2\chi(m+1)^3}.$$

The proof remains the same except that now A. A. Markov's inequality [14, p. 105] $\|Q_l'\|_{I(\tau)}\leqq\tau^{-1}m^2\|Q_l\|_{I(\tau)}$ is required.

The estimate (2.3) is best possible in the sense that κ tends to the strong unicity constant for the extremal signature $\{(x_j^T,(-1)^j)\}_{j=0}^{m+1}$ as $\chi\to\infty$.

For asymptotic results for $m\to\infty$ on a fixed interval, we will need a result with a weaker hypothesis than (2.2).

THEOREM 2.2. *Suppose the assumptions of Theorem* 2.1 *hold except that* (2.2) *is weakened to*

$$\left|\phi_j-\frac{j\pi}{m+1}\right|\leqq\frac{1}{2(m+1)}. \tag{2.7}$$

Then

$$\|P_m-P_m^*\|_{I(\tau)}\leqq K^m\varepsilon$$

for some $K>0$ (*independent of* m).

Sketch of proof. As in the proof of Theorem 2.1, we show that, for each l with $0 \leqq l \leqq m+1$, $\|Q_l\|_{I(\tau)}$ is bounded by K^m. One way to do do this is to write Q_l in Lagrange interpolation form as a sum

$$Q_l(x) = \sigma \sum_{\substack{j=0 \\ j\neq l}}^{m+1} (-1)^j \prod_{\substack{k=0 \\ k\neq j,l}}^{m+1} \left(\frac{x-x_k}{x_j-x_k}\right). \tag{2.8}$$

If $x_k = x_k^T$ for all k, then each denominator $\prod (x_j - x_k)$ in (2.8) is bounded in magnitude from below by $(m+1)2^{-m-1}$. One the other hand, from (2.7) it can be shown that no term $(x_j - x_k)^{-1}$ increases by more than a fixed proportion, independent of m, when $\{x_k^T\}$ is replaced by $\{x_k\}$. Moreover, each numerator $\prod (x - x_k)$ is bounded by 2^m. Therefore each summand of (2.8) is bounded by $K^m/(m+1)$ for some K. □

3. Asymptotic results for small intervals. For the following results we specialize to the case $M = 3m+3$, which turns out to be the minimal appropriate choice for real CF approximation on small intervals. (For practical computation on finite intervals there is no reason to stop at $M = 3m+3$. However, it is an artifact of the theorems here that although the same orders of convergence with respect to τ can be established with any $M \geqq 3m+3$, the associated constants grow worse as M increases.) Therefore let P_m^{cf} denote $P_{m,3m+3}^{cf}$ from now on. Note that $\{a_k\}_{k=0}^{\infty}$, P_m^{cf}, P_m^*, etc. depend on τ. For convenience, we shall assume in all proofs, without loss of generality, that $F_m \equiv 0$.

All of our asymptotic results are based on the following facts from [18] concerning the CF extension q of Theorem 1.1.

LEMMA 3.1. *Let the Taylor coefficients $\{a_k\}$ of f satisfy*

$$|a_{m+1+l}| \leqq \tau^l |a_{m+1}| \neq 0 \quad \text{for } 1 \leqq l \leqq 2m+2. \tag{3.1}$$

Then there exist constants $\tau_0 > 0$, $\alpha > 0$, $\beta > 0$ such that, for all $\tau \leqq \tau_0$,

(i) *q has winding number exactly $m+1$ on ∂D;*

(ii)

$$\|q - a_{m+1} w^{m+1}\|_{\partial D} \leqq 9\sqrt{m+1}\,\tau |a_{m+1}|; \tag{3.2}$$

(iii)

$$|b_{m+1-l}| \leqq \alpha^{2m+2} (\beta\tau)^l |a_{m+1}| \quad \forall l \geqq 1. \tag{3.3}$$

In particular this holds for $\tau_0 = \frac{1}{72}$, $\alpha = 3$, $\beta = 6$.

Proof. (i) and (iii) follow by applying [18, Lemma 7] with $\nu = M - m - 1 = 2m+2$ and $z = w^{-1}$ to $f(w)/(a_{m+1} w^M)$.

To prove (ii), we use the inequality

$$\sum_{k=0}^{m} |b_k| < \sqrt{8(m+1)}\,\tau |a_{m+1}|,$$

from the proof of [18, Thm 9]. From (3.3) we readily derive

$$\sum_{k=-\infty}^{-1} |b_k| \leqq a^{2m+2} (\beta\tau)^{m+2} |a_{m+1}| / (1 - \beta\tau) \leqq 5\tau |a_{m+1}|$$

under the stated bounds for α, β, τ. From (3.1) we also have

$$\sum_{k=m+2}^{3m+3} |b_k| \leqq \frac{72}{71} \tau |a_{m+1}|.$$

Since $5+\sqrt{8(m+1)}+\frac{72}{71}\leqq 9\sqrt{m+1}$, these three inequalities imply (ii). □

Lemma 3.1 implies that $F-P_m^{cf}$ nearly equioscillates on $I(\tau)$.

LEMMA 3.2. *Assume that* (3.1) *holds with* $\tau\leqq\frac{1}{72}$, *and analogously that the remainder term of* (1.1) *satisfies*

$$\|F-F_M\|_{I(\tau)}=\|R_{3m+3}\|_{I(\tau)}\leqq\frac{\tau^{2m+3}|a_{m+1}|}{1-\tau}. \tag{3.4}$$

Then:

(i) *for all* $w\in\partial D$ (*hence all* $x=x(w)\in I(\tau)$),

$$\left|F(x)-P_m^{cf}(x)-\left(1-\frac{b_{-m-1}}{a_{m+1}}\right)\operatorname{Re}q(w)\right|\leqq\frac{1}{2}(22\tau)^{2m+3}|a_{m+1}|; \tag{3.5}$$

(ii) *there exist* $m+2$ *points* $\tau=x_0>x_1>\cdots>x_{m+1}=-\tau$ *in* $I(\tau)$ *on which* $F-P_m^{cf}$ *alternates in sign and such that*

$$\|F-P_m^{cf}\|_{I(\tau)}-\min_j|(F-P_m^{cf})(x_j)|\leqq(22\tau)^{2m+3}|a_{m+1}|; \tag{3.6}$$

(iii)

$$\big|\|F-P_m^{cf}\|_{I(\tau)}-|\lambda|\big|\leqq(22\tau)^{2m+2}|a_{m+1}|; \tag{3.7}$$

(iv) *and if in addition* $\tau\leqq(10\sqrt{m+1})^{-1}$,

$$x_j=\tau\cos\phi_j\quad\textit{with}\ \left|\phi_j-\frac{j\pi}{m+1}\right|<\frac{16\tau}{\sqrt{m+1}}. \tag{3.8}$$

Proof. The idea of the proof is to show that $f-p$ is circular up to $O(\tau^{2m+2})$ on ∂D, hence that $F_M-P_m^{cf}$ equioscillates up to $O(\tau^{2m+2})$ on $I(\tau)$. Such a development leads naturally to an exponent $2m+2$ in (i), (ii) and (iii). However, it turns out that since $\operatorname{Re}w^{-m-1}=\operatorname{Re}w^{m+1}$ on ∂D, the term of degree $k=-m-1$ that dominates the deviation of $f-p$ from a circle fails to introduce any deviation from perfect equioscillation in $F_M-P_m^{cf}$. This is the reason for the higher exponent $2m+3$ in (i) and (ii). Unfortunately, keeping track of this "bonus" will lengthen the proof.

Proof of (i). By (3.3) we get for all $w\in\partial D$

$$\begin{aligned}&|q(w)+p(w)-f(w)-b_{-m-1}w^{-m-1}|\\&\qquad\leqq\sum_{k=-\infty}^{-m-2}|b_k|\leqq|a_{m+1}|\alpha^{2m+2}\sum_{l=2m+3}^{\infty}(\beta\tau)^l\leqq\frac{1}{\alpha}\frac{(\alpha\beta\tau)^{2m+3}}{1-\beta\tau}|a_{m+1}|.\end{aligned} \tag{3.9}$$

Now, due to (1.12) and (1.13),

$$\begin{aligned}F(x)-P_m^{cf}(x)-\left(1-\frac{b_{-m-1}}{a_{m+1}}\right)\operatorname{Re}q(w)\\=[F(x)-F_M(x)]+\operatorname{Re}\{f(w)-p(w)-q(w)+b_{-m-1}w^{-m-1}\}\\+\frac{b_{-m-1}}{a_{m+1}}\operatorname{Re}\{q(w)-a_{m+1}w^{-m-1}\}.\end{aligned}$$

By noting that $\operatorname{Re}w^{-m-1}=\operatorname{Re}w^{m+1}$ and

$$|b_{-m-1}|\leqq(\alpha\beta\tau)^{2m+2}|a_{m+1}|, \tag{3.10}$$

due to (3.3), and by using (3.2), (3.4) and (3.9), we get

$$\left|F(x)-P_m^{cf}(x)-\left(1-\frac{b_{-m-1}}{a_{m+1}}\right)\operatorname{Re} q(w)\right| \leqq\left[\frac{1}{1-\tau}+\frac{\alpha^{2m+2}\beta^{2m+3}}{1-\beta\tau}+9(\alpha\beta)^{2m+2}\sqrt{m+1}\right]\tau^{2m+3}|a_{m+1}|. \tag{3.11}$$

In particular, for $\alpha=3$, $\beta=6$, $\tau\leqq\frac{1}{72}$, one obtains (i) after some numerical estimates that use the bound $\sqrt{m+1}\leqq 2^{m/2}$.

Proof of (ii). Since q has real coefficients and maps ∂D onto a circle of radius $|\lambda|$ about 0 whose winding number is $m+1$, there are $m+2$ ordered points $1=w_0, w_1, \cdots, w_{m+1}=-1$ on the upper half of ∂D such that

$$\operatorname{Re} q(w_j)=q(w_j)=\sigma_q(-1)^j|\lambda|, \qquad j=0,\cdots,m+1, \tag{3.12}$$

where $\sigma_q=\pm1$. Let $x_j=x(w_j)$ be their images under (1.2). If we can show that the right-hand side of (3.5) is smaller than $(1-b_{-m-1}/a_{m+1})|\lambda|$, then (3.6) follows from (3.5). Now, since

$$q(w)-\sum_{k=m+2}^{M} a_k w^k=\sum_{k=-\infty}^{m+1} b_k w^k$$

may be thought of as a CF extension of $a_{m+1}w^{m+1}$, its norm is at least equal to that of the minimal extension, which is $a_{m+1}w^{m+1}$ itself. Consequently, using (3.1) again, we have

$$|\lambda|=\|q\|_{\partial D}\geqq|a_{m+1}|-\sum_{k=m+2}^{M}|a_k|\geqq\left(1-\frac{\tau}{1-\tau}\right)|a_{m+1}|\geqq\frac{70}{71}|a_{m+1}|. \tag{3.13}$$

So for (3.6) is suffices to show that

$$\frac{1}{2}(22\tau)^{2m+3}|a_{m+1}|<\frac{70}{71}\left(1-\frac{b_{-m-1}}{a_{m+1}}\right)|a_{m+1}|.$$

This inequality is easily derived from (3.10) under the assumptions on α, β, τ.

Proof of (iii). By (3.1) and the minimality of q,

$$|\lambda|=\|q\|_{\partial D}\leqq\|f\|_{\partial D}\leqq\left(1+\frac{\tau}{1-\tau}\right)|a_{m+1}|\leqq\frac{72}{71}|a_{m+1}|. \tag{3.14}$$

Hence, from (3.5) and (3.10), we conclude

$$|F(x)-P_m^{cf}(x)-\operatorname{Re} q(w)|\leqq\left[\frac{72}{71}(\alpha\beta\tau)^{2m+2}+\frac{1}{2}(22\tau)^{2m+3}\right]|a_{m+1}| \leqq(22\tau)^{2m+2}|a_{m+1}|,$$

which implies (3.7).

Proof of (iv). Finally, if $\tau\leqq(10\sqrt{m+1})^{-1}$, then by (3.2)

$$\|q-a_{m+1}w^{m+1}\|_{\partial D}\leqq 9\sqrt{m+1}\,\tau|a_{m+1}|\leqq\frac{9}{10}|a_{m+1}|.$$

In particular at $w_j=e^{i\phi_j}$, by (3.12),

$$\left|\frac{\sigma_q(-1)^j|\lambda|}{|a_{m+1}|}-\operatorname{sign}(a_{m+1})\,e^{i(m+1)\phi_j}\right|\leqq 9\sqrt{m+1}\,\tau\leqq\frac{9}{10},$$

which implies $\sigma_q = \text{sign}(a_{m+1})$ and therefore

$$\left|\frac{(-1)^j|\lambda|}{|a_{m+1}|} - e^{i(m+1)\phi_j}\right| \leqq 9\sqrt{m+1}\,\tau \leqq \frac{9}{10};$$

hence, by elementary trigonometry and (3.13),

$$|\sin((m+1)\phi_j)| \leqq \frac{|a_{m+1}|}{|\lambda|} 9\sqrt{m+1}\,\tau < 10\sqrt{m+1}\,\tau \leqq 1.$$

This implies that the arguments ϕ_j satisfy

$$(m+1)\left|\phi_j - \frac{j\pi}{m+1}\right| \leqq 10\sqrt{m+1}\,\tau \sin^{-1}(1) \leqq 16\sqrt{m+1}\,\tau,$$

which proves (iii). □

Lemma 3.2(ii) implies, by the de la Vallée Poussin theorem, that if the Chebyshev coefficients of F fall off faster than powers of $\frac{1}{72}$, then P_m^{cf} achieves near-minimal error:

$$\|F - P_m^{cf}\|_{I(\tau)} - \|F - P_m^*\|_{I(\tau)} \leqq (22\tau)^{2m+3}|a_{m+1}|. \tag{3.15}$$

By means of Lemma 3.2(iv) and the results of § 2, one could go further and derive a similar bound on $\|P_m^{cf} - P_m^*\|_{I(\tau)}$. However, the constants involved here (72, 22, etc.) are much too large for such estimates to be useful or realistic. Instead, our purpose is to apply Lemma 3.2 to derive orders of dependence on τ of $\|P_m^{cf} - P_m^*\|_{I(\tau)}$ and other quantities. The reason for taking such pains with numerical constants has been to make sure that Lemma 3.2 can treat both $\tau \to 0$ and $m \to \infty$.

We will need a lemma relating Taylor and Chebyshev coefficients. The proof, which we omit, is based on the formula

$$a_k = 2\sum_{l=0}^{\infty} c_{k+2l}\left(\frac{\tau}{2}\right)^{k+2l}\binom{k+2l}{l},$$

which holds for sufficiently small τ if F is analytic at $x = 0$.

LEMMA 3.3. *Let F be at least $k+2$ times differentiable at $x = 0$, with* kth *Taylor coefficient $c_k \equiv F^{(k)}(0)/k!$. Then as $\tau \to 0$*

$$a_k = 2^{1-k}\tau^k c_k + O(\tau^{k+2}). \tag{3.16}$$

If F is entire and

$$\lim_{k\to\infty} \sqrt{k}\left|\frac{c_{k+1}}{c_k}\right| = 0, \tag{3.17}$$

then (3.16) *holds uniformly in k as $\tau \to 0$.* □

Here is our main theorem on approximation on small intervals:

THEOREM 3.4 ($\tau \to 0$). *Let F be any function with a Lipschitz continuous* $(3m+3)$rd *derivative at $x = 0$ and satisfying $F^{(m+1)}(0)/(m+1)! \equiv c_{m+1} \neq 0$. For each $\tau > 0$, let P_m^{cf} be the* CF *approximation to F on $I(\tau)$ defined by* (1.9) *with some fixed $M \geqq 3m+3$, and let E^* denote $\|F - P_m^*\|_{I(\tau)}$. Then as $\tau \to 0$ E^* decreases according to*

(i) $E^* = |c_{m+1}|2^{-m}\tau^{m+1}(1 + O(\tau^2))$;

the CF *method is accurate to the orders*

(ii) $|\lambda| = E^*(1 + O(\tau^{2m+2}))$,

(iii) $\|F - P_m^{cf}\|_{I(\tau)} = E^*(1 + O(\tau^{2m+3}))$,

(iv) $\|P_m^{cf} - P_m^*\|_{I(\tau)} = O(E^*\tau^{2m+3})$;

and the best approximation error curve has the property

(v) $\|(F-P_m^*)(x)-\operatorname{Re}\tilde{q}(w)\|_{I(\tau)}=O(E^*\tau^{2m+3})$,

where $\tilde{q}$ *is a scalar multiple of some* $(m+1)$*-winding Blaschke product that is real on the real axis, is analytic in* $1 \leqq |w| < \infty$, *and has winding number* $m+1$ *on* ∂D.

Proof (cf. [18, Thm. 10]). We continue to assume $M=3m+3$, so that Lemma 3.2 is directly applicable. For larger M the constants grow worse but the orders are unchanged.

Statement (i) is well known and not difficult to prove; see Nitsche [12, Thm. 4].

Lemma 3.3 and the Lipschitz assumption imply that, although the assumptions of Lemmas 3.1 and 3.2 may not hold for F, for some sufficiently small $\gamma>0$ they must hold for the function $\hat{F}$ defined by $\hat{F}(x)=F(\gamma x)$. Since approximation of $\hat{F}$ on $I(\tau)$ is equivalent to approximation of F on $I(\gamma\tau)$, and since $O(\tau^{2m+3})$ is equivalent to $O((\gamma\tau)^{2m+3})$, it follows that to prove (ii)–(v) we may assume, without loss of generality, that (3.1) and (3.4) are satisfied. Lemmas 3.1 and 3.2 are therefore applicable. (ii) and (iii) then follow from (3.7) and (3.6) together with the de la Vallée Poussin theorem. Because of (3.8), Theorem 2.1 or 2.2 can also be applied to (3.6) to give (iv). Finally, (v) follows from (iv), Lemma 3.1(i), and (3.5), taking

$$\tilde{q}=(1-(b_{-m-1}/a_{m+1}))q. \qquad \square$$

Essentially the same argument yields an almost equally powerful theorem for approximation as $m\to\infty$ on a fixed interval.

THEOREM 3.5 ($m\to\infty$). *Let F be an entire function whose Taylor series coefficients satisfy* (3.17). *Let* $\rho>0$ *be arbitrary. For each* $m\geqq 0$, *let* P_m^{cf} *be the* CF *approximation to F on I defined by* (1.9) *with* $M=3m+3$, *and let* E^* *denote* $\|F-P_m^*\|_I$. *Then as* $m\to\infty$, E^* *decreases according to*

(i) $E^*=|c_{m+1}|2^{-m}(1+O(\rho^2))$;

the CF *method is accurate to the orders*

(ii) $|\lambda|=E^*(1+O(\rho^{2m+2}))$,

(iii) $\|F-P_m^{cf}\|_I=E^*(1+O(\rho^{2m+3}))$,

(iv) $\|P_m^{cf}-P_m^*\|_I=O(E^*\rho^{2m+3})$;

and the best approximation error curve has the property

(v) $\|(F-P_m^*)(x)-\operatorname{Re}\tilde{q}(w)\|_I=O(E^*\rho^{2m+3})$,

where $\tilde{q}$ *is a scalar multiple of some* $m+1$*-winding Blaschke product that is real on the real axis, is analytic in* $1\leqq|w|<\infty$, *and has winding number* $m+1$ *on* ∂D. *Statements* (i), (ii) *and* (iii) *still hold if the factor* $\sqrt{k}$ *is removed from the hypothesis.*

Proof (cf. [18, Thm. 11]). Given $\tau>0$, approximating $F(x)$ on I is equivalent to approximating $\hat{F}(x)\equiv F(x/\tau)$ on $I(\tau)$, as in the last proof. Lemma 3.3 and (3.17) imply that for any ρ, there is some $m_0>0$ such that for all $m\geqq m_0$, all of the assumptions of Lemma 3.1 and 3.2 are satisfied with $\tau=\rho/(22K)$, where K is the constant in Theorem 2.2. The proof now runs like the last one except that only Theorem 2.2, not Theorem 2.1, can be used, as the right-hand side of (3.8) is only guaranteed to decrease like $(m+1)^{-1}$, not $(m+1)^{-2}$.

If the factor $\sqrt{k}$ is eliminated, then the extra assumption for Lemma 3.2(iv) may no longer be satisfied, so (iv) and (v) need no longer hold. $\square$

Theorem 3.5 does not apply to functions with some Taylor coefficients that are zero or very small, such as even or odd functions, but it is an easy matter to generalize it in this direction. Let (3.17) be modified to assert that for some increasing sequence of indices $\{m_j\}$, $|c_{m_j+1}|$ dominates its successors according to

$$\limsup_{j\to\infty}\sup_{l\geqq 1}\left|\frac{c_{m_j+l+1}}{c_{m_j+1}}\right|(m_j)^{1/2}=0.$$

Then the conclusions of Theorem 3.5 hold for $m = m_j$ as $j \to \infty$.

4. Numerical examples. Given F as in (1.1), our procedure for constructing $P^{cf}_{m,M}$ consists of four steps:

(1) Compute the first M Chebyshev coefficients $\{a_k\}$ by projecting F to ∂D by (1.2), then applying the fast Fourier transform (FFT).

(2) Find the largest eigenvalue of A in absolute value and a corresponding eigenvector by means of routines from EISPACK [16].

(3) Compute $b_m, b_{m-1}, \cdots, b_{-m}$ recursively with (1.8). A few additional coefficients $b_{-m-1}, b_{-m-2}, \cdots$ can also be computed for an estimate of how much the error curve of $P^{cf}_{m,M}$ will deviate from a circle.

(4) Construct $P^{cf}_{m,M}$ as a sum of Chebyshev polynomials with (1.9), and convert it to a single polynomial if desired.

Details of this procedure for the problem of complex rational CF approximation are given in [19]. In practice the most time consuming step is (2). The speed of this eigenvalue computation depends critically on the dimension of A, which in turn depends on what value M is large enough to make the difference between F and F_M negligible. Therefore CF approximation is fastest for smooth functions F whose Chebyshev series decrease rapidly. Step (1) is also fastest in this case, as fewer points may be taken in the FFT. Moreover, the CF method is itself most accurate when F is smooth.

For a given near-best approximation P_m to F, define

$$\Delta E = \|F - P_m\|_I - \min_i |(F - P_m)(x_i)|, \tag{4.1}$$

where the points x_i belong to an alternant of length $m+2$ that yields the smallest such difference. The de la Vallée Poussin theorem then gives the bound

$$\|F - P^*_m\| \leqq \|F - P_m\| \leqq \|F - P^*_m\| + \Delta E. \tag{4.2}$$

If $P_m = P^{cf}_{m,M}$, this bound is generally very tight.

For example, Tables 1 and 2 summarize the behavior of the CF approximant $P^{cf}_{m,M}$ for the functions $F(x) = e^x$ and $F(x) = \ln((x+3)/2)$. M has been taken here to be 25, which is more than enough to make $F_M - F$ (hence $P^{cf}_{m,M} - P^{cf}_{m,\infty}$) negligible in both cases. Each computation took around 0.05 sec. on an IBM 370/168, excluding the time spent searching for extrema to apply (4.1). These tables show how extraordinarily accurate the real CF method can be.

Table 3 shows comparable numbers for approximation of a function that is not smooth, $F(x) = |x|$. Now $M = 120$ has been used, leading to computation times of roughly 2 sec. Even so, for the final entry of the table $P^{cf}_{m,M} - P^{cf}_{m,\infty}$ was not negligible, so that ΔE is greater there than it would have been had M been larger.

Figure 1 shows error curves for various CF approximations on $[-1, 1]$. These give further indication that the method is best suited to approximation of smooth functions.

The asymptotic behavior predicted in Theorem 3.4 can readily be observed numerically. For example, consider approximation of e^x on $[-\tau, \tau]$ with $m = 1$. Table 4 shows $\|F - P^{cf}_m\|_{I(\tau)}$ and the quantity ΔE for $\tau = 4, 2, 1, \frac{1}{2}, \frac{1}{4}$. As $\tau \to 0$ one should observe $\|F - P^{cf}_m\|_{I(\tau)} = O(\tau^2)$ and hence $\Delta E = O(\tau^{2m+3})\|F - P^{cf}_m\|_{I(\tau)} = O(\tau^7)$. The final column indicates that in this example ΔE behaves like

$$\Delta E \approx (0.12840\tau)^7 + O(\tau^9).$$

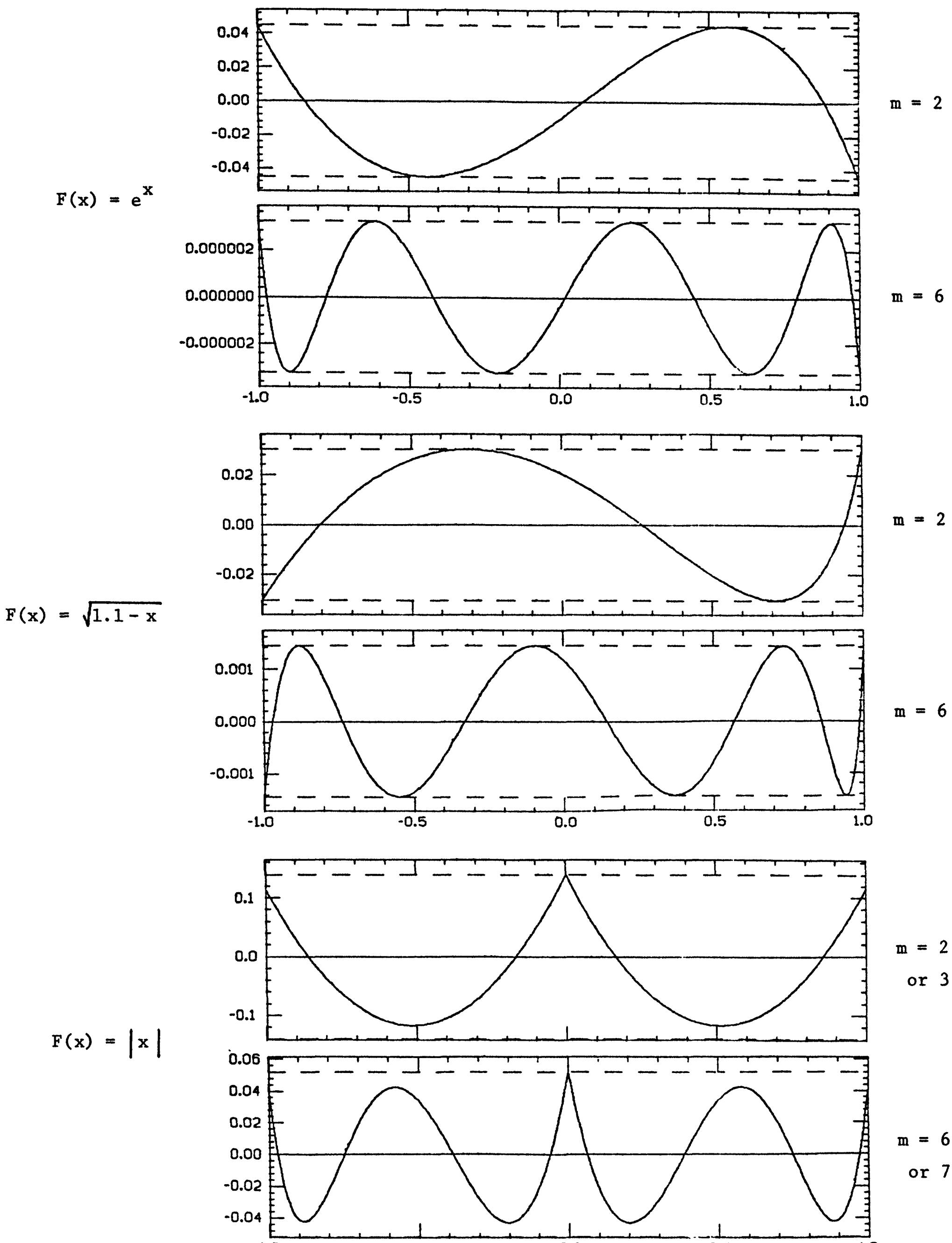

FIG. 1. *Error curves for various real* CF *approximations on* $[-1, 1]$.

TABLE 1.
CF *approximation to* $F(x)=e^x$ *on* $[-1, 1]$.

m	$\|\lambda\|$	$\|F-P_m^{cf}\|_I$	ΔE
0	1.19608 42668	1.17540 99930	4.2 (−4)
1	.27879 94302	.27880 18479	6.0 (−7)
2	.04501 73878	.04501 73884	1.8 (−11)

TABLE 2.
CF *approximation to* $F(x)=\ln(x+3)/2$ *on* $[-1, 1]$.

m	$\|\lambda\|$	$\|F-P_m^{cf}\|_I$	ΔE
0	.34571 10782	.34664 79871	1.5 (−4)
1	.02982 95424	.02983 01138	1.3 (−7)
2	.00342 39799	.00342 39808	2.1 (−10)
3	.00044 16161	.00044 16161	<7 (−13)

TABLE 3.
CF *approximation to* $F(x)=\|x\|$ *on* $[-1, 1]$.

m	$\|\lambda\|$	$\|F-P_m^{cf}\|_I$	ΔE
0 or 1	.44827	.53396	6.8 (−2)
2 or 3	.11359	.13901	2.3 (−2)
4 or 5	.06161	.07587	1.4 (−2)
6 or 7	.04185	.05179	9.6 (−3)

TABLE 4.
Degree 1 CF *approximation to* $F(x)=e^x$ *on* $[-\tau, \tau]$ *for various* τ.

τ	$\|F=P_1^{cf}\|$	ΔE	$\sqrt[7]{\Delta E}/R$
4	16.79618 25729	1.4 (−2)	.13515
2	1.51410 48013	8.1 (−5)	.13025
1	.27880 18479	6.0 (−7)	.12908
$\frac{1}{2}$	.06425 18670	4.5 (−9)	.12856
$\frac{1}{4}$	.01573 37522	3.5 (−11)	.12844

Thus not only is the convergence of the order predicted, but the constant term is very small.

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Note added in proof. The authors have become aware that some aspects of the "real CF method" appear in the work of Bernstein, Achieser, Talbot, Clenshaw, Darlington, Lam, D. Elliott, Hollenhorst, and G. H. Elliott (in historical order). References will be given in [8]. In particular, M. Hollenhorst derives in his dissertation (Universität Erlangen–Nürnberg, 1976) nonasymptotic error bounds for the polynomial approximation defined by (1.9) with the lower bound $-m$ in the sum replaced by 0. Because of this replacement, his approximation is considerably further from the best approximation than ours.

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