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REAL VS. COMPLEX RATIONAL CHEBYSHEV APPROXIMATION ON AN INTERVAL

BY

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ABSTRACT. If $f \in C[-1,1]$ is real-valued, let $E^r(f)$ and $E^c(f)$ be the errors in best approximation to f in the supremum norm by rational functions of type (m, n) with real and complex coefficients, respectively. It has recently been observed that $E^c(f) < E^r(f)$ can occur for any $n \geq 1$, but for no $n \geq 1$ is it known whether $\gamma_{mn} = \inf_f E^c(f)/E^r(f)$ is zero or strictly positive. Here we show that both are possible: $\gamma_{01} > 0$, but $\gamma_{mn} = 0$ for $n \geq m + 3$. Related results are obtained for approximation on regions in the plane.

1. Introduction. Let I be the unit interval $[-1, 1]$, C^r the set of continuous real functions on I, and $\|\cdot\|$ the supremum norm $\|f\| = \sup_{x \in I}|f(x)|$. For nonnegative integers m and n, let R_{mn} and $R^r_{mn} \subseteq R_{mn}$ be the spaces of rational functions of type (m, n) with coefficients in $\mathbf{C}$ and $\mathbf{R}$, respectively. For $f \in C^r$, let $E^c(f)$ and $E^r(f)$ denote the infima

$$(1) \qquad E^c(f) = \inf_{r \in R_{mn}} \|f - r\|, \qquad E^r(f) = \inf_{r \in R^r_{mn}} \|f - r\|.$$

It is known that both limits are attained, and a function that does so is called a *best approximation* (*BA*) to f. In the real case the BA is unique **[8]**, and in the complex case for $n \geq 1$ in general it is not **[7, 10, 11, 14, 15]**.

Obviously $E^c \leq E^r$ for any f, but since f is real, it is not at first obvious whether a strict inequality can occur. However in 1971 Lungu **[7]**, following a proposal of Gončar **[16]**, published a class of examples showing that $E^c(f) < E^r(f)$ is indeed possible if $n \geq 1$. Independently, Saff and Varga **[10, 11]** made the same discovery in 1977, and obtained more general sufficient conditions for $E^c(f) < E^r(f)$ and also a sufficient condition for $E^c(f) = E^r(f)$. The former was later sharpened by Ruttan **[18]** to the following statement: $E^c(f) < E^r(f)$ must hold if the best real approximation to f attains its maximum error on no alternation set of length greater than $m + n + 1$ points. For a survey of such results, see **[14]**.

But is E^c ever *much* less than E^r? If γ_{mn} denotes the infimum

$$(2) \qquad \gamma_{mn} = \inf_{f \in C^r \setminus R^r_{mn}} E^c(f)/E^r(f),$$

then one would like to know whether γ_{mn} can be zero or is always positive, and if the latter, how small it is. In all of the examples devised to date, $E^c(f)/E^r(f)$ has fallen

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in the range $(\frac{1}{2}, 1]$, suggesting that $\gamma_{mn} = \frac{1}{2}$ might be the minimum value. Saff and Varga posed in particular the question, is γ_{nn} positive or zero **[10, 11]**? Ellacott has suggested that $\gamma_{mn} = \frac{1}{2}$ may hold for $m \geqslant n$ **[3]**. (For more on his argument see §2.) Some partial results for $(m, n) = (1, 1)$ have been obtained by Bennet, et al. **[1, 2]** and by Ruttan **[9]**.

In this paper we resolve some of these questions, as follows. First, not only can $\gamma_{mn} < \frac{1}{2}$ occur, but $\gamma_{mn} = 0$ *for all* $m \geqslant 0$, $n \geqslant m + 3$ (Theorem 1). Second, $\gamma_{01} > 0$ (Theorem 2). We conjecture that $\gamma_{mn} > 0$ holds whenever $n < m + 3$. Finally, at least some of our arguments extend to approximation on complex regions, and we show: $\gamma_{0n}^{\Delta} = 0$ *for* $n \geqslant 4$ *in approximation on the unit disk* Δ (Theorem 3). A similar result is obtained for approximation on a symmetric Jordan region.

2. $\gamma_{mn} = 0$ **for** $n \geqslant m + 3$.

THEOREM 1. $\gamma_{mn} = 0$ *for all* $m \geqslant 0$, $n \geqslant m + 3$.

PROOF. The idea of the construction is indicated in Figure 1, where crosses represent poles and circles represent zeros.

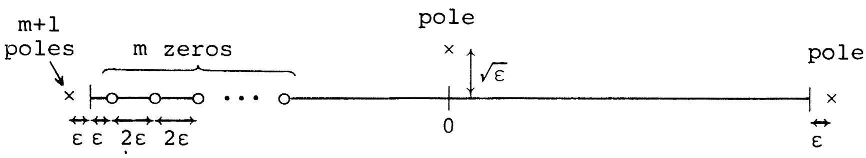

FIGURE 1

Given $m \geqslant 0$, let $\phi \in R_{m,n+3}$ be defined by

$$\phi(x) = \frac{\varepsilon\prod_{j=1}^{m}[(-1 + (2j - 1)\varepsilon) - x]}{[x + (1 + \varepsilon)]^{m+1}\left[i\sqrt{\varepsilon} - x\right][(1 + \varepsilon) - x]} \tag{3}$$

and as the function in C^r to be approximated take $f(x) = \operatorname{Re}\phi(x)$. We will show that f has the following two properties:

(a) $\|f - \phi\| = \|\operatorname{Im}\phi\| = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

(b) There exists a constant $C > 0$ such that for all sufficiently small ε,

$$(-1)^j f(-1 + 2j\varepsilon) \geqslant C, \qquad 0 \leqslant j \leqslant m, \tag{4}$$

and

$$(-1)^{m+1} f(1) \geqslant C. \tag{5}$$

Condition (b) states that the error function for the zero approximation to f approximately equioscillates at $m + 2$ points, and by the de la Vallée Poussin theorem for real rational approximation **[8**, Theorem 98], this implies $E^r \geqslant C$. (For the purposes of this theorem $r \equiv 0$ has rational type $(\mu, \nu) = (-\infty, 0)$, so the "defect" $d = \min\{m - \mu, n - \nu\}$ is n, which means one needs approximate equioscillation at $m + n + 2 - d = m + 2$ points.) On the other hand if $n \geqslant m + 3$, then $\phi \in R_{mn}$, so (a) implies $E^c = O(\sqrt{\varepsilon})$. Thus since ε can be arbitrarily small, the theorem will be proved once (a) and (b) are established.

PROOF OF (a). Let us write ϕ as a product of three functions ϕ_1, ϕ_2, ϕ_3 corresponding to the poles and zeros near $-1, 0,$ and 1, respectively. Of these functions only ϕ_2 has a nonzero imaginary part on I, and we bring this into the numerator. The factor ϕ_1 gets the constant ε from (3):

$$(6)\quad \phi(x) = \phi_1(x)\phi_2(x)\phi_3(x)$$
$$= \left(\frac{\varepsilon\prod_{j=1}^{m}[(-1+(2j-1)\varepsilon)-x]}{[x+(1+\varepsilon)]^{m+1}}\right)\left(\frac{-i\sqrt{\varepsilon}-x}{x^2+\varepsilon}\right)\left(\frac{1}{(1+\varepsilon)-x}\right).$$

Since $(f-\phi)(x) = -i\,\mathrm{Im}\,\phi(x)$, we compute

$$(f-\phi)(x) = -i\phi_1(x)\mathrm{Im}\,\phi_2(x)\phi_3(x) = \phi_1(x)\frac{i\sqrt{\varepsilon}}{x^2+\varepsilon}\phi_3(x).$$

It is not hard to see that on $[-1, -\frac{1}{2}]$ these factors have magnitude $O(1)$, $O(\sqrt{\varepsilon})$, and $O(1)$, so their product is $O(\sqrt{\varepsilon})$. Similarly in $[-\frac{1}{2}, \frac{1}{2}]$ one has $O(\varepsilon)O(1/\sqrt{\varepsilon})O(1) = O(\sqrt{\varepsilon})$, and in $[\frac{1}{2}, 1]$, $O(\varepsilon)O(\sqrt{\varepsilon})O(1/\varepsilon) = O(\sqrt{\varepsilon})$. Together these estimates give $(f-\phi)(x) = O(\sqrt{\varepsilon})$ for all $x \in I$, as claimed.

PROOF OF (b). Again we use the factorization $\phi = \phi_1\phi_2\phi_3$ of (6). Let $\{x_j\}_{j=0}^{m}$ be the set of points $x_j = -1 + 2j\varepsilon$ that appear in condition (4). At each x_j, ϕ_1 evidently takes the form $\alpha_j\varepsilon^{m+1}/\beta_j\varepsilon^{m+1}$ for some constants α_j and β_j, and thus $\phi_1(x_j)$ is *independent of* ε. Moreover these quantities obviously alternate in sign, i.e.

$$\phi_1(x_0) = \tau_0 > 0,\ -\phi_1(x_1) = \tau_1 > 0, \ldots, (-1)^m\phi_1(x_m) = \tau_m > 0,$$

with τ_j independent of ε. In addition since all of the points x_j are contained in $[-1, -1+2m\varepsilon]$, we have $\phi_2(x_j) = 1 + O(\sqrt{\varepsilon})$, $\phi_3(x_j) = \frac{1}{2} + O(\varepsilon)$ on $\{x_j\}$. Together these facts establish (4) for some $C = C_1 > 0$.

For condition (5) we compute

$$\phi(1) = \phi_1(1)\phi_2(1)\phi_3(1)$$
$$= \left(\frac{\varepsilon}{2}(-1)^m(1+O(\varepsilon))\right)\left(-1+O\left(\sqrt{\varepsilon}\right)\right)\frac{1}{\varepsilon} = \frac{1}{2}(-1)^{m+1} + O\left(\sqrt{\varepsilon}\right),$$

which implies that (5) holds for $C = C_2$ with any $C_2 < \frac{1}{2}$. Taking $C = \min\{C_1, C_2\}$ now yields (b). $\square$

REMARK ON AN ARGUMENT OF ELLACOTT. As alluded to in the Introduction, Ellacott has observed that one can conclude from the *CF method* **[13, 4]** that if p is a polynomial of degree $m+1$, then

$$(7)\qquad E^c(p)/E^r(p) \geq \tfrac{1}{2}$$

for $n \leq m$ **[3]**. This is one of his arguments for suggesting that $\gamma_{mn} = \frac{1}{2}$ or at least $\gamma_{mn} > 0$ may hold for $n \leq m$. However we claim that (7) is valid in fact for all $n \leq 2m+1$, which by Theorem 1 means that it holds even in many cases with $\gamma_{mn} = 0$. Therefore although Ellacott's conjecture is plausible, it appears that (7) does not provide very strong support for it.

To demonstrate that (7) holds for $n \leqslant 2m+1$, let p be transplanted to the unit circle by defining a function $\hat{p}$ for $z \in \mathbf{C}$ as follows:

$$x = \frac{1}{2}(z + z^{-1}), \qquad \hat{p}(z) = p(x) = p\left(\frac{1}{2}z + \frac{1}{2}z^{-1}\right) = \sum_{k=-m-1}^{m+1} \alpha_k z^k.$$

For $n \leqslant 2m+1$, the BA to p in R^r_{mn} on I was obtained explicitly by Talbot **[12, 5]**, and its deviation from p is

$$E^r(p) = 2\sigma_n, \tag{8}$$

where σ_n is the smallest singular value of the $(n+1) \times (n+1)$ Hankel matrix $(\alpha_{m-n+1+i+j})^n_{i,j=0}$. On the other hand if $r \in R_{mn}$ is any complex approximation to p on I, consider the transplanted function $\hat{r}$ defined by $\hat{r}(z) = r(x)$. It is readily verified that $\hat{r}$ has $\nu \leqslant n$ poles in $1 < |z| < \infty$ and is of order $O(z^{m-\nu})$ at ∞. Therefore $\hat{r}$ lies in the space $\tilde{R}_{mn}$ defined in **[13, 4]**, and by the theory given there this implies

$$\sigma_n \leqslant \sup_{|z|=1} |(\hat{p} - \hat{r})(z)| = \sup_{|x|=1} |(p - r)(x)|.$$

Thus

$$E^c(p) \geqslant \sigma_n, \tag{9}$$

which together with (8), establishes (7).

By applying [**4**, Lemma 5.1 in Part II] (7) can be seen to hold even for some rational functions f, namely for those of exact type (M, N) where either $M \leqslant m+1$, $N = n+1$, $n \leqslant m$ or $M = m+1$, $N \leqslant n+1$, $n \leqslant 2m+1-N$; details will be given in **[5]**.

3. $\gamma_{01} > 0$.

THEOREM 2. $\gamma_{01} > 0$.

PROOF. Let $f \in C^r$ be arbitrary, and let c^* be a BA to f in R_{mn}. Then for any $r \in R^r_{mn}$ one has $\|\operatorname{Im} c^*\| \leqslant \|f - c^*\| = E^c(f)$ and $E^r(f) \leqslant E^c(f) + \|c^* - r\|$, and therefore

$$E^r(f) \leqslant E^c(f) + \|\operatorname{Im} c^*\| \frac{\|c^* - r\|}{\|\operatorname{Im} c^*\|} \leqslant E^c(f)\left(1 + \frac{\|c^* - r\|}{\|\operatorname{Im} c^*\|}\right). \tag{10}$$

Now suppose that for any $c \in R_{mn} \setminus R^r_{mn}$ with no poles on I, one can find $r^{(c)} \in R^r_{mn}$ such that

$$\|c - r^{(c)}\| / \|\operatorname{Im} c\| \leqslant M \tag{11}$$

for some fixed M. Then $r^{(c^*)}$ can be inserted in (10), independent of f, and one obtains $\gamma_{mn} \geqslant 1/(1+M)$. Our proof of $\gamma_{01} > 0$ consists of exhibiting a mapping $c \mapsto r^{(c)}$ for the case $(m, n) = (0, 1)$ that satisfies (11).

Thus let $c(z) = a/(1 - z/z_0)$ be given, where z_0 lies in the region $C^0 = \mathbf{C} \cup \{\infty\} \setminus I$. Let $\theta \in (0, \pi/2)$ and $\rho \in (1, \infty)$ be arbitrary fixed constants (say,

$\theta = \pi/4$, $\rho = 2$). Our choice of $r^{(c)}$ depends on which of four domains A^+, A^-, B, C the pole lies in:

$$A^{\pm} = \{z \in C: |\arg(-1 \pm z)| < \theta\},$$
$$B = \{z \in C - A^+ - A^-: |z| \leq \rho\},$$
$$C = C^0 - A^+ - A^- - B.$$

The configuration is indicated in Figure 2.

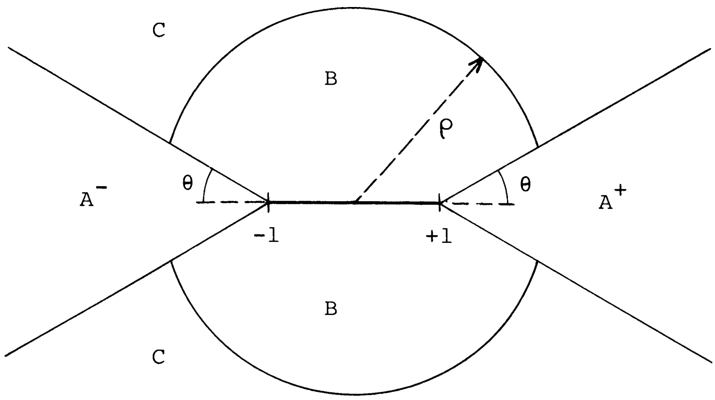

FIGURE 2

We define $r^{(c)}$ as follows:

$$\text{For } z_0 \in A^{\pm}: \quad r^{(c)}(z) = \frac{1 - 1/|z_0|}{1 \mp z/|z_0|} \operatorname{Re} c(\pm 1).$$

$$\text{For } z_0 \in B: \quad r^{(c)} \equiv 0.$$

$$\text{For } z_0 \in C: \quad r^{(c)} \equiv \operatorname{Re} a.$$

The proof can now be completed by showing that there exist constants M_A, M_B, M_C such that (11) holds for z_0 restricted to each domain $A^+ \cup A^-$, B, C. The global constant M can then be taken as $M = \max\{M_A, M_B, M_C\}$. The algebra involved is unfortunately quite tedious, so we will omit these verifications. However, details of a similar argument for the case of approximation on certain Jordan regions in **C** are given in **[17]**. □

4. $\gamma_{0n}^{\Delta} = 0$ **for** $n \geq 4$.

Let Δ be the closed unit disk $\{z \in \mathbf{C}: |z| \leq 1\}$, and let f be continuous in Δ and analytic in the interior and satisfy $f(\bar{z}) = \overline{f(z)}$. Let $\|f\|_{\Delta}$ denote $\sup_{z \in \Delta}|f(z)|$, and define $E^c(f; \Delta)$, $E^r(f; \Delta)$, and γ_{mn}^{Δ} as in (1) and (2). Until recently it was not even known whether $\gamma_{mn}^{\Delta} < 1$ is possible, but in a separate paper we show that this inequality holds at least for all pairs (m, n) with $m = 0$, $n \geq 1$ or $m \geq 0$, $n = 1$ **[6]**.

By a variation of the argument of §2, we will now prove

THEOREM 3. $\gamma_{0n}^{\Delta} = 0$ *for* $n \geq 4$.

PROOF. Let $\zeta = e^{i\theta}$ for some fixed $\theta \in (0, \pi)$, and for any $\varepsilon > 0$, define

$$\phi(z) = \frac{\varepsilon(1-\zeta)^2}{[z + (1+\varepsilon)][(1+\varepsilon) - z][z - (1+\varepsilon^{1/3})\zeta]^2}$$

and

$$f(z) = \tfrac{1}{2}\left(\phi(z) + \overline{\phi(\bar{z})}\right).$$

In analogy to the proof of Theorem 1, $\gamma_{0n}^{\Delta} = 0$ for $n \geqslant 4$ will follow from the properties

(a) $\|f - \phi\|_{\Delta} = O(\varepsilon^{1/3})$;

(b) there exists a constant $C > 0$ such that for all sufficiently small ε, $f(-1) \leqslant -C$, $f(1) \geqslant C$.

Both (a) and (b) can be readily derived by observing that the term

$$(1-\zeta)^2/[z - (1+\varepsilon^{1/3})\zeta]^2$$

behaves like $1 + O(\varepsilon^{1/3})$ near $z = 1$ and like $-|(1-\zeta)/(1+\zeta)|^2 + O(\varepsilon^{1/3})$ near $z = -1$. We omit the details. □

This argument can be extended to show $\gamma_{0n}^{\Omega} = 0$ for $n \geqslant 4$ for approximation on any Jordan region Ω with $\Omega = \overline{\Omega}$, provided $\partial\Omega$ is differentiable at its two points of intersection with $\mathbf{R}$, say z_1 and z_2, hence forms a right angle to $\mathbf{R}$ at these points. Again one introduces a complex double pole, slightly above the point z_1 (analogous to taking $\xi = e^{i\theta}$ with θ small above), and this generates an approximate sign change between $\phi(z_1)$ and $\phi(z_2)$.

One can also prove $\gamma_{01}^{\Omega} > 0$ for the same class of regions Ω. See [**17**].

Note added in proof. After studying the present paper, E. Saff has pointed out to us that the existence of arbitrarily small numbers γ_{mn} is implied by a result of Walsh in 1934 [**19**, Theorem IV], although this consequence was never recognized. Walsh showed that for any $m \geqslant 0$, the family $\bigcup_{n=0}^{\infty} R_{mn}$ is dense in $C[I]$ (or indeed in the space of continuous functions on any Jordon arc in $\mathbf{C}$), so that $\lim_{h\to\infty} E_{mn}(f) = 0$ for $f \in C[I]$. On the other hand, as we have seen, if f has $m+1$ zeros, then it cannot be approximated arbitrarily closely in $\bigcup_{n=0}^{\infty} R_{mn}^{r}$, i.e. $\lim_{n\to\infty} E_{mn}^{r}(f) > 0$. It follows that for any $m \geqslant 0$, $\lim_{n\to\infty} \gamma_{mn} = 0$.

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