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THE CARATHÉODORY-FEJÉR METHOD FOR REAL RATIONAL APPROXIMATION*

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Abstract. A "Carathéodory–Fejér method" is presented for near-best real rational approximation on intervals, based on the eigenvalue (or singular value) analysis of a Hankel matrix of Chebyshev coefficients. In approximation of a smooth function F, the CF approximant R^{cf} frequently differs from the best approximation R^* by only one part in millions or billions. To account for this we show here under weak assumptions that if F is approximated on $[-\varepsilon, \varepsilon]$, then as $\varepsilon \to 0$, $\|F-R^*\| = O(\varepsilon^{m+n+1})$ while $\|R^{cf}-R^*\| = O(\varepsilon^{3m+2n+3})$. In contrast, the latter figure would be $O(\varepsilon^{m+n+2})$ for the Chebyshev economization approximant of Maehly or the Chebyshev–Padé approximant of Gragg. It follows that as $\varepsilon \to 0$, best approximation error curves approach the real parts of $m+n+1$-winding rational functions of constant modulus to within $O(\varepsilon^{3m+2n+3})$. Numerical examples are given, including applications to e^x on $[-1, 1]$ and e^{-x} on $[0, \infty)$. For the latter problem we conjecture that the errors in (n, n) approximation decrease with each n by a ratio approaching a fixed constant $9.28903 \cdots$.

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Key words. Chebyshev approximation, rational approximation, small intervals, Meinardus conjecture, Hankel matrix, CF method

Introduction. The purpose of this paper is to describe and analyze a new analytical method for near-best rational Chebyshev approximation on an interval, which we call the Carathéodory–Fejér (CF) method, that is based on an eigenvalue analysis of a Hankel matrix of Chebyshev coefficients. The CF method achieves an extraordinary degree of optimality in approximating many smooth functions. Let R^* be the best (Chebyshev) approximation of rational type (m, n) on $[-1, 1]$ to a continuous function $F(x)$, let R^{cf} be the corresponding CF approximation, and let $E^* = \|F-R^*\|_\infty$ and $E^{cf} = \|F-R^{cf}\|_\infty$ be the associated errors. Then Table 1 shows how close E^{cf} and E^* turn out to be for the case $F(x) = e^x$. Such extremely strong agreement demands explanation. In this paper it is shown that if a smooth function $F(x)$ satisfying a simple normality condition is approximated on $[-\varepsilon, \varepsilon]$, then $\|R^{cf}-R^*\| = O(\varepsilon^{3m+2n+3})$ as $\varepsilon \to 0$ (Thm. 6). As a corollary it is also shown that to the same order as $\varepsilon \to 0$, $F-R^*$

TABLE 1
Errors in best and CF approximation of e^x on $[-1, 1]$ by rational functions of type (n, n), $0 \leqq n \leqq 4$

(m, n)	E^*	$E^{cf} - E^*$ (approx.)
(0, 0)	1.1752	10^{-4}
(1, 1)	2.0970 (−2)	10^{-6}
(2, 2)	8.6900 (−5)	10^{-12}
(3, 3)	1.5507 (−7)	10^{-20}
(4, 4)	1.5381 (−10)	$<10^{-27}$

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equals the real part of an $(m+n+1)$-winding rational function of constant modulus (i.e., of a quotient of finite Blaschke products) on the complex unit circle (Thm. 7).

Before describing the method in § 1, we will make some fairly extensive historical remarks, because there are various earlier papers with connections to CF approximation, but they are scattered and not well known. In outline, the present real, rational CF method is connected to complex rational CF approximation on the unit disk, but the connection is not at all trivial, and it has not been seen before. The CF method is also related to earlier real approximation work by Lam and Elliott, but it extends their technique in two ways. First, it works for arbitrary m and n, not just $m \geqq n$. As is often the case with near-best approximation methods [17], the extension to $m<n$ is the most difficult point. Second, it is asymptotically much more accurate.

Our research on CF approximation began with the study of error curves in approximation of analytic functions on the complex unit disk. If $r^*(z)$ is the best approximation of type (m, n) to $f(z)$ on $|z| \leqq 1$, it turns out that the error curve $(f-r^*)(|z|=1)$ often approximates extremely closely a perfect circle about the origin of winding number $m+n+1$. This phenomenon for polynomial approximation was discussed by Trefethen in [40], where by an analysis based on the Carathéodory–Fejér theorem [2], [5], it was shown that in best approximation on the disk $|z| \leqq \varepsilon$, the error curve is circular to $O(\varepsilon^{2m+3})$ as $\varepsilon \to 0$. By means of an extension of the CF theorem due originally to Takagi [19], [37] and generalized by Adamjan, Arov, and Krein [1], this result was extended to $O(\varepsilon^{2m+2n+3})$ for rational approximation in [41]. At the same time Gutknecht found that the CF technique could be transplanted by the Joukowski map $x=\frac{1}{2}(z+z^{-1})$ from $|z|=1$ to $x \in [-1, 1]$, and the resulting real CF method was analyzed for polynomial approximation in [21]. The present paper completes this series by presenting and analyzing asymptotically a CF method for real, rational approximation. However, this paper can be read independently.

The Joukowski transplantation has previously been applied for near-best real approximation by Frankel, Gragg, and Johnson [14], [17], who derived a Chebyshev–Padé approximation on $[-1, 1]$ based on Padé expansions at $x=0$. This Chebyshev–Padé approximation is related to, but not the same as, the earlier rational economization fraction of Maehly [6, p. 178], [17]. Our fraction might be called the *Chebyshev–CF* approximant, for it fits directly into the framework of Gragg and his colleagues. Indeed, corresponding to their Fourier–Padé and Laurent–Padé approximations, one can develop Fourier–CF and Laurent–CF approximations for real periodic and complex meromorphic functions, respectively [19]. In general the CF approximations will be more complicated but, for smooth functions, much closer to optimal.

In the area of real rational approximation, various ideas have appeared previously that are related to the CF method. Eigenvalues of Hankel matrices were used half a century ago for estimating the error of the best approximation and for solving certain special problems exactly by Bernstein, Achieser, and Mirakyan; see [31, p. 166] and [2, App. D] for references. The use of such a device for near-best approximation was apparently first proposed by Darlington in 1970 [10] for the real polynomial case, and the first (and only previous) extension to rational near-best approximation is due to Lam and D. Elliott in 1972 [12], [27], [28]. The connection between the CF method and approximation on the disk was first pointed out in the excellent dissertation of Hollenhorst [25] (for the polynomial case), and this was also the first work to contain error estimates. Further related contributions have also been made by C. Clenshaw, G. H. Elliott [13], A. Reddy, and A. Talbot [38], [39].

One of our own contributions in previous papers [21], [40], [41] has been to connect CF methods with the Carathéodory–Fejér theory and the related results of

Takagi and Adamjan, Arov, and Krein. This makes it possible to fill various theoretical gaps. A second innovation in our papers has been that by means of arguments related to strong uniqueness, we apply results of the CF method to get estimates on the behavior of the best approximation itself. In our view, the CF idea is not just a method for generating near-best approximations, but a theory that should reveal hitherto unrecognized properties of real and complex Chebyshev approximation and of the relation between the two.

As mentioned above, the present paper also differs from the work of Lam and Elliott in two practical ways. First, our method applies for arbitrary m and n, rather than just $m \geqq n$. The idea behind this extension is derived from an example by Talbot [38]. Second, its asymptotic order of accuracy on small intervals is $O(\varepsilon^{3m+2n+3})$ rather than $O(\varepsilon^{2m+2n+3})$. This is more than just an improvement in degree, for there appears to be a maximum possible order of accuracy that CF methods can attain, due to an intrinsic limitation on how regular the behavior of best approximation error curves is. Whereas the Lam and Elliott method is not optimal in this respect, we believe that ours probably is.

Our arguments proceed in two steps. First, one shows that the CF method yields an error curve that nearly equioscillates; this implies $E^{cf} \approx E^*$. Second, one shows by an argument related to strong uniqueness or Lipschitz continuity of best approximations that this behavior further implies $R^{cf} \approx R^*$. All of our estimates are asymptotic, pertaining only to the interval $[-\varepsilon, \varepsilon]$ in the limit $\varepsilon \to 0$. (Equivalently, one could consider increasingly smooth functions $F(\varepsilon x)$ on the fixed interval $[-1, 1]$.) This is effectively the same limit considered in the past in various papers on Chebyshev approximation, notably [29], [34] for real polynomial, [7], [30], [43] for real rational, [32] for complex polynomial, and [42] for complex rational approximation. Where these papers obtain one term of an asymptotic expansion of the best approximation (two, in the case of [34]), the CF method gets many.

Ideas related to the CF method are currently attracting much attention in the theories of digital filtering, control, and linear systems. This work has been mainly stimulated by the paper of Adamjan, Arov, and Krein [1], and is being carried out by (among others) M. Bettayeb, A. Bultheel, P. Dewilde, Y. Genin, S. Kung, and L. Silverman. See the book by Kailath [26] for some references, and also [20].

1. The Carathéodory–Fejér method. Let the unit disk and circle be denoted $D = \{z : |z| < 1\}$ and $\partial D = \{z : |z| = 1\}$, let ∂D^+ be the upper semicircle $\partial D \cap \{z : \operatorname{Im} z \geqq 0\}$, and let $I_\varepsilon = [-\varepsilon, \varepsilon]$. Let $\|\cdot\|_{I_\varepsilon}$, $\|\cdot\|_{\partial D}$, etc. be the corresponding supremum norms, but let $\|\cdot\|$ be an abbreviation for $\|\cdot\|_{I_\varepsilon}$. In what follows x will always denote a real and z a complex variable; upper case letters will be used for functions of x and lower case for functions of z.

We begin with a real function $F(x)$ that is continuous on I_ε and with a pair of fixed integers $m, n \geqq 0$. Let V_{mn} be the set of rational functions of type (m, n) with real coefficients, and let $R^*(x)$ denote the best approximation to F on I_ε out of V_{mn}. (R^* exists and is unique; see [2], [6], or [31].) For any finite $M \geqq 0$, F possesses a partial Chebyshev expansion

$$F(x) = F_M(x) + G_M(x) = \sum_{k=0}^{M}{}' a_k T_k(x/\varepsilon) + G_M(x), \tag{1.1}$$

with T_k denoting the kth Chebyshev polynomial, where the prime indicates that the term with $k = 0$ should be multiplied by $\frac{1}{2}$. Here a_k is defined by the inner product

[6, p. 117]

$$a_k = \frac{2}{\pi}\int_{-\varepsilon}^{\varepsilon} F(x)T_k(x/\varepsilon)\frac{dx}{\sqrt{\varepsilon^2-x^2}}.$$

Our fundamental transplantation is the map

(1.2) $$x(z) = \varepsilon \operatorname{Re} z = \tfrac{1}{2}\varepsilon(z+z^{-1}),$$

a bijection of ∂D^+ onto I_ε, which for $x \in I_\varepsilon$, $z \in \partial D$ leads to the formula

(1.3) $$T_k(x/\varepsilon) = \tfrac{1}{2}(z^k + z^{-k}).$$

In particular, let us set $a_{-k} = a_k$ and define

(1.4a) $$f_M(z) = \sum_{k=-M}^{M} a_k z^k, \qquad f^+(z) = \sum_{k=m-n+1}^{M} a_k z^k,$$

(1.4b) $$f^0(z) = \begin{cases} \sum_{k=n-m}^{m-n} a_k z^k & \text{if } m \geqq n, \\ -\sum_{k=m-n+1}^{n-m-1} a_k z^k & \text{if } m < n. \end{cases}$$

Then

(1.5) $$F_M(x) = \tfrac{1}{2}f_M(z) = \tfrac{1}{2}[f^+(z) + f^+(z^{-1}) + f^0(z)].$$

The idea of the CF method is to first approximate the analytic function f^+ on ∂D by considering an infinite-dimensional space $\tilde{V}_{mn}$ in which a best approximation can be found exactly, then derive from this a near-best approximation R^{cf} to F on I_ε. Let $\tilde{V}_{mn}$ be the set of functions that can be written with real coefficients in the form

$$\tilde{r}(z) = \sum_{k=-\infty}^{m} d_k z^k \Big/ \sum_{k=0}^{n} e_k z^k,$$

where the terms of negative degree in the numerator converge to a bounded analytic function in $|z| > 1$ and the denominator has no zeros in $D \cup \partial D$. Let H be the real symmetric Hankel matrix

$$H = \begin{pmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_M \\ a_{m-n+2} & & ⋰ & \\ \vdots & ⋰ & & 0 \\ a_M & & & \end{pmatrix}.$$

(A *Hankel matrix* is a matrix with $a_{ij} = a_{|i+j|}$.) Let

$$H = U\Lambda U^T$$

be a real orthogonal eigenvalue decomposition of H—i.e., U, Λ are square real matrices with $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_{M+n-m})$ and U is orthogonal ($U^T = U^{-1}$). We assume the eigenvalues are ordered by absolute magnitude: $|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_{M+n-m}|$. (If H had complex coefficients it would become clear that a singular value decomposition rather than an eigenvalue decomposition is most appropriate, but since H is real symmetric, the two are the same here except for the possibility of negative signs.) Let λ abbreviate λ_{n+1} and let $(u_1, \cdots, u_{M+n-m})^T$ be the corresponding right eigenvector, namely the $(n+1)$st column of U. The following result was proved by Carathéodory and Fejér in 1911 for the polynomial case $n = 0$ [5], extended to rational approximation by Takagi in 1924 [37], and generalized further by Adamjan, Arov, and Krein in 1971 [1]. A presentation and partial proof can be found in [41]. A full discussion of degenerate cases will be given in [19].

THEOREM 1. *f^+ has a unique best approximation $\tilde{r}^*$ on ∂D out of $\tilde{V}_{mn}$, which is given by*

$$f^+ - \tilde{r}^* = b, \tag{1.6a}$$

where

$$b(z) = \lambda z^M \frac{u_1 + \cdots + u_{M+n-m} z^{M+n-m-1}}{u_{M+n-m} + \cdots + u_1 z^{M+n-m-1}}. \tag{1.6b}$$

The error is

$$\|f^+ - \tilde{r}^*\|_{\partial D} = |\lambda|,$$

and the error curve $(f^+ - \tilde{r}^*)(\partial D)$ is a perfect circle about the origin whose winding number is $m+n+1$ if $|\lambda_n| > |\lambda| > |\lambda_{n+2}|$.

The function b is λ times a quotient of finite Blaschke products, which is why $f^+ - \tilde{r}^*$ maps ∂D onto a circle, and the optimality of $\tilde{r}^*$ for the complex approximation problem can be seen to follow from this by Rouché's theorem. Now this optimality is not of use to us. However, let us transplant to I_ε by defining

$$\tilde{R}(x) = \tfrac{1}{2}[\tilde{r}^*(z) + \tilde{r}^*(z^{-1}) + f^0(z)], \tag{1.7a}$$

or by (1.5) and (1.6a),

$$\tilde{R}(x) = \tfrac{1}{2}[f_M(z) - b(z) - b(z^{-1})]. \tag{1.7b}$$

Then by (1.5) again,

$$F_M(x) - \tilde{R}(x) = \operatorname{Re} b(z), \tag{1.8}$$

and if b has winding number $m+n+1$, it follows that $F_M - \tilde{R}$ equioscillates on I_ε at $m+n+2$ points $\varepsilon = x_0 > x_1 > \cdots > x_{m+n+1} = -\varepsilon$:

$$\|F_M - \tilde{R}\|_{I_\varepsilon} = |\lambda|, \qquad (F_M - \tilde{R})(x_j) = (-1)^j \lambda. \tag{1.9}$$

If $\tilde{R}$ belonged to V_{mn}, this equioscillation would imply $\tilde{R} = R^*$ and $|\lambda| = E^*$, and we would have solved our original approximation problem (for F_M) exactly. Unfortunately, this is in general not the case. (The main exception occurs when $M = m+1$, and this gives rise to some of the examples of Achieser, Talbot, and others mentioned in the Introduction.) But the key to the CF method is that for smooth functions F, $\tilde{R}$ turns out be very close to V_{mn}.

Let $q = \sum_{k=0}^{n} \tilde{e}_k^* z^k$ denote the normalized denominator of $\tilde{r}^*$—the polynomial of degree $\partial q \leqq n$ with constant term $\tilde{e}_0^* = 1$ whose zeros are the finite poles of $\tilde{r}^*$ lying outside ∂D. Define

$$Q(x) = q(z) q(z^{-1})/\tau, \tag{1.10}$$

where τ is the scalar $q(i)q(-i)$, inserted to make $Q(x)$ have constant term 1. Note that $Q(x) = |q(z)|^2/\tau > 0$ on I_ε. Now since $\tilde{r}^* \in \tilde{V}_{mn}$,

$$f^+(z) - b(z) = \tilde{r}^*(z) = O(z^{m-\partial q}) \quad \text{as } z \to \infty,$$

hence since $\partial q \leqq n$ and $f_M - f^+ = O(z^{m-n})$,

$$f_M(z) - b(z) = O(z^{m-\partial q}) \quad \text{as } z \to \infty. \tag{1.11}$$

Let us consider the Laurent series with respect to ∂D of the product

$$\tilde{R}(x)Q(x)=\tfrac{1}{2}[f_M(z)-b(z)-b(z^{-1})]q(z)q(z^{-1})/\tau. \tag{1.12}$$

By (1.11) and the definition of q, $[f_M(z)-b(z)]q(z)q(z^{-1})/\tau$ must be analytic outside ∂D except for a pole of order at most m at ∞, and therefore all terms of order greater than m in the Laurent series of (1.12) are due to $b(z^{-1})q(z)q(z^{-1})/\tau$. By symmetry, all terms of order less than $-m$ are due to $b(z)q(z)q(z^{-1})/\tau$. Hence, if we define

$$\hat{b}(z)=b(z)q(z)q(z^{-1})/\tau=\sum_{k=-\infty}^{M+n}\hat{b}_k z^k, \tag{1.13a}$$

$$\hat{b}^T=\sum_{k=-m}^{M+n}\hat{b}_k z^k, \qquad \hat{b}^R(z)=\hat{b}(z)-\hat{b}^T(z), \tag{1.13b}$$

then the function

$$P_1(x)=\tfrac{1}{2}[f_M(z)q(z)q(z^{-1})/\tau-\hat{b}^T(z)-\hat{b}^T(z^{-1})]$$

is a polynomial of degree m in x. If we further set

$$R_1^{cf}(x)=\frac{P_1(x)}{Q(x)}, \qquad B^R(x)=\frac{1}{2}[\hat{b}^R(z)+\hat{b}^R(z^{-1})], \tag{1.14a}$$

then we obtain

$$\tilde{R}(x)=R_1^{cf}(x)+\frac{B^R(x)}{Q(x)}, \tag{1.14b}$$

hence

$$\tilde{R}(x)Q(x)=R_1^{cf}Q(x)+O(T_{m+1}(x)). \tag{1.15}$$

We will call $R_1^{cf}\in V_{mn}$ the *type* 1 or *Maehly type* CF approximation of F, because as in Maehly's generalization of Padé approximation (cf. [6, p. 118] and [17]), truncation of higher-order terms in $\tilde{R}$ is done after multiplying through by the denominator Q. There is a second, probably superior way to truncate $\tilde{R}$, namely by using a Chebyshev–Padé kind of approximation with fixed denominator Q. That is, one may take the *type* 2 or *Gragg type* CF approximant as

$$R_2^{cf}(x)=\frac{P_2(x)}{Q(x)}, \tag{1.16a}$$

with P_2 defined by the condition

$$\tilde{R}(x)=R_2^{cf}(x)+O(T_{m+1}(x)). \tag{1.16b}$$

One could go further, in complete analogy with the Chebyshev–Padé approximation defined by Gr

One could go further, in complete analogy with the Chebyshev–Padé approximation defined by Gragg, and define a third type of CF approximation by permitting the denominator of R^{cf} to be free as well as the numerator. However, one might then end up with a fraction having a pole on I_ε. For this reason, and on the basis of numerical experiments and the analogy with the Chebyshev–Padé situation, we believe that R_2^{cf} is the best of these three possibilities, and from now on we will drop the subscripts and assume $R^{cf}\equiv R_2^{cf}$.

To obtain the polynomial P ($\equiv P_2$) satisfying (1.16), one proceeds as follows. Let $\tilde{c}_k$, β_k, and γ_k denote the kth Chebyshev coefficients of $\tilde{R}(x)$, $P(x)$, and $1/Q(x)$, respectively. Then the coefficients $\{\beta_k\}$ satisfy the Toeplitz system of equations

$$(1.17)\qquad \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{2m} \\ \gamma_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma_1 \\ \gamma_{2m} & \cdots & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix}\beta_m\\ \vdots \\ \beta_0 \\ \vdots \\ \beta_m\end{pmatrix} = 2 \begin{pmatrix}\tilde{c}_m \\ \vdots \\ \tilde{c}_0\\ \vdots\\ \tilde{c}_m\end{pmatrix}$$

Since $1/Q(x)>0$ on I_ε, the infinite symmetric Toeplitz matrix $(\gamma_{|i-j|})_{i,j=-\infty}^{\infty}$ is known to be positive definite [18], and hence the principal submatrix appearing in (1.17) is positive definite also, hence nonsingular. Moreover, since this submatrix is symmetric about its anti-diagonal, a symmetric right-hand-side $(\tilde{c}_k)$ leads to a symmetric solution (β_k), as indicated in (1.17). Indeed, in practice one may reduce (1.17) to a system of size $m+1$ instead of $2m+1$. Consequently, P is always well defined by (1.16).

In summary, here is the real rational (type 2) CF approximation method. We have indicated four points at which a numerical implementation can naturally be based upon the Fast Fourier Transform. For further information on uses of the FFT in complex analysis, see [22]. The FFT method indicated for the polynomial factorization of Step 4 is that proposed in [22, § 3.2]; see also [41].

Step 1. Given F, find its Chebyshev coefficients $a_0, \cdots, a_M$ for some large M (FFT).

Step 2. Construct the Hankel matrix H and find its $(n+1)$st eigenvalue (in absolute value) and eigenvector.

Step 3. Find the Laurent series on the circle of the rational function $b(z)$ defined by (1.6b) (FFT). Subtract this plus its conjugate from $f_M(z)$ to obtain the Chebyshev coefficients $\{\tilde{c}_k\}$ for $\tilde{R}(x)$ by (1.7b).

Step 4. Factor the denominator of (1.6b) to obtain the polynomial $q(z)$ and construct $Q(x)$ from (1.10) (FFT). Find the Chebyshev coefficients $\{\gamma_k\}$ for $1/Q(x)$ (FFT).

Step 5. Determine the polynomial $P(x)$ satisfying (1.16) by solving (1.17), and define $R^{cf}=P/Q$.

Step 6. To get a bound on $E^{cf}-E^*$, examine how close the error curve of R^{cf} comes to equioscillating.

Remark. This somewhat obscure construction of R^{cf} can be made much more transparent in the case $m \geqq n$. The theory of complex CF approximation shows that $\tilde{r}^*$ in Theorem 1 is close to V_{mn}, and for $m \geqq n$, (1.7a) then implies that $\tilde{R}$ is close to V_{mn} also. See [46].

2. Asymptotic results for small intervals. The basis of our results for small intervals is the theory worked out in [41] for complex approximation on small disks. For these results a normality assumption is needed. Let F be given, and let $M \geqq 3m+2n+2$ be a fixed integer.

ASSUMPTION A. *The Mth derivative of $F(x)$ exists and is Lipschitz continuous at $x=0$. Moreover, if $F(x)=\sum_{k=0}^{M}\alpha_k x^k+O(x^{M+1})$, with $\alpha_k\equiv 0$ for $k<0$, then*

$$\det\begin{pmatrix} \alpha_{m-n+1} & \alpha_{m-n+2} & \cdots & \alpha_m \\ \alpha_{m-n+2} & & & \vdots \\ \vdots & & & \alpha_{m+n-2} \\ \alpha_m & \cdots & \alpha_{m+n-2} & \alpha_{m+n-1} \end{pmatrix} \neq 0.$$

Equivalently, the Padé approximation of F of type (m, n) *has a full n finite poles—see* [23, *Thms.* 7.5e–f] *and* [16, § 3].

The nonvanishing of this Hankel determinant is a standard assumption that appears also, for example, in [42], [43], and [31, p. 170]. For many functions, including e^x, it is satisfied for all (m, n).

Here is the main result from [41] that we require:

THEOREM 2. *Let F satisfy Assumption* A, *and for each* $\varepsilon > 0$, *let* $\tilde{r}^*$ *be the extended best approximation in* $\tilde{V}_{mn}$ *of the function* f^+ *defined by* (1.4a). *Then for all sufficiently small* ε, $b = f^+ - \tilde{r}^*$ *has winding number exactly* $m+n+1$ *on* ∂D, *it approximates a monomial according to*

$$(f^+ - \tilde{r}^*)(z) = \lambda z^{m+n+1}(1+O(\varepsilon)) \tag{2.1}$$

uniformly on ∂D, *and its Laurent coefficients on* ∂D *satisfy*

$$b_k = [O(\varepsilon)]^{2m+2n+2-k} \quad \forall k \leqq m+n+1 \tag{2.2a}$$

uniformly in k. In addition, the coefficients of the denominator q of $\tilde{r}^*$ *satisfy*

$$\tilde{e}_k^* = O(\varepsilon^k) \qquad (0 \leqq k \leqq n). \tag{2.2b}$$

Proof. It can be seen that as $\varepsilon \to 0$, $\{a_k\}$ and $\{\alpha_k\}$ are related by

$$a_k = \alpha_{|k|} 2^{1-|k|} \varepsilon^{|k|} + O(\varepsilon^{|k|+1}), \qquad -M \leqq k \leqq M$$

(cf. [21, Lemma 3.3]), hence since $\alpha_k = 0$ for $k < 0$,

$$a_k = \alpha_k 2^{1-k} \varepsilon^k + O(\varepsilon^{k+1}), \qquad -M \leqq k \leqq M. \tag{2.3}$$

Now if the $O(\varepsilon^{k+1})$ term were zero, the extended approximation problem for f^+ would be that of approximating $\sum_{k=0}^{M} \alpha_k 2^{1-k} (\varepsilon z)^k$, and for this the theory of [41] applies. From [41, Lemmas 4.1, 4.3 and 4.4] one would obtain (2.2b), (2.2a), and (2.1), respectively, on the basis of Assumption A. In fact the term $O(\varepsilon^{k+1})$ is not zero, but it is of size $O(\varepsilon)$ relative to the terms just considered. This is enough to make the arguments of [41] still go through; we omit the details. In particular, Assumption A and (2.3) imply that for all sufficiently small ε, the corresponding Hankel matrix made up of coefficients a_k will also have nonzero determinant. □

We will need a lemma on the behavior of the denominators of R^{cf} and R^*. Let us write

$$R^{cf} = P/Q, \qquad R^* = P^*/Q^*,$$

where P and P^* are polynomials of degree at most m, and Q and Q^* are polynomials of degree at most n with constant terms 1.

LEMMA 3 (cf. [41, Lemma 6.1]). *As* $\varepsilon \to 0$,

$$Q = 1 + O(\varepsilon), \tag{2.4a}$$

$$Q^* = 1 + O(\varepsilon), \tag{2.4b}$$

and hence

$$QQ^* = 1 + O(\varepsilon) \tag{2.4c}$$

uniformly on I_ε.

Proof. Equation (2.4a) follows from (2.2b). Equation (2.4b) is a corollary of the known fact [7], [43] that as $\varepsilon \to 0$, R^* approaches coefficientwise the Padé approximant $R^p \in V_{mn}$, whose normalized denominator, having coefficients independent of ε, obviously satisfies $Q^p = 1 + O(\varepsilon)$ uniformly on I_ε as $\varepsilon \to 0$. (In fact [7] and [43] show

$R^* \to R^p$ for $[0, \varepsilon]$, not $[-\varepsilon, \varepsilon]$. However, C. Chui of [7] assures us that the result also holds for the latter problem (private communication, September 1981).) □

We now show that from Theorem 2 it follows that $F - R^{cf}$ equioscillates at $m+n+2$ points on I_ε to within $O(\varepsilon^{3m+2n+3})$. Let $\{z_j^T\}$ be the set of $(2m+2n+2)$th roots of unity on ∂D^+, and let $\{x_j^T\}$ be the corresponding set of Chebyshev abscissae for I_ε, $x_j^T = \varepsilon \operatorname{Re} z_j^T$.

THEOREM 4. *Let F satisfy Assumption* A, *and for each $\varepsilon > 0$, let R^{cf} be its* CF *approximation out of V_{mn}. Then for all sufficiently small ε, there is a set of points $\varepsilon = x_0 > x_1 > \cdots > x_{m+n+1} = -\varepsilon$ satisfying*

$$x_j = x_j^T(1 + O(\varepsilon)) \tag{2.5}$$

at which

$$\|F - R^{cf}\| - \operatorname{sgn} \lambda (-1)^j (F - R^{cf})(x_j) = O(\varepsilon^{3m+2n+3}). \tag{2.6}$$

Proof. Let $\{z_j\}$ be the set of $m+n+2$ points on ∂D^+ at which $f^+ - \tilde{r}^*$ is real, numbered in counterclockwise order from $z_0 = 1$ to $z_{m+n+1} = -1$, and take $x_j = \varepsilon \operatorname{Re} z_j$. The bound (2.5) follows from (2.1). By (1.9), $F_M - \tilde{R}$ exactly equioscillates on the set $\{x_j\}$ with error $|\lambda|$, so to prove (2.6), it is enough to show

$$\|F - F_M\| = O(\varepsilon^{3m+2n+3}) \tag{2.7}$$

and

$$\|\tilde{R} - R^{cf}\| = O(\varepsilon^{3m+2n+3}). \tag{2.8}$$

The first bound follows from the Lipschitz continuity statement of Assumption A. To establish (2.8), we observe that from (1.13a) and (2.2a, b), we have

$$\hat{b}_k = [O(\varepsilon)]^{2m+2n+2-k} \quad \forall k \leqq m+1.$$

This implies by (1.13b),

$$\hat{b}^R = O(\varepsilon^{3m+2n+3}).$$

From this and (1.14) and (2.4a) it follows that the analogue of (2.8) holds for type 1 CF approximation:

$$\|\tilde{R} - R_1^{cf}\| = O(\varepsilon^{3m+2n+3}). \tag{2.9}$$

Now $\hat{b}^R$ has terms only of degree $\leqq -m-1$, each of order $O(\varepsilon^{3m+2n+3})$, and from (1.10) and (2.2b) it is straightforward to see further that the Chebyshev coefficients of $1/Q(x)$ satisfy

$$|\gamma_k| = [O(\varepsilon)]^{|k|} \quad \forall k.$$

From these facts and (1.14) it follows that the degree-m part of the Chebyshev series of $\tilde{R} - R_1^{cf}$ has magnitude $O(\varepsilon^{3m+2n+4})$. In other words, if the β-coefficient vector for P_1 is inserted in the left hand side of (1.17), then that system is satisfied up to an error of magnitude $O(\varepsilon^{3m+2n+4})$. But as $\varepsilon \to 0$, the matrix in (1.17) approaches the identity, so it follows that the coefficients of P_1 agree with those of P up to $O(\varepsilon^{3m+2n+4})$. By a final application of (2.4a), this implies

$$\|R_1^{cf} - R^{cf}\| = O(\varepsilon^{3m+2n+4})$$

and with (2.9) this yields (2.8). □

Thus the error curve of R^{cf} equioscillates up to $O(\varepsilon^{3m+2n+3})$. By the de la Vallée Poussin theorem for rational approximation [31, Thm. 98], this implies

COROLLARY. *As* $\varepsilon \to 0$

$$\|F-R^{cf}\|-\|F-R^*\|=O(\varepsilon^{3m+2n+3})$$

and

$$|\lambda|-\|F-R^*\|=O(\varepsilon^{3m+2n+3}).$$

We wish to show further that $\|R^{cf}-R^*\|=O(\varepsilon^{3m+2n+3})$. Now by definition $\|F-R^*\|\leqq\|F-R^{cf}\|$, so (2.6) implies that at the $m+n+2$ points $\{x_j\}$, $R^{cf}-R^*$ must satisfy an alternating sequence of constraints

$$-\operatorname{sgn}\lambda\,(-1)^j(R^{cf}-R^*)(x_j)\leqq\eta, \tag{2.10}$$

where $\eta=O(\varepsilon^{3m+2n+3})$. Does this imply that $R^{cf}-R^*$ is small? In fact it does, and this question was taken up directly in the paper on rational approximation on small intervals by Maehly and Witzgall [30, Lemma 4.6]. However, for our needs (2.10) can be reduced to a similar set of constraints on a polynomial instead of a rational function, which will be easier to deal with. Let us write

$$R^{cf}-R^*=\frac{P}{Q}-\frac{P^*}{Q^*}=\frac{PQ^*-P^*Q}{QQ^*},$$

and let S denote PQ^*-P^*Q, a polynomial of degree at most $m+n$. Then by (2.4c),

$$R^{cf}-R^*=S(1+O(\varepsilon)) \tag{2.11}$$

uniformly on I_ε as $\varepsilon\to 0$. Therefore (2.10) leads to the sequence of $m+n+2$ constraints

$$-\operatorname{sgn}\lambda\,(-1)^jS(x_j)\leqq\eta \tag{2.12}$$

for some new $\eta=O(\varepsilon^{3m+2n+3})$. We want to deduce that $\|S\|=O(\varepsilon^{3m+2n+3})$.

This is a commonly occurring problem in approximation theoretic proofs. If $\{x_j\}$ were a fixed set of points (i.e., not dependent on ε), then the argument that would be required is the key step in proofs of strong uniqueness or Lipschitz continuity for polynomial Chebyshev approximation. Essentially the same reasoning for this has appeared in (at least) papers of Freud [15], Maehly and Witzgall [29], and Cline [8]; Maehly and Witzgall even give a figure illustrating (2.12) graphically. For a general discussion see [6]. In our application the near-alternation points are not fixed, but by (2.5) they are close to Chebyshev abscissae for small ε, hence uniformly separated from each other. This uniform separation is what is needed to make the argument go through, and the same is the case for applications to strong uniqueness and Lipschitz continuity [11], [24].

LEMMA 5 (cf. [21, Thm. 2.1]). *Let $S(x)$ be a real polynomial of degree at most μ on I_ε. Suppose there exist $\mu+2$ points $\varepsilon\geqq x_0>x_1>\cdots>x_{\mu+1}\geqq-\varepsilon$ at which*

$$(-1)^jS(x_j)\leqq\eta \tag{2.13}$$

for some $\eta\geqq 0$, and suppose that $x_j=\varepsilon\cos\phi_j$ with

$$\left|\phi_j-\frac{j\pi}{\mu+1}\right|\leqq\frac{\delta}{\mu(2\mu+1)} \tag{2.14}$$

for some $\delta<1$. Then

$$\|S\|\leqq\frac{(2\mu+1)}{1-\delta}\eta. \tag{2.15}$$

Proof. $S(\varepsilon \cos \phi)$ is a trigonometric polynomial of degree μ in ϕ, so by Bernstein's inequality [6, p. 91], one has

$$\left|\frac{dS(\varepsilon \cos \phi)}{d\phi}\right| \leqq \mu \|S\|.$$

If $\{x_j^T\}$ are the Chebyshev points $x_j^T = \varepsilon \cos (j\pi/(\mu+1))$, then this bound together with (2.13) and (2.14) implies

$$(-1)^j S(x_j^T) \leqq \kappa \tag{2.16}$$

where

$$\kappa = \eta + \frac{\delta\|S\|}{2\mu+1}.$$

Now according to a computation of Cline in [8, § 4], (2.16) implies

$$\|S\| \leqq (2\mu+1)\kappa. \tag{2.17}$$

Therefore

$$\|S\| \leqq (2\mu+1)\left(\eta + \frac{\delta\|S\|}{2\mu+1}\right) = (2\mu+1)\eta + \delta\|S\|,$$

hence (2.15). □

Lemma 5 provides all that is needed to prove our first main theorem:

THEOREM 6. *Let F satisfy Assumption* A, *and for each* $\varepsilon > 0$ *let* R^{cf} *and* R^* *be its CF and Chebyshev approximations in* V_{mn}. *Then as* $\varepsilon \to 0$,

$$\|R^{cf} - R^*\| = O(\varepsilon^{3m+2n+3}).$$

Proof. Applying Lemma 5 to (2.12) with $\mu = m+n$ gives

$$\|S\| = O(\varepsilon^{3m+2n+3}).$$

The result follows from (2.11). □

Together with (2.8) Theorem 6 implies

$$\|R^* - \tilde{R}\| = O(\varepsilon^{3m+2n+3}). \tag{2.18}$$

We can interpret this as a statement about the geometry of optimal error curves, analogous to the theorems in [40] and [41] showing that error curves in complex Chebyshev approximation are close to perfect circles:

THEOREM 7. *Let F satisfy Assumption* A . *Then for all sufficiently small* ε, *there exists a rational function* $b(z)$ *that is analytic in* $1 \leqq |z| < \infty$ *except for at most n poles and has constant modulus and winding number* $m+n+1$ *on* ∂D, *satisfying*

$$\|(F-R^*)(x) - \operatorname{Re} b(z)\| = O(\varepsilon^{3m+2n+3}).$$

Proof. Follows from (1.8), (2.7), and (2.18) □

Theorems 6 and 7 are the extensions to rational approximation of results given in Theorem 3.4 of [21]. Theorem 3.5 of that paper also proves analogous estimates for $m \to \infty$ on a fixed interval, but we have not extended these results.

If F satisfies Assumption A, then $\|F - R^*\|$ has size $O(\varepsilon^{m+n+1})$ but not $o(\varepsilon^{m+n+1})$ as $\varepsilon \to 0$. Relative to this scale, therefore, our results have strength $O(\varepsilon^{2m+n+2})$. It appears that these orders are best possible, except that in the case $n = 0$, a certain "bonus" cancellation makes it possible to increase $3m+3$ to $3m+4$ in Theorems 6 and 7 (but not in the estimate on $|\lambda|$ in the corollary to Theorem 4). See the proof of [21, Lemma 3.2] for details.

3. Numerical examples. The CF method is not difficult to implement numerically, and the techniques we have used are described in [41, § 7]. In outline, we rely heavily on the fast Fourier transform as indicated here at the end of § 1, and use EISPACK routines based on Sturm sequencing for the eigenanalysis of Step 2. The bottleneck is the eigenvalue computation, which takes time $O(M^3)$, for unfortunately no way is known to take advantage of the Hankel structure of H. Ideally one wants M to be large enough so that the Chebyshev coefficients a_k for $k > M$ are negligible, hence $F_M \approx F$. For most of the examples considered below, such as those involving e^x, this is achieved with $M = 35$, leading to computation times on the order of 0.1 sec on our IBM 370/168. With $F(x) = |x|$, on the other hand, $M = 120$ is only barely large enough to get approximations with $m, n \leqq 2$ accurately, and the computation time increases to 2 sec. Thus the CF method is not only more accurate, but also much faster if the function to be approximated is smooth. For certain high-precision numbers below we have resorted to quadruple precision.

In general the CF method will yield an approximation satisfying

$$|\lambda| \approx E^*,$$

and by the de la Vallée Poussin result one has

$$E^{cf}_{\min} \leqq E^* \leqq E^{cf}, \tag{3.1}$$

where

$$E^{cf}_{\min} = \min_j |(F - R^{cf})(x_j)|,$$

with the minimum taken over that set of $m+n+2$ nearly alternating points which maximizes its value. In our experiments R^{cf} and λ were computed to close to machine precision, and by means of a minimization routine (FMIN, by Richard Brent), E^{cf} and $E^{cf}_{\min}$ were also found to this accuracy (Step 6). The quantity not so precisely known is E^*, for we do not have a high-accuracy rational Chebyshev approximation routine at hand. Therefore in what follows we report $|\lambda|$ rather than E^*.

As a first example, let us give more details related to e^x. Table 2 shows the eigenvalue $|\lambda|$ in approximation on $[-1, 1]$ for $0 \leqq m, n \leqq 3$. Each digit known to agree with the corresponding digit of E^* after both are rounded has been underlined. In most cases this knowledge is based on (3.1). The agreement is excellent, and we believe that it will get steadily better as $m, n \to \infty$ in any fashion. (The table leaves some doubt as to whether this is true for, say, $m = 0$ and $n \to \infty$, but further experiments

TABLE 2.
The eigenvalue $|\lambda|$ in CF approximation to e^x on $[-1, 1]$ for various (m, n). Underlined digits are known to agree with corresponding digits of E^*, after rounding. Doubly underlined digits agree with the conjectured limit formula (3.2) of Meinardus.

	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	1.1961 (−0)	2.787994 (−1)	4.5017 38776 (−2)	5.5283 70108 71194 (−3)
$n=1$	2.1724 (−1)	2.096982 (−2)	1.7890 66755 (−3)	1.3461 23369 20018 (−4)
$n=2$	3.5288 (−2)	1.677017 (−3)	8.6899 91075 (−5)	4.3991 63371 96896 (−6)
$n=3$	4.5235 (−3)	1.239861 (−4)	4.2766 46704 (−6)	1.5506 69053 97117 (−7)

show that by type (0, 9), five significant digits can be underlined, and the number is growing with n.) This example relates to a conjecture of Meinardus [31, p. 168] which proposes that as $m, n \to \infty$,

$$E^*_{mn}(e^x) = \frac{m!n!}{2^{m+n}(m+n)!(m+n+1)!}(1+o(1)). \tag{3.2}$$

At this writing the conjecture is unproved, but it is known to be valid up to a constant factor [4], [33] of less than 40. The double underlinings in Table 2 mark digits where (3.2) agrees with $|\lambda|$, and it is evident that $|\lambda|$ is much closer to E^* than (3.2) is. This suggests that it might be possible to resolve the Meinardus conjecture by means of the CF method. (Chebyshev–Padé approximation, it turns out, is not strong enough [17].) Two key proofs would be enough to settle the issue:

(i) $|\lambda| = E^*_{mn}(1+o(1))$ as $m, n \to \infty$,

(ii) $|\lambda|$ satisfies (3.2) as $m, n \to \infty$.

Unfortunately, these claims are not at all easy to establish, and the Meinardus conjecture will probably we proved true before long by some simpler technique. Nevertheless, it would be very interesting to know that asymptotically, best approximation errors agree with the eigenvalues of an infinite Hankel matrix of Chebyshev coefficients.

Incidentally, it is likely that for the problem of complex approximation of e^z on the unit disk, formula (3.2) holds with the factor 2^{m+n} removed. Saff [44] has established such a result for the limit $n = \text{const}$, $m \to \infty$. The best approximation errors computed in [41, Table 3] by the complex CF method show that the agreement of this conjecture with exact best approximation errors for $m, n \leqq 3$ is about as close as in the real case. The disappearance of the power of 2 is natural in the light of (2.3).

Table 3 shows some results of the CF method for functions besides e^x. In each case $|\lambda|$ is given for approximation of type (0, 1), (1, 1) and (2, 1) on $[-1, 1]$, with underlinings as before. Evidently CF approximation really works, even for as low a degree as (1, 1). It is apparent that it performs relatively poorly for $|x|$, and this is not surprising in view of the nondifferentiability of this function.

TABLE 3.

The eigenvalue $|\lambda|$ in CF approximation of types (0, 1), (1, 1) and (2, 1) to various functions on $[-1, 1]$. Underlined digits are known to agree with corresponding digits of E^*.

$F(x)$	$\|\lambda\|$: (0, 1)	$\|\lambda\|$: (1, 1)	$\|\lambda\|$: (2, 1)
$\|x\|$	4.483 (−1)	4.4827 (−1)	1.1359 (−1)
x^6	5.397 (−1)	5.3970 (−1)	1.9257 (−1)
$\sqrt{1.1-x}$	2.238 (−1)	1.6331 (−2)	2.9709 (−3)
arctan (x)	8.312 (−1)	4.7889 (−2)	4.7889 (−2)
$1/\Gamma(x+1)$	4.041 (−1)	1.1955 (−1)	2.1045 75498 (−2)
e^x	2.172 (−1)	2.0970 (−2)	1.7890 66755 (−3)
$\log(x+3)/2$	1.598 (−1)	8.6079 41336 (−4)	4.9591 1561392 (−5)

Now let us confirm that the asymptotic orders of accuracy predicted in § 2 are valid, and sharp. Table 4 shows $|\lambda|$ and $E^{cf} - E^{cf}_{\min}$ in (1, 1) approximation of e^x on I_ε for $\varepsilon = 1, \frac{1}{2}, \cdots, \frac{1}{16}$. With each factor of 2 reduction in ε, we expect $|\lambda|$ to decrease by approximately $2^{m+n+1} = 8$, and $E^{cf} - E^{cf}_{\min}$ to decrease by approximately $2^{3m+2n+3} = 256$. The table confirms these predictions. Such asymptotic behavior, however, is dependent on the smoothness of F. With $F(x) = |x|$, for example, cutting ε in half is equivalent to dividing F by 2, and therefore the effect will be to cut λ and $E^{cf}-E^{cf}_{\min}$ exactly in half, no matter what m and n are. Thus for $F(x)=|x|$ there is nothing to be gained by shrinking the interval, although the CF method still improves if m and n are increased.

TABLE 4.
Type (1, 1) *approximation of* e^x *on* $[-\varepsilon, \varepsilon]$ *for various* ε.

ε	$\lvert\lambda\rvert$	Ratio	$E^{cf}-E^{cf}_{\min}$	Ratio
1	2.097 (−2)	–	2.03 (−6)	–
$\frac{1}{2}$	2.605 (−3)	8.04	9.18 (−9)	221
$\frac{1}{4}$	3.255 (−4)	8.00	3.73 (−11)	246
$\frac{1}{8}$	4.069 (−5)	8.00	1.47 (−13)	253
$\frac{1}{16}$	5.086 (−6)	8.00	5.77 (−16)	255

Our final example is associated with some further conjectures about asymptotic degree of approximation. In a paper of Cody, Meinardus, and Varga [9], the problem of approximating e^{-t} on the semi-infinite interval $[0, \infty)$ was studied. They proved that in rational approximation of type $(0, n)$ or (n, n), the error decreases geometrically as $n \to \infty$, but did not determine an asymptotic rate of decrease. For approximation of type $(0, n)$, they gave numerical results that suggested the limiting behavior

$$\lim_{n\to\infty} (E^*_{0n})^{1/n} = \tfrac{1}{3}$$

and this equality was later proved valid by A. Schönhage [35]. For approximation of type (n, n), their numerical results reported for $n \leqq 14$ suggest to us

$$\lim_{n\to\infty} (E^*_{nn})^{1/n} = \lim_{n\to\infty} \frac{E^*_{n+1,n+1}}{E^*_{nn}} = \frac{1}{9.28\cdots}, \tag{3.3}$$

and a limit 1/9 has also been conjectured [45]. But no result of this kind has been established.

Despite appearances, this problem can be approached by the CF method. The function

$$t = \frac{1-x}{1+x}$$

maps $[0, \infty]$ bijectively onto $[-1, 1]$, inducing a one-to-one correspondence between rational functions $R(t)$ and $R(x)$ in V_{nn}. Under this transplantation an equioscillating curve on one domain maps to an equioscillating curve on the other, and as a consequence it can be shown that the given problem is equivalent to the problem of approximating

$$F(x) = e^{(x-1)/(x+1)}$$

on $[-1, 1]$, which can be treated by the CF method. (For justification see the theory of [3]; this transplantation only works when $m = n$.) Here F is C^∞ on $[-1, 1]$, but analytic only on $(-1, 1]$, so its Chebyshev series decreases faster than any polynomial but not geometrically. We took $M = 200$, leading to truncated terms $|a_k| < 10^{-23}$ for $k > M$, and calculated eigenvalues in quadruple precision on the IBM 370/168. This gave us λ_{n+1} for $0 \leqq n \leqq 18$ accurate to many places, and for $n \geqq 5$, $|\lambda|$ agrees with the value E^* reported by Cody, Meinardus and Varga to all four places that they give. We have little doubt that $|\lambda| = E^*(1+o(1))$ as $n \to \infty$.

On the basis of these numbers we conjecture

$$\left.\begin{matrix} |\lambda_{n+1}| \\ E^*_{nn} \end{matrix}\right\} = \frac{A}{B^n}\left(1+O\left(\frac{1}{n}\right)\right)$$

for some constants $A \approx .656$, $B \approx 9.28903$. The evidence is summarized in Table 5. Remarkably, in the course of this writing Schönhage [36] has independently conjectured that (3.3) approaches a limit

$$\frac{3}{2}(2-\sqrt{3})^2 \approx \frac{1}{9.28547}.$$

In fact he proves that lim inf $(E^*_{nn})^{1/n}$ is at least two-thirds of this value. The closeness but inequality of these two conjectured limits is striking.

TABLE 5.

Eigenvalues $|\lambda|$ *in* (n, n) *approximation of* $\exp((x-1)/(x+1))$ *on* $[-1, 1]$ *for various n. This is equivalent to approximation of* e^{-t} *on* $[0, \infty)$. *Digits agreeing with the values of* E^* *given by Cody, Meinardus, and Varga* [9] *have been underlined. (The results of* [9] *were given to four places and for* $n \leqq 14$ *only.) The ratio appears to approach a limiting value* 9.28903 . . . .

n	$\|\lambda_{n+1}\|$	Ratio $\rho_n = \|\lambda_n/\lambda_{n+1}\|$	Richardson extrapolant $\rho_n + \frac{1}{3}(\rho_n - \rho_{n/2})$
0	.560172 (−0)		
1	.668057 (−1)	8.38508	
2	.735558 (−2)	9.08232	9.31473
3	.799452 (−3)	9.20078	
4	.865210 (−4)	9.23998	9.29253
5	.934574 (−5)	9.25779	
6	.100845 (−5)	9.26740	9.28961
7	.108750 (−6)	9.27316	
8	.117227 (−7)	9.27689	9.28920
9	.126329 (−8)	9.27944	
10	.136112 (−9)	9.28127	9.28910
11	.146631 (−10)	9.28262	
12	.157946 (−11)	9.28364	9.28905
13	.170119 (−12)	9.28444	
14	.183217 (−13)	9.28507	9.28904
15	.197314 (−14)	9.28558	
16	.212485 (−15)	9.28600	9.28904
17	.228815 (−16)	9.28635	
18	.246392 (−17)	9.28664	9.28903

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