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CIRCULARITY OF THE ERROR CURVE AND SHARPNESS OF THE CF METHOD IN COMPLEX CHEBYSHEV APPROXIMATION*

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Dedicated to Peter Henrici on the occasion of his 60th birthday

Abstract. Let $f(z)$ be analytic at the origin, and for $\varepsilon > 0$, let $f(\varepsilon z)$ be best approximated in the Chebyshev sense on the unit disk by a rational function of type (m, n). It has been shown previously by the CF method that the error curve for this approximation deviates from a circle by at most $O(\varepsilon^{2m+2n+3})$ as $\varepsilon \to 0$. We prove here that this bound is sharp in two senses: the error curve for a given function cannot be asymptotically more circular than the CF method predicts; moreover there exist functions for which the near-circularity is of order $\varepsilon^{2m+2n+3}$ but no smaller.

Key words. Chebyshev approximation, CF method, error curve

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1. Introduction and statement of results. Let S denote the complex unit circle $\{z : |z| = 1\}$, Δ the closed unit disk $\{z : |z| \leqq 1\}$, and $A = A(\Delta)$ the set of functions continuous in Δ and analytic in the interior. Let m, $n \geqq 0$ be fixed integers, and let R_{mn} be the set of rational functions in A of type (m, n) (i.e. no poles in Δ). Let $\|\cdot\|$ denote the supremum norm $\|\phi\| = \sup_{z\in S} |\phi(z)|$, which for $\phi \in A$ is identical to $\sup_{z\in\Delta} |\phi(z)|$. Here is the *rational Chebyshev approximation problem* for $f \in A$: find a *best approximation* (BA) $r^* \in R_{mn}$ such that $\|f - r^*\| = \inf_{r\in R_{mn}} \|f - r\|$ [1], [5]. Approximations of this kind are useful in various contexts in numerical analysis, and have a particularly important and natural application in the problem of the *design of digital filters* [2], [7]. It is known that a BA r^* always exists, but that it need not be unique unless $n = 0$ [3]. We will write $E^* = \|f - r^*\|$.

The *error curve* for an approximation r of f is the image $(f - r)(S)$. In typical examples (see [9], [10]) the error curve for a BA r^* often closely approximates a perfect circle about the origin of winding number $m + n + 1$. That is to say, if we define

$$\eta^* = E^* - \min_{z\in S} |(f - r^*)(z)|,$$

then often $\eta^* \ll E^*$. This near-circularity is an important consideration in the design of algorithms to compute BAs numerically; for example, it causes the well-known *Lawson algorithm* [8] to converge asymptotically very slowly [9]. The following result justifies the claim $\eta^* \ll E^*$ in an asymptotic sense. First we require that f satisfy a normality condition:

ASSUMPTION A. *The Padé approximation to f of type (m, n) has n finite poles, counted with multiplicity, and its Taylor series agrees with that of f exactly through the term of degree $m + n$.*

Then one has:

THEOREM 1 ([10, Thm. 6.3]). *Let $\hat{f} \in A$ satisfy Assumption* A, *and for any $\varepsilon \in (0, 1)$, let r^* be a* BA *to $f(z) = \hat{f}(\varepsilon z)$ on Δ in R_{mn}. Then*

$$\eta^* = O(\varepsilon^{2m+2n+3}) \tag{1}$$

as $\varepsilon \to 0$, uniformly for all BAs *r^*.*

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Let us agree to write $f \equiv O(g)$ (as opposed to $f = O(g)$) if both $f = O(g)$ and $g = O(f)$ hold, that is, if there exist c, $C > 0$ such that for all sufficiently small ε, $c|f(\varepsilon)| < |g(\varepsilon)| < C|f(\varepsilon)|$. It can be seen that Assumption A implies $E^* \equiv O(\varepsilon^{m+n+1})$ [10]. Thus Theorem 1 implies $\eta^*/E^* = O(\varepsilon^{m+n+2})$: as $\varepsilon \to 0$, the error curve for any BA deviates from a perfect circle in relative radius by no more than $O(\varepsilon^{m+n+2})$.

The first purpose of this paper is to establish a bound on near-circularity by showing that Theorem 1 is sharp in the following sense:

THEOREM 1′. *For each pair* (m, n), *there exists a function* $\hat{f}$ *as in Theorem* 1 *for which*

$$\eta^* \equiv O(\varepsilon^{2m+2n+3}) \tag{2}$$

as $\varepsilon \to 0$, *uniformly for all* BAs r^*.

This result establishes that whereas in real Chebyshev approximation error curves equioscillate exactly, in the complex case there is a definite limit to the degree to which they approximate circles.

Our second purpose relates to the analytic procedure for computing near-best approximations that was developed in [9], [10] and called the *Carathéodory–Fejér* (CF) *method*. Based on the calculation of a singular value decomposition of a Hankel matrix of Taylor coefficients, this procedure delivers a (unique) CF *approximant* $r^{\mathrm{cf}} \in R_{mn}$ which is near-best in the following sense:

THEOREM 2 ([10, Lemma 5.1, Thm. 6.2]). *Let* $\hat{f}$ *and* f *be as in Theorem* 1, *and for any* $\varepsilon \in (0, 1)$, *let* r^{cf} *be the* CF *approximant to* f. *Then*

$$\eta^{\mathrm{cf}} = O(\varepsilon^{2m+2n+3}) \tag{3}$$

and

$$\|r^{\mathrm{cf}} - r^*\| = O(\varepsilon^{2m+2n+3}) \tag{4}$$

as $\varepsilon \to 0$, *uniformly for all* BAs r^*.

(We define quantities η^{cf} and E^{cf} in the obvious way in analogy to η^* and E^*.)

Theorem 1 is clearly implied by Theorem 2, and in [10] this is how it is derived. Here we will apply the same argument as in the proof of Theorem 1′ to establish the sharpness of Theorem 2, as follows:

THEOREM 2′. *For any* $\hat{f}$ *as in Theorem* 1, *suppose*

$$\eta^{\mathrm{cf}} \equiv O(\varepsilon^{2m+2n+3}) \tag{5}$$

as $\varepsilon \to 0$. *Then one has also*

$$\eta^* \equiv O(\varepsilon^{2m+2n+3}) \tag{6}$$

as $\varepsilon \to 0$, *uniformly for all* BAs r^*. *Moreover whether or not* (5) *holds, one has*

$$E^{\mathrm{cf}} - E^* = O(\varepsilon^{2m+2n+4}) \tag{7}$$

as $\varepsilon \to 0$, *uniformly for all* BAs r^*.

Note that the estimate (7) is one order in ε higher than the more obvious bound $E^{\mathrm{cf}} - E^* = O(\varepsilon^{2m+2n+3})$ that follows from (4), which was given as [10, Prop. 5.2].

All of these results suggest that the CF method perhaps captures as many terms in an asymptotic description of r^* as can be obtained from any analytic procedure. For the case $n = 0$, *P.* Henrici has shown that explicit algebraic formulas for these terms can be derived systematically [4]. However, at present it is not even known whether E^* and the coefficients of r^* depend analytically on ε for small ε, although such a result is available in real Chebyshev approximation [5], [6].

2. Polynomial approximation. In the polynomial case $n=0$, Theorem 1′ can be proved by an explicit example. We discuss this case separately because it is so simple, and because a similar idea forms the basis of the subsequent proofs.

For clarity let us write p, P_m instead of r, R_{m0}. Given $\varepsilon \in (0, 1)$ and $m \geqq 0$, consider

$$f(z) = (\varepsilon z)^{m+1} + (\varepsilon z)^{2m+3}. \tag{8}$$

We will show that 0 is the (necessarily unique) BA to f in P_m. Obviously this will imply Theorem 1′ for the case $n=0$, since this example will then have $E^* = \varepsilon^{m+1} + \varepsilon^{2m+3}$, $\eta^* = 2\varepsilon^{2m+3}$. The following argument is based implicitly on the Kolmogorov criterion [1], [5].

Suppose that 0 is not the BA, so that $\|f-p^*\| < \|f\|$, or equivalently, $\|z(f-p^*)\| < \|zf\|$. Now $zf(z)$ achieves its maximum modulus at precisely the $(m+2)$nd roots of unity $\zeta_k = e^{2\pi ik/(m+2)}$, $0 \leqq k \leqq m+1$, and at each of these points it is positive and real. Therefore we must have

$$\operatorname{Re} zp^*(z) > 0 \quad \text{at } z = \zeta_k, \quad 0 \leqq k \leqq m+1. \tag{9}$$

Since $zp^*(z)$ is a polynomial of degree at most $m+1$, it is determined by its values at any $m+2$ points. If these are the roots of unity $\{\zeta_k\}$, the coefficients of $zp^*(z)$ are given by a discrete Fourier transform of the values $\zeta_k p^*(\zeta_k)$, and in particular, the coefficient of degree 0 is the mean of these quantities,

$$(zp^*)(0) = \frac{1}{m+2} \sum_{k=0}^{m+1} \zeta_k p^*(\zeta_k). \tag{10}$$

But since $(zp^*)(0) = 0$, (9) and (10) are inconsistent, contradicting the assumption $p^* \neq 0$. QED.

In this example the CF approximant p^{cf} is also identically zero, so the results of Theorem 2′ are verified.

3. Rational approximation. The obvious generalization of (8) to $n>0$ would be

$$f(z) = (\varepsilon z)^{m+n+1} + (\varepsilon z)^{2m+2n+3}. \tag{11}$$

However, the (m, n) Padé approximant to this function is identically 0, so (11) does not satisfy Assumption A for $n>0$. Moreover, it can be shown that for all $m \geqq 0$ and $n \geqq 1$, the BA to (11) is not 0 [3], and what it is is unclear. Nor have we been able to devise any other function for which an exact BA can be exhibited and satisfies (2).

Therefore we resort to an indirect argument for the proof of Theorem 1′. The following theorem shows that for a certain function $\hat{f}$, the BAs r^* must satisfy (2) as $\varepsilon \to 0$, even though we do not know exactly what they are. Theorem 1′ then follows as a consequence of this result.

THEOREM 3. *Let $\hat{r} \in R_{mn}$ have n finite poles, and define*

$$\hat{f}(z) = \hat{r}(z) + z^{m+n+1} + z^{2m+2n+3} \tag{12}$$

and $r(z) = \hat{r}(\varepsilon z)$. Then one has

$$\eta^* \equiv O(\varepsilon^{2m+2n+3}) \tag{13}$$

and

$$\|f-r\| - E^* = O(\varepsilon^{2m+2n+4}) \tag{14}$$

as $\varepsilon \to 0$, uniformly for all BAs r^*.

Proof. For each $\varepsilon \in (0, 1)$, the function to be approximated is

$$f(z) = r(z) + (\varepsilon z)^{m+n+1} + (\varepsilon z)^{2m+2n+3}.$$

Let $\{\zeta_k^+\}$ and $\{\zeta_k^-\}$ denote the $(2m+2n+4)$th roots of unity, as follows:

$$\zeta_k^+ = e^{2\pi i k/(m+n+2)}, \qquad \zeta_k^- = e^{2\pi i (k+1/2)/(m+n+2)}, \qquad 0 \leqq k \leqq m+n+1,$$

and for simplicity write $\Gamma = \varepsilon^{m+n+1}$, $\gamma = \varepsilon^{2m+2n+3}$. Then $f - r$ maps S onto a near-circle with winding number $m+n+1$ about the origin, which attains a maximum (resp. minimum) modulus of $\Gamma + \gamma$ (resp. $\Gamma - \gamma$) at the points ζ_k^+ (resp. ζ_k^-):

$$(f-r)(\zeta_k^\pm) = \frac{\pm\Gamma + \gamma}{\zeta_k^\pm}. \tag{15}$$

This near-circularity (together with Rouche's theorem; see [10, § 2]) implies that r is a nearly optimal approximation in R_{mn} to f. The key to our proof is that because f satisfies Assumption A (trivially), it follows further that r must be nearly equal to r^*. Let r and r^* be written as quotients p/q and p^*/q^*, respectively, normalized by $q(0) = q^*(0) = 1$. Then from [10, Thm. 6.2, Lemma 6.1] one has

$$\|r - r^*\| = O(\gamma) \tag{16}$$

and

$$q, q^* = 1 + O(\varepsilon) \tag{17}$$

uniformly on Δ as $\varepsilon \to 0$. ([10, Thm. 6.2] actually shows $\|r^{cf} - r^*\| = O(\gamma)$, as stated already in (4), but the same argument given there suffices to establish (16).) Equation (17) indicates that as $\varepsilon \to 0$, the Chebyshev approximation problem becomes less and less nonlinear, a circumstance which depends essentially on Assumption A (cf. [3, § 4]).

Equations (16) and (17) imply

$$r - r^* = \frac{pq^* - p^*q}{qq^*} = s + O(\varepsilon\gamma)$$

uniformly on Δ, where s is a polynomial in P_{m+n}. Now apply the discrete mean value formula as in (10) to $zs(z)$ on both $\{\zeta_k^+\}$ and $\{\zeta_k^-\}$, in succession. The results are

$$\frac{1}{m+n+2} \sum_{k=0}^{m+n+1} \zeta_k^\pm (r - r^*)(\zeta_k^\pm) = O(\varepsilon\gamma),$$

or in particular,

$$\frac{1}{m+n+2} \sum_{k=0}^{m+n+1} \operatorname{Re}\{\zeta_k^\pm (r - r^*)(\zeta_k^\pm)\} = O(\varepsilon\gamma). \tag{18}$$

The reason why we are interested in the real part in (18) is that because of (15), a small correction Δr to r will affect the moduli $|(f-r)(\zeta_k^\pm)|$ by essentially $\pm\operatorname{Re}\{\zeta_k^\pm \Delta r(\zeta_k^\pm)\}$. In particular, (16) guarantees that the correction $r^* - r$ has magnitude $O(\gamma/\Gamma)$ relative to $\|f - r\|$, which for our purposes can be weakened to $O(\varepsilon)$, and by simple geometry we obtain

$$|(f - r^*)(\zeta_k^\pm)| = \Gamma \pm \gamma \pm \operatorname{Re}\{\zeta_k^\pm (r - r^*)(\zeta_k^\pm)\} + O(\varepsilon\gamma). \tag{19}$$

Combining this with (18) shows for some indices k_+ and k_-, one must have

$$|(f-r^*)(\zeta^+_{k_+})| \geqq \Gamma + \gamma - O(\varepsilon\gamma) \tag{20}$$

and

$$|(f-r^*)(\zeta^-_{k_-})| \leqq \Gamma - \gamma + O(\varepsilon\gamma). \tag{21}$$

These two bounds, together with (1), yield (13) directly. Equation (20) also implies $E^* \geqq \Gamma + \gamma - O(\varepsilon\gamma)$, and since $E^* \leqq \|f-r\| = \Gamma + \gamma$, this implies (14). Q.E.D.

It should be mentioned that it is possible that r is itself a BA to f in this problem, at least for all sufficiency small ε, but we have not been able to establish this.

4. Sharpness of the CF method. The argument of the last proof can be applied not only to the function $\hat{f}$ of (12), but to a general function that satisfies Assumption A. Theorem 2′ is thus established as follows:

Proof of Theorem 2′. In the CF method, r^{cf} is obtained by discarding terms of negative degree in z of a meromorphic function $\tilde{r}^*$ defined in $1 \leqq |z| < \infty$ whose error curve is exactly circular and has winding number exactly $m+n+1$, for all sufficiently small ε; see [10] for details. Let us again write $\Gamma = \varepsilon^{m+n+1}$ and $\gamma = \varepsilon^{2m+2n+3}$, and also $\tilde{E}^* = \|f - \tilde{r}^*\|$. Then from [10, Lemmas 4.4, 5.1, 6.1] one has

$$f - \tilde{r}^* = \alpha \Gamma z^{m+n+1} + O(\varepsilon\Gamma)$$

and

$$\tilde{r}^* - r^{\mathrm{cf}} = \beta\gamma z^{-1} + O(\varepsilon\gamma)$$

as $\varepsilon \to 0$, uniformly on S, where α is a fixed nonzero complex constant and β is a complex number that may depend on ε, but by assumption (5), has magnitude $\beta \equiv O(1)$.

Following the proof of Theorem 3, let τ denote $e^{i\arg(\beta/\alpha)/(m+n+2)}$, and define $2m+2n+4$ equally spaced points $\{\zeta^+_k\}$ and $\{\zeta^-_k\}$ on S by

$$\zeta^+_k = \tau e^{2\pi i k/(m+n+2)}, \qquad \zeta^-_k = \tau e^{2\pi i (k+1/2)/(m+n+2)}, \qquad 0 \leqq k \leqq m+n+1.$$

Then $f - r^{\mathrm{cf}}$ maps S onto a near-circle with winding number $m+n+1$ about the origin, and its modulus satisfies

$$E^{\mathrm{cf}} = \tilde{E}^* + |\beta|\gamma + O(\varepsilon\gamma). \tag{22}$$

Moreover the sets $\{\zeta^\pm_k\}$ have been defined in such a way that $f - \tilde{r}^*$ and $\tilde{r}^* - r^{\mathrm{cf}}$ are in phase up to $O(\varepsilon\gamma)$ at $\{\zeta^+_k\}$ and out of phase up to $O(\varepsilon\gamma)$ at $\{\zeta^-_k\}$, and therefore one has

$$|(f - r^{\mathrm{cf}})(\zeta^\pm_k)| = \tilde{E}^* \pm |\beta|\gamma + O(\varepsilon\gamma).$$

As before, we get now

$$r^{\mathrm{cf}} - r^* = \frac{p^{\mathrm{cf}}q^* - p^*q^{\mathrm{cf}}}{q^{\mathrm{cf}}q^*} = s + O(\varepsilon\gamma)$$

for some polynomial $s \in P_{m+n}$. By the discrete mean value formula there follows in analogy to (18)

$$\frac{1}{m+n+2} \sum_{k=0}^{m+n+1} \mathrm{Re}\,\{e^{-i\arg\beta} \zeta^\pm_k (r^{\mathrm{cf}} - r^*)(\zeta^\pm_k)\} = O(\varepsilon\gamma),$$

and the analogue to (19) is

$$|(f-r^*)(\zeta^\pm_k)| = \tilde{E}^* \pm |\beta|\gamma \pm \mathrm{Re}\,\{e^{-i\arg\beta}\zeta^\pm_k (r^{\mathrm{cf}} - r^*)(\zeta^\pm_k)\} + O(\varepsilon\gamma).$$

Equation (6) follows as before from these two formulas together with (1). They also imply as before $E^* \geqq \tilde{E}^* + |\beta|\gamma + O(\varepsilon\gamma)$, which together with (22) and the fact $E^* \leqq E^{\mathrm{cf}}$ establishes (7) in the case where (5) holds.

If (5) does not hold, the same derivation of (7) is still valid. (If $\beta = 0$ for some ε, $\arg\beta$ and τ can be defined arbitrarily for these ε.) However if $\eta^{\mathrm{cf}} = O(e^{2m+2n+4})$, (7) can be obtained much more easily by Rouché's theorem. Q.E.D.

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