CHEBYSHEV APPROXIMATION ON THE UNIT DISK

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Abstract. We consider the problem of rational Chebyshev approximation of an analytic function on the unit disk, and survey known results related to nearly-circular error curves and to the Carathéodory-Fejér (CF) method for near-best approximation. The real CF (or Chebyshev-CF) method for approximation of a continuous real function on an interval is also described.

1. THE PROBLEM

This paper will present a relatively easygoing account of some recent developments in complex Chebyshev approximation that relate to "the CF method", to "nearly circular error curves", and to "the AAK theory". This should be a good introduction to the second lecture of Gutknecht in this volume [4], where the *CF table* and related matters are studied carefully. Also closely related are the third lecture of Henrici [10], where the ideas described here are applied to analyze the asymptotic behavior of best approximations, and the lectures of Meinguet [12], where the AAK theory is developed in an elegant way at a higher level. Most of the material presented here appears in greater detail in [16] and [17].

Let S be the unit circle $|z| = 1$, D the unit disk $|z| < 1$, and P_m the set of complex polynomials of degree at most m. If f is a function continuous on $\overline{D}$ and analytic in D, it is natural to ask, how well can f be approximated on $\overline{D}$ by a polynomial $p \in P_m$ with respect to the supremum norm? Since

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f and p are analytic, the maximum modulus principle ensures that it is enough to consider the boundary circle, so we define

$$\|\phi\| = \sup_{z\in S} |\phi(z)| .$$

The *polynomial Chebyshev approximation problem* is this: find a *best approximation (BA)* $p^* \in P_m$ to f such that

$$\|f - p^*\| = \inf_{p\in P_m} \|f - p\| .$$

More generally, we can consider approximation by rational functions of type (m,n) that are constrained to have no poles in $\bar{D}$. Let R_{mn} be the set of such functions. The *rational Chebyshev approximation problem* is then: find a BA $r^* \in R_{mn}$ such that

$$\|f - r^*\| = \inf_{r\in R_{mn}} \|f - r\| .$$

We let E^*, or E^*_{mn}, denote this infimum.

In polynomial approximation, p^* exists and is unique and is characterized by the Kolmogorov criterion, although in practice this characterization is not very useful for computing BAs. In contrast, although r^* exists too, it need *not* be unique for $n > 0$ — a fact first established at this NATO meeting! [8] In the rational case no characterization of BAs is available, and no very satisfactory algorithms for their computation are known. In fact the CF method that we will describe, although in principle it delivers only a near-best approximation, often comes closer to best than can practicably be achieved by other means. For details about the general theory of complex Chebyshev approximation, see the first paper of Gutknecht in this volume [3].

Unlike real Chebyshev approximations, complex BAs on the unit disk are not very important for the construction of function evaluation procedures for computers, perhaps because it is rare that a function defined on all of **C** can be reduced to a representation on a disk. However, there are other applications in which the unit circle comes into its own, particularly in the areas of *linear systems theory* [11] and *digital signal processing* [14]. The reason is that both of these problems involve linear processes with constant coefficients that act on a discrete time variable; they are therefore naturally analyzed by Fourier methods, but because the time variable is discrete, its dual variable has a bounded domain, which is conveniently reduced to S through an application of the *z-transform*. The paper of Meinguet in this volume is motivated by systems theory applications [13], while for applications of CF ideas to digital filter design, see [6].

2. THE NEAR-CIRCULARITY PHENOMENON

Our approximation problem has a simple geometric interpretation. Given f and an approximation r, consider the image $(f-r)(S)$, which is called the *error curve* for r. The Chebyshev approximation problem is obviously equivalent to the following: *find* r^* *so that the error curve can be contained in a circle about* 0 *of minimal radius.* See Fig. 1.

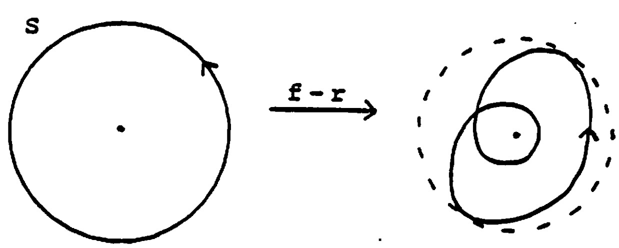
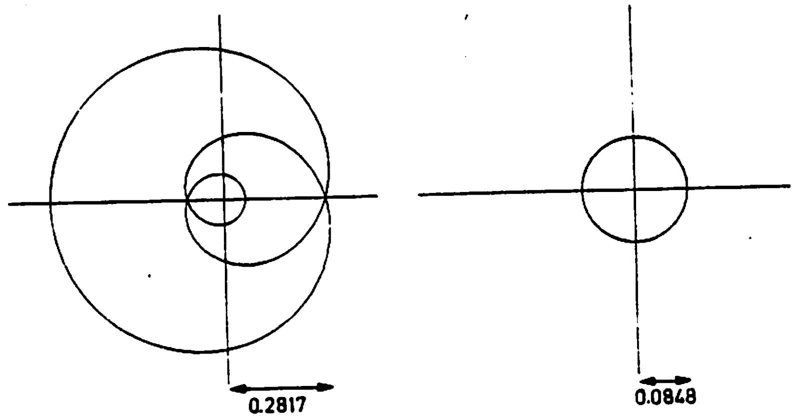


Figure 1. The error curve

A priori, we do not know much about what the error curve for r^* will look like, except that it must touch the boundary circle at at least $m+2$ points. But when best approximations are computed numerically, a remarkable fact emerges: their error curves are often very nearly perfect circles with winding number $m+n+1$. For illustration, consider the plots shown in Fig. 2. In Fig. 2a, the error curve for the type (1,1) Padé approximant to e^z is plotted, and it is evidently a closed loop with winding number $m+n+1 = 3$. Fig. 2b shows on the same scale the error curve for the BA r^*. This curve also has winding number 3, but one can no longer see this, because its modulus varies as z traverses S by less than 1%. In approximation of type (2,1), this figure becomes 0.01%, and it decreases rapidly further as m and n are increased.

The question is, how can this near-circularity phenomenon be accounted for?



(a) Padé (b) best

Figure 2. Error curves for type (1,1) approximation of e^z.

To begin with, we can observe that if an approximation r happens to have a nearly circular error curve, then it is nearly best. The proof is essentially Rouché's theorem.

THEOREM 1. Suppose the error curve for $r \in R_{mn}$ *has winding number at least* $m+n+1$ *about the origin. Then*

$$\min_{z \in S} |(f-r)(z)| \leq E^* \leq \|f-r\| .$$

In particular, if the error curve is a perfect circle, then $r = r^*$.

Proof. The second inequality is nothing more than the definition of E^* . For the first, suppose to the contrary $\|f-r^*\| < \min_{z \in S} |(f-r)(z)|$. Then simple geometry shows that the funcion $(f-r)-(f-r^*) = r^*-r$ must have the same winding number as $f-r$ on S , which by assumption is at least $m+n+1$. But this is impossible, for r^*-r belongs to $R_{m+n,2n}$, and hence has at most $m+n$ zeros in D . $\square$

Thus "nearly circular implies nearly best", which makes the near-circularity phenomenon at least plausible. Nevertheless, the implication runs in the wrong direction, so we will need additional ideas to see why the phenomenon is in practice so pronounced.

3. THE CARATHÉODORY-FEJÉR METHOD

Another clue to near-circularity can be obtained from the following observation: in many extremal problems in function theory where a supremum norm is involved, solutions occur that involve circles or arcs of circles. For example, a standard proof of the Riemann mapping theorem for a Jordan region Ω containing 0 considers the set of all analytic functions on Ω normalized by $f(0)=0$, $f'(0)=1$. Among these, that function with minimal norm $\|f\|$ is precisely the conformal map from Ω to a disk — which means, it maps the boundary $\partial\Omega$ onto a circle αS . Another well-known example of an extremal problem whose solution involves circles is the *Nevanlinna-Pick* problem of interpolation with minimal norm in the disk ([20], chap. 10).

These problems in effect contain an infinite number of unknown parameters — the Taylor coefficients of an analytic function. In contrast, the Chebyshev approximation problem has a finite number of unknown parameters. This is the essential reason why near-circles rather than perfect circles appear.

Our analysis is based on an extended problem that is closely related to both the above proof of the Riemann mapping theorem and to Nevanlinna-Pick, namely the *Carathéodory-Fejér problem.* Let q be a polynomial of degree d , and consider the set of extensions of q to a Taylor series $\tilde{q}$ that is analytic and bounded in D . Carathéodory and Fejér asked in 1911: what extension $\tilde{q}$, if any, attains minimal norm $\|\tilde{q}\|$? They found the following solution: a minimal extension $\tilde{q}$ exists, it is unique, and it is the Taylor series of a function that is a constant times a *finite Blaschke product* of order $\nu \le d$,

$$\tilde{q}(z) = \sigma \prod_{i=1}^{\nu} \frac{z - z_i}{\bar{z}_i z - 1} \qquad \sigma \in \mathbf{C},\ z_i \in D .$$

Thus $\tilde{q}$ maps S onto a perfect circle with winding number at most d . Moreover, Schur showed in 1917 that $\tilde{q}$ can be computed analytically by solving a certain matrix eigenvalue problem.

Our original polynomial approximation problem can be viewed as follows: we are given coefficients $a_{m+1}, a_{m+2}, \ldots$, and asked to find $a_0, \ldots, a_m$ so that $\Sigma_0^\infty a_k z^k$ has minimal norm on $\bar{D}$. (Without loss of generality we have assumed that the Taylor series of the given function f begins at degree $m+1$.) The CF problem reverses the prescription: given $a_0, \ldots, a_m$, what infinite set of coefficients $a_{m+1}, a_{m+2}, \ldots$ leads to a series $\Sigma_0^\infty a_k z^k$ of minimal norm? Since the number of unknown parameters is now infinite, circular error curves become possible.

The idea of the *CF method* is perhaps now obvious. First, truncate the given function f at some high order M, so that it takes the form $f(z) = \Sigma_{m+1}^{M} a_k z^k$. Next, apply the CF theorem to construct an extension of f *backwards* to a Laurent series $\tilde{f}(z) = \Sigma_{-\infty}^{M} a_k z^k$ that has minimal norm on S. Finally, with a little luck, the coefficients of negative index in this expansion will be small, so that the coefficients $a_0, \ldots, a_m$ determine a polynomial p^{cf} that is close to p^*.

To describe the CF method more precisely, let us generalize to the rational approximation problem. Given f analytic in D and continuous on $\bar{D}$, we seek a BA $r^* \varepsilon R_{mn}$. Now let $\tilde{R}_{mn}$ denote the set of functions that are meromorphic in $1 \leq |z| \leq \infty$, have $\nu \leq n$ poles in $1 \leq |z| < \infty$ and at most $m - \nu$ poles at ∞ (i.e. a zero at ∞ of order at least $\nu - m$, if $m < \nu$), and are bounded in $1 \leq |z| \leq \infty$ except near these poles. Equivalently, $\tilde{R}_{mn}$ is the set of functions that can be represented in the form

$$\tilde{r}(z) = \sum_{k=-\infty}^{m} d_k z^k \Big/ \sum_{k=0}^{n} e_k z^k ,$$

where the numerator converges in $C - \bar{D}$ and is bounded there except near ∞. Consider then:

EXTENDED APPROXIMATION PROBLEM. Find $\tilde{r}^* \varepsilon \tilde{R}_{mn}$ *such that*

$$\|f - \tilde{r}^*\| = \inf_{\tilde{r}\varepsilon\tilde{R}_{mn}} \|f - \tilde{r}\| .$$

Like the CF problem, this extended problem has a solution that can be explicitly constructed. For general f of the class considered (in fact for $f \varepsilon L^{\infty}(S)$), the procedure for this is exactly what is taken up in the *AAK theory* published in 1971 by Adamjan, Arov, and Krein [1,13]. Let us however again assume $f \varepsilon P_M$ for some large M, in which case the solution is simpler and can be worked out on the basis of an extension of the Carathéodory-Fejér theorem to rational approximation published by T. Takagi in 1924 [5,15]. (For the intermediate possibility $f \varepsilon R_{MN}$, see the paper [4] by Gutknecht in this volume.)

Let H_{m-n} denote the $(M-m+n) \times (M-m+n)$ Hankel matrix

$$H_{m-n} = \begin{bmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_M \\ a_{m-n+2} & & \cdot & \\ \vdots & \cdot & & 0 \\ a_M & & & \end{bmatrix} .$$

Let

$$H_{m-n} = U \Sigma \overline{U}^{H}$$

be a *singular value decomposition (SVD)* of H_{m-n} — i.e. U is a square unitary matrix ($U^{-1} = U^{H}$), and Σ is a square diagonal matrix of nonnegative real *singular values* arranged in non-increasing order, $\Sigma = \mathrm{diag}(\sigma_0, \ldots, \sigma_{M-m+n-1})$, $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{M-m+n-1}$. In particular, let σ_n be the *nth singular value* of H_{m-n} (starting from the 0th), and let $u^{(n)}$ be the corresponding *nth singular vector,* namely the nth column of U , which we write in the form

$$u^{(n)} = (u_0, u_1, \ldots, u_{M-m+n-1})^T .$$

(If the coefficients a_k are real, the SVD reduces to an eigenvalue decomposition.) Then the following is the basic theorem, due to Carathéodory and Fejér, Schur, Takagi, and Trefethen, that the rational CF method is based on:

THEOREM 2 [17]. *The function f has a unique BA $\tilde{r}^*$ in $\tilde{R}_{mn}$, and it is given by*

$$(f - \tilde{r}^*)(z) = \sigma_n z^{m-n+1} \cdot \frac{u_0 + u_1 z + \ldots + u_{M-m+n-1} z^{M-m+n-1}}{\overline{u}_0 + \overline{u}_1 z^{-1} + \ldots + \overline{u}_{M-m+n-1} z^{-M+m-n+1}} .$$

The error curve $(f - \tilde{r}^*)(S)$ is a perfect circle about the origin with radius $\|f - \tilde{r}^*\| = \sigma_n$, and if σ_n is a simple singular value of H_{m-n} , it has winding number exactly $m+n+1$. □

The *rational CF method* now consists of constructing $\tilde{r}^*$ and then truncating it to obtain an approximation $r^{cf} \in R_{mn}$ that is hopefully near-best. Here is the recipe. In practice, many of these computations are best performed numerically with the Fast Fourier Transform, as indicated (see [9]).

Step 1. Given f analytic in $\overline{D}$, compute its Taylor coefficients $a_0, \ldots, a_M$ for some large M (FFT).

Step 2. Set up the Hankel matrix H_{m-n} and compute its nth singular value and vector.

Step 3. Factor the denominator of the Blaschke product in Thm. 2 to determine its n (or fewer) zeros outside $\overline{D}$ (FFT). The polynomial with these zeros is the denominator q^{cf} of r^{cf} .

Step 4. Multiply by q^{cf} to obtain the numerator $\tilde{p}^*$ in the representation

$$\tilde{r}^*(z) = \sum_{k=-\infty}^{m} d_k z^k \Big/ \sum_{k=0}^{n} e_k z^k .$$

Compute coefficients $d_0, \ldots d_m$ of this function (FFT).

Step 5. Truncate terms of negative degree in the numerator of $\tilde{r}^*$ to obtain the *CF approximant*

$$r^{cf}(z) = \sum_{k=0}^{m} d_k z^k \Big/ \sum_{k=0}^{n} e_k z^k .$$

The truncation $\tilde{r}^* \to r^{cf}$ described in Step 5 is only one ("Type 1") of several reasonable methods for obtaining an approximation in R_{mn} from $\tilde{r}^*$. Various others are mentioned in [5], and it is not yet clear which is best.

4. ACCURACY OF THE CF METHOD

The CF method construction leads immediately to a beautiful result:

THEOREM 3 [17]. $\sigma_n(H_{m-n}) \le E^*_{mn}$.

Proof. Theorem 2 implies $\sigma_n = \|f - \tilde{r}^*\|$, and since $R_{mn} \subseteq \tilde{R}_{mn}$, one has also $\|f - \tilde{r}^*\| \le \|f - r^*\| = E^*$. □

This theorem gives some algebraic insight into best approximations that is far from trivial.

Now for all we know a priori, the bound of Thm. 3 could be very crude. But the following table of results for $f(z) = e^z$ shows that in practice, just the opposite can be the case:

(m,n)	$\sigma_n(H_{m-n})$	E^*_{mn}
(0,0)	1.258	1.344
(1,0)	.5575	.5586
(2,0)	.177374	.177377
(3,0)	.043368927	.0433689...

TABLE 1

Obviously the inequality in Thm. 3 is sometimes virtually an equality. Indeed the ellipsis in the last line of Table 1 reflects the fact that we have not succeeded in computing the error E^*_{30}

for e^z more accurately than by applying Thm. 1 to r^{cf}. Although the examples in this paper involve only e^z, similar results hold for many other functions.

For more insight let us look in detail at the problem $f(z) = e^z$, $(m,n) = (1,1)$. Here the SVD construction leads to the following extended BA in $\tilde{R}_{11}$:

$$\tilde{r}^*(z) = \frac{\cdots + 0.00000983z^{-2} + 0.00024668z^{-1} + 0.99613054 + 0.58955195z}{1 - 0.43416584z}$$

Obviously the terms of negative degree in the numerator are very small. Truncating them gives

$$r^{cf}(z) = \frac{0.99613054 + 0.58955195\,z}{1 - 0.43416584z},$$

and since $\tilde{r}^* - r^{cf}$ is small, r^{cf} must have a nearly circular error curve, which by Thm. 1 means it is near best. In fact, numerical computation gives the following representation for r^*, which is obviously almost the same as r^{cf}:

$$r^*(z) = \frac{0.99625 + 0.58952\,z}{1 - 0.43414\,z}.$$

Ideally, a general theory would be available that would show exactly why the CF method is so accurate for a function like e^z, and delineate just what functions the method performs well for. Unfortunately, Thm. 3 is the only fully general theorem regarding CF approximation that we have. However, the following *asymptotic* results have been obtained in [17] and [18], and show that at least in a limiting sense, the CF method is highly accurate, and error curves are nearly circular.

We will need a normality condition.

ASSUMPTION A. The Padé approximant r^p to f of order (m,n) has n finite poles, and its Taylor series agrees with f exactly through the term of degree $m+n$.

Essentially this assumption requires that the determinant of a certain Hankel matrix section of H_{m-n} be nonzero, which is a standard hypothesis in Padé approximation; see the lectures in this volume by Brezinski [2].

Now for any $\varepsilon > 0$, consider approximation of $f(\varepsilon z)$ on $\overline{D}$. (Or equivalently, consider approximation of $f(z)$ on $|z| \le \varepsilon$.) By a fairly lengthy but elementary argument, one derives the following estimates for the accuracy of the CF approximant:

THEOREM 4 [17,18]. *As* $\varepsilon \to 0$,

$$\|f - r^*\| - \sigma_n = O(\varepsilon^{2m+2n+3})$$

and

$$\|r^{cf} - r^*\| = O(\varepsilon^{2m+2n+3})$$

uniformly for all BAs r^* *, and*

$$\|f - r^{cf}\| - \|f - r^*\| = O(\varepsilon^{2m+2n+4}) . \square$$

These orders of accuracy are remarkably high, when one considers that $\|f - r^*\|$ has order ε^{m+n+1} as $\varepsilon \to 0$. Thus the CF method has a *relative* accuracy of $O(\varepsilon^{m+n+2})$. Padé approximation, by contrast, has relative accuracy $O(\varepsilon)$.

Furthermore, the CF method also gives information about the BA (or BAs) r^* . Since the error curve of $\tilde{r}^*$ is perfectly circular, Thm. 4 together with the associated bound $\|\tilde{r}^* - r^{cf}\| = O(\varepsilon^{2m+2n+3})$ implies that the error curve of r^* must be nearly circular:

THEOREM 5 [17,18]. *As* $\varepsilon \to 0$,

$$\|f - r^*\| - \min_{z \in S} |(f - r^*)(z)| = O(\varepsilon^{2m+2n+3}) .$$

Moreover, this estimate is sharp in that there exist functions for which the left hand side has magnitude at least const. $\varepsilon^{2m+2n+3}$ ***for some fixed constant.*** $\square$

Naturally one would also like estimates for the limit $m, n \to \infty$ with a fixed value of ε . A theorem in this direction for the case $n = 0$ is given in [16], but the general rational approximation problem has not been treated.

5. REAL CF APPROXIMATION

The CF method and the phenomenon of near-circularity can be transplanted to a more general domain Ω by means of a conformal map of the exterior of $\overline{\Omega}$ onto the exterior of $\overline{D}$. The resulting procedure makes use of a Hankel matrix of Faber series coefficients, and is called the *Faber-CF method*. For details see the second lecture by Gutknecht in this volume [4]. Gutknecht has also developed *Laurent-CF* and *Fourier-CF* methods for related problems, which extend the CF idea in analogy to the work of Gragg, et al. on Laurent-Padé and Fourier-Padé approximation [4,5].

Here we will discuss the particularly interesting case of transplantation to a real interval, where a *real CF* or *Chebyshev-CF* method is obtained that turns out to be even more powerful than on the disk. (Related ideas were developed earlier by S. Darlington, D. Elliott, B. Lam, and others; see [19] for a discussion and references.) The real CF method is presented at length in [7] and [19].

Let $F(x)$ be a continuous real function on $I = [-1,1]$, which for simplicity as before we will assume has a finite expansion in Chebyshev polynomials,

$$F(x) = \sum_{k=0}^{M} a_k T_k(x) , \quad a_k \in \mathbb{R} .$$

If $x \in I$ and $z \in S$ are related by $x = \operatorname{Re} z = \frac{1}{2}(z+z^{-1})$, then a widely known but underutilized formula expresses $T_k(x)$ in terms of z :

$$T_k(x) = \operatorname{Re} z^k = \frac{1}{2}(z^k + z^{-k}) .$$

To apply a CF method to F, we consider the analytic function f defined by

$$f(z) = \sum_{k=0}^{M} a_k z^k ,$$

so that one has

$$F(x) = \operatorname{Re} f(z) = \frac{1}{2}(f(z) + f(z^{-1})) .$$

If the CF/Takagi construction is carried out for f — so that we are working with the singular value (or eigenvalue) decomposition of a Hankel matrix of Chebyshev coefficients of F — we obtain the optimal approximation $\tilde{r}^* \in \tilde{R}_{mn}$ to f on S,

$$\tilde{r}^*(z) = \sum_{k=-\infty}^{m} d_k z^k \Big/ \sum_{k=0}^{n} e_k z^k .$$

By cross-multiplication, the real part of $\tilde{r}^*$ can be written in the form

$$\operatorname{Re} \tilde{r}^*(z) = \frac{1}{2}(\tilde{r}^*(z) + \tilde{r}^*(z^{-1}))$$

$$= \frac{\sum_{k=0}^{\infty} \hat{d}_k (z^k + z^{-k})}{\sum_{k=0}^{n} \hat{e}_k (z^k + z^{-k})} = \frac{\sum_{k=0}^{\infty} \hat{d}_k T_k(x)}{\sum_{k=0}^{n} \hat{e}_k T_k(x)}$$

for some new sets of coefficients $\{\hat{d}_k\}$, $\{\hat{e}_k\}$. Assuming $m \geq n$, we now truncate this expression in the obvious way to obtain the *Chebyshev-CF approximant*

$$R^{cf}(x) = \frac{\sum_{k=0}^{m} \hat{d}_k T_k(x)}{\sum_{k=0}^{n} \hat{e}_k T_k(x)} .$$

(If $m < n$, the definition of R^{cf} is more complicated. See [19].)

Now by construction, R^{cf} belongs to R_{mn} and has real coefficients. But beyond this, why should we expect it to be close to the real BA R^* to F? To see the answer, observe that if $R^{cf}(x) \approx \operatorname{Re} \tilde{r}^*(z)$, then

$$(F - R^{cf})(x) \approx \operatorname{Re} (f - \tilde{r}^*)(z) .$$

By Thm. 2, $(f - \tilde{r}^*)(S)$ is a perfect circle with winding number $m+n+1$ (assuming $\sigma_n(H_{m-n})$ is simple), and therefore $\operatorname{Re}(f-\tilde{r}^*)(z)$ equioscillates $m+n+2$ times between values $\pm\sigma_n$ as z traverses the upper half of S. Therefore R^{cf}, correspondingly, must have a nearly-equioscillating error curve on I, which by the well-known theorem of de la Vallée Poussin [12], implies that R^{cf} is near-best.

Thus the justification of the real CF construction is a matter of showing $R^{cf}(x) \approx \operatorname{Re} \tilde{r}^*(z)$, that is, the truncated terms $\hat{d}_{m+1}, \hat{d}_{m+2}, \ldots$ are small. Again, it would be nice to have a non-asymptotic theory for this, but none is available; in fact even Thm. 3 fails to extend (except with the loss of a factor of two) to Chebyshev-CF approximation. In analogy to Thm. 4, on the other hand, one can derive a very satisfactory asymptotic result. The following theorem relates to approximation of $F(\varepsilon x)$ on $[-1,1]$, and requires again that F satisfy Assumption A. Note that the powers of ε here are even higher than in Thm. 4, a result of the fact that fewer terms are truncated here in going from $\tilde{r}^*$ to R^{cf} than in the complex CF method.

THEOREM 6 [19]. *As* $\varepsilon \to 0$,

$$\|F - R^*\|_I - \sigma_n = O(\varepsilon^{3m+2n+3})$$

and

$$\|R^{cf}-R^*\|_I = O(\varepsilon^{3m+2n+3}) .$$

The second estimate can be replaced by $O(\varepsilon^{3m+2n+4})$ *in the special case* $n=0$. □

As in the complex case, the asymptotic accuracy of the real CF method is overwhelmingly apparent in many computational examples where ε is not at all small. Table 2 shows results for polynomial approximation of $F(x) = e^x$. A comparison with the numbers of Table 1 confirms that, as the asymptotic results suggest, the CF method is even more powerful on I than on $\overline{D}$. (From a computational point of view, however, the CF method for $\overline{D}$ remains probably the more important, because for approximation on I rapidly convergent alternatives such as the Remes algorithm are available [12].)

(m,n)	$\sigma_n(H_{m-n})$	E^*_{mn}
(0,0)	1.196	1.175
(1,0)	.2787994	.27880...
(2,0)	.0450173878	.04501738...
(3,0)	.005528370108712	.0055283701087...

TABLE 2

Since Thm. 6 is the analog for real CF approximation of Thm. 4 for the complex case, one naturally wonders, what about a real analog of Thm. 5? Sure enough, it is easy to derive by CF methods the following result: as $\varepsilon \to 0$, best approximation error curves equioscillate up to $O(\varepsilon^{3m+2n+3})$. This would be very interesting, were it not of course well known that error curves in real Chebyshev approximation equioscillate exactly! In other words the sharpness statement in Thm. 5 is very significant, and does not extend to real approximation.

Nevertheless, it seems that the high accuracy of the real CF method must reveal something about the structure of BAs in real Chebyshev approximation. One can, for example, state a somewhat awkward theorem to the effect that as $\varepsilon \to 0$, real BA error functions approach the real parts of Blaschke products up to $O(\varepsilon^{3m+2n+3})$ [7,19]. It remains to be seen whether this fact can be recast in some way that makes its significance for approximation on the interval as clear as that of the near-circularity results for the unit disk.

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