SIAM REVIEW
Vol. 26, No. 3, July 1984

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BOOK REVIEWS

EDITED BY GEORGE HANDELMAN

Publishers are invited to send books for review to Professor George Handelman, Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181.

Nonlinear Equivalence, Reduction of PDEs to ODEs and Fast Convergent Numerical Methods. By E. E. ROSINGER. Pitman, Boston, 1982. 247 pp. $21.95. Paper. ISBN 0-273-08570-0.

In 1956, Lax and Richtmyer published their well-known theorem on difference schemes for time-dependent problems, relating the consistency and stability of the scheme to the convergence of the approximate solution to the exact solution. The present monograph provides a generalization of the Lax–Richtmyer theorem to nonlinear evolution equations. The theory is developed in a way that is independent of the type of equation that is being developed (e.g., parabolic or hyperbolic). The theory includes the case of nonlinear equations with "finite blowup time"; that is, equations for which a solution may fail to exist, or to have a given regularity, after a finite amount of time. The results apply especially to smooth solutions of the original problem. To handle the estimates of derivatives that inevitably occur, the author defines and makes considerable use of a "level V of relative numerical smoothness" of a function. This is a concept very similar to the "inverse assumptions" that occur in finite element analysis; the number V is the constant that occurs in the inverse assumption.

The author uses his theory to formulate and analyse some new versions of the method of lines for approximating evolution equations. The book concludes with some new discretizations that are said to perform better than the traditional ones in certain circumstances. Some numerical examples are given for soliton solutions of Burgers' equation.

This work contains original results, due almost entirely to the author. The exposition is very clear and easy to read. The book should be of interest to those working in evolution equations and their numerical approximation.

R. B. KELLOGG
Institute for Physical Science and Technology
University of Maryland

Fourier Analysis of Numerical Approximations of Hyperbolic Equations. By R. VICHNEVETSKY and J. B. BOWLES. Society for Industrial and Applied Mathematics, Philadelphia, 1982. xii + 140 pp. $21.50. ISBN 0-89871-181-9.

It is well known that the Fourier transform is a valuable tool for the analysis of discrete numerical models of partial differential equations. But traditionally, Fourier ideas have been applied mainly to the single purpose of testing the von Neumann condition: does the discrete model admit any sinusoidal solutions with amplification factors greater than 1 in modulus? The classic book by Richtmyer and Morton presents a wealth of material relating the answer to this question to stability and convergence, and may be said to be the culmination of the "amplification factors" style of Fourier analysis of numerical methods.

But Fourier analysis can reveal much more than just the presence or absence of growing modes. Consider the Crank–Nicolson or trapezoidal approximation to the model hyperbolic equation $u_t = u_x$,

$$v_j^{n+1} = v_j^n + \frac{k}{4h}(v_{j+1}^n + v_{j+1}^{n+1} - v_{j-1}^n - v_{j-1}^{n+1}), \tag{1}$$

where h and k are the step sizes in space and time. The differential equation admits solutions $\exp(i(\omega t + \xi x))$, where ω is the *frequency* and ξ is the *wave number*, if and only if ω and ξ satisfy the dispersion relation $\omega = \xi$. But an easy substitution shows that Crank-Nicolson imposes instead the *numerical dispersion relation*

$$\tan\frac{\omega k}{2} = \frac{k}{2h}\sin \xi h. \tag{2}$$

Thus in Fourier space a discrete numerical model takes the form of a trigonometric approximation to the ideal dispersion relation. Unlike its ideal counterpart, the numerical dispersion relation is periodic in both ξ and ω.

By proceeding from here in Fourier space a great deal can be learned, and this is the purpose of the new book by Vichnevetsky and Bowles. The *order of accuracy* of the discrete model is the order of contact of the numerical and ideal dispersion relations at the origin $\xi = \omega = 0$. The *dissipativity* of the model, if any, depends on whether ω has a nonzero imaginary part for $\xi \neq 0$. The *phase velocity* $c = -\omega/\xi$ reveals something about numerical errors. These and related matters occupy the first five chapters of the book. The authors emphasize that discrete models can be viewed as *digital filters*, a topic from electrical engineering whose analysis in the Fourier domain is already well established.

The use of the Fourier transform becomes particularly interesting when Vichnevetsky and Bowles turn in the second half of the book to the concepts of *dispersion* and *group velocity*. The group velocity, defined by $C = -d\omega/d\xi$, describes the propagation of wave energy in any dispersive but nondissipative linear system—such as a solid crystal, the surface of a pond, or a finite difference model. In the case of Crank–Nicolson, differentiation of (2) gives the result

$$C = -\cos \xi h \cos^2 \frac{\omega k}{2}, \tag{3}$$

a trigonometric approximation to the ideal function $C \equiv -1$ for $u_t = u_x$. For the past decade Prof. Vichnevetsky has been one of a few voices advocating the importance of group velocity to the behavior of numerical methods. This book gives several impressive numerical demonstrations that it is this quantity that determines how a discrete model really behaves.

Vichnevetsky and Bowles are particularly interested in the influence of group velocity on the propagation of *parasitic waves*, that is, sawtoothed numerical solutions with $\xi \approx \pi/h$ that are physically spurious. From (2) and (3) one sees that for each sufficiently small $\omega > 0$, Crank–Nicolson admits one wave solution with $\xi \in [0, \pi/2h)$ and another with $\xi \in (\pi/2h, \pi/h]$; the first has $C < 0$, but the second has $C > 0$. Similar results hold for many nondissipative discrete models. Thus parasitic wave energy often travels in the physically wrong direction, and this explains why spurious wiggles often appear upstream of a boundary or discontinuity in numerical calculations. The authors show that the amplitude of such wiggles can be predicted by calculating numerical reflection coefficients at the boundary, and derive many interesting consequences. This kind of analysis is quite easy and should have been done long ago, but was not; much of it is new with Vichnevetsky.

Thus the present book is truly original, presenting a novel set of ideas of considerable importance. Unfortunately, its execution is disappointing. The book seems more a succession of examples than a unified treatise, exhibiting considerable repetition but not much depth. Various small points of wording, notation, mathematical precision, and graphics have been carelessly handled. The design of the whole looks rather like a somewhat hasty concatenation of Vichnevetsky's earlier papers.

A more serious weakness in this work is that it communicates very little perspective about its subject matter. The authors spend too much time with the particular example of the space-centered, semi-discrete model of $u_t = cu_x$, drawing specific conclusions that obscure the generality of what they are doing. It is particularly distressing that they repeatedly fail to connect their ideas to related topics in the numerical analysis literature. The following are some topics intimately related to the themes of this book, but which the authors completely fail to mention:

- "Modified equations" for asymptotic analysis of difference formulas;
- Quantitative mathematical results on numerical oscillations about discontinuities;
- Periodic dispersion relations as used in solid state physics for analyzing sound and light vibrations in crystals;
- "Absorbing" or "radiation" boundary conditions for artificial boundaries;
- Stability for discrete models of initial boundary value problems.

In failing to apply their ideas to shed new light on established subjects such as these, Vichnevetsky and Bowles have missed the opportunity to make their book a truly major contribution to numerical analysis. In failing to even give appropriate references, they have limited its value as a text.

To summarize, I encourage anyone working with computational methods for partial differential equations to take a look at this book for a valuable exposure to the fascinating phenomena of numerical wave propagation. But in this book the "waves" style of Fourier analysis of numerical methods has not yet reached its culmination.

LLOYD N. TREFETHEN
Courant Institute of Mathematical Sciences

Optimization Theory and Applications. By LAMBERTO CESARI. Springer-Verlag, New York, 1983. xiv + 542 pp. $68.00. ISBN 0-387-90676-2.

Modern optimal control theory was developed in the early 1950's as can be seen from the dates of the published pioneering research (Bellman et al. (1956)[1], Bushaw (thesis 1953, paper in the Lefschetz series [9], 1958)[2], Flügge Lotz (1953)[3], Gamkrelidze (1958)[4], Hestenes (1950)[5], Krasovskii (1957)[6], LaSalle (1959)[7], Lefschetz (1950)[9], Pontryagin et al. (1960)[10], Wazewski (1961)[13]). In the 1960's it became sufficiently formalized so that complete books on the subject started to appear. It was, of course, realized that optimal control theory is a part of the calculus of variations, a subject almost as old as the infinitesimal calculus itself, but the relation between the classical and the modern theory was not at all clear at the beginning. Indeed, the classical calculus of variations treats mainly "interior" maxima and minima, while the modern version puts much more emphasis on those extrema which lie on the boundary of some constraint set. Therefore equations become inequalities, the classical results have to be changed appropriately and the need arises to use different methods of approach. A book representing these new methods in all their power and elegance is Lee and Markus [8].

The gap between the classical and the modern approach was clearly bridged in the research literature, but most books on the subject still remained on one or the other side of