Conformal mapping solution of Laplace's equation on a polygon with oblique derivative boundary conditions

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Abstract: We consider Laplace's equation in a polygonal domain together with the boundary conditions that along each side, the derivative in the direction at a specified oblique angle from the normal should be zero. First we prove that solutions to this problem can always be constructed by taking the real part of an analytic function that maps the domain onto another region with straight sides oriented according to the angles given in the boundary conditions. Then we show that this procedure can be carried out successfully in practice by the numerical calculation of Schwarz-Christoffel transformations. The method is illustrated by application to a Hall effect problem in electronics, and to a reflected Brownian motion problem motivated by queueing theory.

Keywords: Laplace equation, conformal mapping, Schwarz-Christoffel map, oblique derivative, Hall effect, Brownian motion, queueing theory.

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1. The oblique derivative problem and conformal mapping

Let Ω by a polygonal domain in the complex plane \mathbb{C} , by which we mean a possibly unbounded simply connected open subset of \mathbb{C} whose boundary $\partial\Omega$ consists of a finite number of straight lines, rays, and line segments. Let these sides be denoted by $\Gamma_1, \ldots, \Gamma_n, n \ge 1$, where Γ_k is the complex open interval (z_k, z_{k+1}) , for some sequence of vertices $z_k \in \mathbb{C} \cup \{\infty\}$, and let Ω lie everywhere to the left of $\partial\Omega$ as $\partial\Omega$ is traversed in the order $\Gamma_1, \Gamma_2, \ldots$. For convenience we set $z_0 = z_n$ and $z_{n+1} = z_1$. We permit geometries in which one or more slits are cut into Ω by viewing the two sides of such a slit as distinct boundary segments, e.g. Γ_k and Γ_{k+1} . The closure $\overline{\Omega}$ is then

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a manifold that cannot be embedded in the plane (although it can be reduced to an embeddable region by a conformal map), and a function in $C(\overline{\Omega})$ will have distinct values on Γ_k and Γ_{k+1} corresponding to limits on the slit taken from different sides. One could also go further and let Ω be a polygonal Riemann surface, but we will not do this.

Let a set of real numbers $\theta_1, \ldots, \theta_n$ be given. At any point $z \in \Gamma_k$, let $u_n(z) = (\partial u/\partial n)(z)$ denote the inward normal derivative of a function u, and $u_{\sigma} = \partial u/\partial \sigma$ its tangential derivative along Γ_k in the positive (counterclockwise) sense with respect to Ω . This paper is concerned with the following *oblique derivative problem*:

Problem O. Find functions $u \in C^2(\Omega) \cap C^1(\Omega \cup \bigcup_k \Gamma_k)$ satisfying Laplace's equation

$$\Delta u(z) = 0, \quad z \in \Omega \tag{1.1}$$

together with the homogeneous oblique derivative boundary conditions

$$\cos \theta_k u_n(z) + \sin \theta_k u_n(z) = 0, \quad z \in \Gamma_k.$$
(1.2)

The situation is illustrated in Fig. 1(a). If $\theta_k \equiv 0 \pmod{\pi}$, we have a Neumann condition on side Γ_k , while $\theta_k \equiv \frac{1}{2}\pi \pmod{\pi}$ gives a tangential condition, which we do not rule out.

Obviously any constant function is a solution to Problem O. If Ω were a smooth domain with a continuous single-valued obliquity funciton $\theta(\sigma)$ for $\sigma \in \partial \Omega$, the constants would be the only solutions, as can be shown by consideration of Riemann-Hilbert problems [12,13,20]. More generally, if Ω were smooth and $\theta(\sigma)$ changed continuously by $K\pi$ as σ traversed $\partial \Omega$, there would be a solution space of finite dimension max{1, K}. But Problem O is quite different from the analogous problem on a smooth domain, for no boundary conditions have been specified at the vertices, nor have we required regularity or boundedness there. As a result the space of solutions has infinite dimension, so that in any particular application, additional conditions will be needed to ensure uniqueness. We have intentionally omitted such conditions from the statement of Problem O for the reason that their most natural form varies from problem to problem. The statements of Problems O_H and O_B in Sections 2 and 3 below incorporate appropriate additional conditions for the two applications considered there.

The purpose of this paper is to construct nonconstant solutions to Problem O by the following process of conformal mapping, illustrated in Fig. 1. Suppose an analytic function f in $\Omega \cup \bigcup_k \Gamma_k$



Fig. 1. Problem O and its solution by conformal mapping. The derivative of u at the boundary of Ω in the direction given by the arrow must be zero.



Fig. 2. Problem O in a rectangle with $\theta_2 = \theta_3 = 0$, $\theta_4 = \frac{1}{4}\pi$, $\theta_1 = -\frac{1}{4}\pi$.

is found that maps Ω onto another polygonal domain $f(\Omega)$. (f may fail to be globally or locally one-to-one, in which case we view $f(\Omega)$ not as a subset of \mathbb{C} but as a Riemann surface.) Suppose further that each image $f(\Gamma_k)$ consists of one or more sides of $f(\Omega)$ oriented at an angle θ_k (mod π) counterclockwise from the real axis. For each point $z \in \Gamma_k$, either f'(z) = 0, or f is conformal at z. Assuming the latter, we see that f maps any curve that meets Γ_k at z at an angle θ_k clockwise from the inward normal (Fig. 1(a)) onto a curve that meets $f(\Gamma_k)$ at f(z) with a vertical slope (Fig. 1(b)). In other words, the derivative of f at z with respect to arc length in the direction given by θ_k is pure imaginary. Now set

$$u(z) = \operatorname{Re} f(z). \tag{1.3}$$

Then the derivative of u in the direction given by θ_k is zero. Of course the same is true if f'(z) = 0. Thus in either case (1.3) satisfies (1.1) and (1.2), and we have found a solution to Problem O.

Conversely, every solution to Problem O is the real part of an analytic function that maps Ω onto a domain with straight sides. The following theorem is proved in the Appendix.

Theorem 1. A function u(z) defined in $\Omega \cup \bigcup_k \Gamma_k$ is a solution to Problem O if and only if $u(z) = \operatorname{Re} f(z)$, where f is an analytic function in $\Omega \cup \bigcup_k \Gamma_k$ that maps each interval Γ_k onto a subset of a straight line in \mathbb{C} oriented at the angle θ_k counterclockwise from the real axis.

To illustrate the conformal mapping construction we will now look at an example, which appears again in Section 3. Consider the oblique derivative problem shown in Fig. 2(a). The domain Ω is a rectangle bounded by four sides $\Gamma_1, \ldots, \Gamma_4$. On Γ_2 and Γ_3 we have Neumann conditions, and on Γ_4 and Γ_1 , oblique derivative conditions at 45° from the normal in the indicated directions. To apply the conformal mapping method, we must find a polygonal image domain $f(\Omega)$ whose sides are oriented as suggested in Fig. 2(b).

It can be shown that for this particular geometry, Problem O has no bounded non-constant solutions that are continuous throughout $\overline{\Omega}$ (see Theorem 3 below). Instead we must look for solutions that are unbounded near one or more of z_1, \ldots, z_4 . Figure 3 illustrates some of the possibilities.



Fig. 3. Various possible solution domains $f(\Omega)$ for the problem of Fig. 2. In each case $f(\Omega)$ lies to the left as the boundary is traversed in the direction 1-2-3-4-1.

First, Fig. 3(a) shows the domain $f(\Omega)$ for a simple solution with $f(z_3) = \infty$. Here each $f(\Gamma_k)$ is oriented as in the first column of Fig. 2(b). Figure 3(b) shows a similar domain with different orientations of $f(\Gamma_1)$ and $f(\Gamma_2)$. Figure 3(c) shows a domain with $f(z_2)$ instead of $f(z_3)$ at ∞ . (Any or all of the vertices of Ω may map to ∞ , but the C^1 hypothesis of Problem O implies that no other points of Ω or $\partial\Omega$ may map there.) Figure 3(d) shows a case with f'(z) = 0 at a point $z \in \Gamma_3$, so that the boundary component $f(\Gamma_3)$ bends back on itself. Figures 3(e) and 3(f) show domains with two vertices at ∞ . Figure 3(g) shows a domain with all four vertices at ∞ , bounded

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Fig. 4. Construction of f as a composition of two Schwarz-Christoffel maps.

by four slits, the endpoint of each of which is the image of a point as in Fig. 3(d) with f'(z) = 0. Figure 3(h) shows a case in which f is not one-to-one: $f(\Omega)$ is a two-sheeted polygon.

Obviously the possibilities of Fig. 3 could be multiplied ad infinitum. But if we exclude functions with essential singularities at the vertices, the set of all solutions is a vector space of countably infinite dimension that can be characterized in an orderly way by means of formulas related to the Schwarz-Christoffel transformation. This is true not just for the problem of Fig. 2 but for the general instance of Problem O. We will now present these formulas and indicate how they can be used to construct solutions numerically.

The first step, illustrated in Fig. 4, is to transplant the problem from Ω onto the upper half *t*-plane by a conformal map ϕ with $\phi(z_{k_{x}}) = \infty$ for some vertex $z_{k_{x}}$. Is is easy to see that Problem O is invariant under this transplantation in the sense that ϕ induces a one-to-one correspondence between solutions to Problem O on Ω and solutions to Problem O (with the same parameters θ_{k}) on the upper half plane viewed as a polygon with degenerate vertices at each $\phi(z_{k})$. It is well known that ϕ must have the Schwarz-Christoffel form

$$\phi^{-1}(t) = C_1 + C_2 \int_{T_{\phi}}^{t} \prod_{\substack{k=1\\k \neq k_{\infty}}}^{n} (t' - t_k)^{-\beta_k} \mathrm{d}t'$$
(1.4)

for some constants C_1 , $C_2 \in \mathbb{C}$, $T_{\phi} \in \mathbb{R}$ and prevertices $t_k = \phi(z_k)$ on the real axis [14]. Here $\beta_k \pi$ is the external angle of $\partial \Omega$ at z_k , with $-1 \leq \beta_k \leq 3$. From a practical point of view the difficulty is that the values $\{t_k\}$ are not known a priori. Determining ϕ numerically therefore requires an iteration to find these values. An efficient method for this has recently been developed by the first author and implemented in a Fortran package called SCPACK [16,17]. All of the computations of the next two sections were performed with SCPACK. A machine-readable copy of this package can be obtained by writing to Trefethen.

The second step, also illustrated in Fig. 4, is to map the upper half *t*-plane onto $f(\Omega)$ by a function $\psi(t)$. Essentially this is just another Schwarz-Christoffel map involving the same prevertices t_k and new angle parameters β'_k , but there are several new features. First, ψ may have branch points in the upper half plane; to handle these we must permit additional factors $(t' - s_k)(t' - \overline{s_k})$ in (1.4) [4]. Second, additional factors $t' - s_k$ with $s_k \in \mathbb{R}$ will have to be

included whenever there are points along some Γ_k with f'(z) = 0, as in Fig. 3(d) and 3(g). Both types of additional factors can be conveniently absorbed into a single polynomial p(t') with real coefficients. Finally, since $f(\Omega)$ is not required to be embeddable in the plane, the parameters β'_k are no longer constrained to lie in [-1, 3]. All together, the general map ψ with algebraic singularities only can be written in the form

$$\psi(t) = C_3 + e^{i\theta_{k_{\infty}-1}} \int_{T_{\psi}}^{t} p(t') \prod_{\substack{k=1\\k \neq k_{\infty}}}^{n} (t'-t_k)^{m_k-\beta'_k} dt', \quad \beta_k = \frac{1}{\pi} (\theta_k - \theta_{k-1}), \quad (1.5)$$

where p is a polynomial with real coefficients, each m_k is an integer, T_{ψ} is any point in (t_{k_x-1}, ∞) , C_3 is a complex constant, and the branch of each factor in the product is taken so that its value at T_{ψ} is positive. Note that rather than introduce a complex constant C_4 in front of the integral in analogy with (1.4), we have prescribed the phase of this constant explicitly and absorbed its modulus in p.

The following theorem, whose proof is given in the Appendix, confirms that f can always be constructed from (1.4) and (1.5) provided that the growth of u near the vertices is at most algebraic.

Theorem 2. Let ϕ be a conformal map (1.4) of Ω onto the upper half t-plane, and let ψ be the function (1.5) defined by an arbitrary choice of C_3 , T_{ψ} , p, and $\{m_k\}$ subject to the restrictions listed above. Then $u = \operatorname{Re} f$ with $f = \psi \circ \phi$ is a solution to Problem O. Conversely, if u is a solution to Problem O that satisfies $|u(z)| \leq \operatorname{const} |z - z_k|^{-M}$ for some M in an Ω -neighborhood of each finite vertex z_k , and $|u(z)| \leq \operatorname{const} |z|^M$ near any vertex z_k at ∞ , then $u = \operatorname{Re} f$ with $f = \psi \circ \phi$ for some ψ of the form (1.5).

From Theorem 2 one can obtain as a corollary, also derived in the Appendix. a complete description of all bounded, continuous solutions to Problem O:

Theorem 3. Let u be a solution to Problem O that is continuous on $\overline{\Omega}$, and let each θ_k be normalized to lie in $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$. If u is nonconstant, then it attains both its maximum and its minimum value at vertices z_k with

$$-\frac{1}{2}\pi < \theta_k < \theta_{k-1} \leqslant \frac{1}{2}\pi, \quad i.e. \ \beta_k' < 0 \tag{1.6}$$

(and possibly elsewhere also). If K is the number of vertices at which (1.6) is satisfied, then the set of solutions to Problem O in $C(\overline{\Omega})$ is the vector space of dimension $\max\{1, K\}$ obtained from (1.5) by taking $m_k = -1$ at vertices of type (1.6) and $m_k = 0$ at other vertices, and letting p be an arbitrary real polynomial of degree at most K-2.

These theorems make it clear that the numerical construction of solutions to Problem O should be manageable. As a practical observation, note that once ϕ has been found, the prevertices $\{t_k\}$ for ψ are known. Thus in situations with deg p = 0 and $m_k \equiv 0$ in (1.5), there is no parameter problem to be solved for ψ . Our Brownian motion application in Section 3 is essentially of this type. In such cases the lengths of the sides of the image domain $f(\Omega)$ cannot be chosen arbitrarily, but are constrained implicitly by Ω and $\{\theta_k\}$. In Fig. 3 we have acknowledged this situation by making each domain asymmetrical even when a more attractive symmetrical shape suggests itself (e.g., let the two horizontal lines in Fig. 3(a) lie at the same height). Such symmetrical solutions will be possible in that example only when the rectangle of Fig. 2(a) is a square.

When nontrivial terms p and m_k enter into the map ψ , on the other hand, their values are normally not known a priori, but are determined by extra conditions involving, for example, lengths of sides of $f(\Omega)$. Our Hall effect application in Section 2 is of this type. In these cases ψ is no longer known immediately, but it can be determined by solving a generalized Schwarz-Christoffel parameter problem. Since p enters (1.5) linearly, the generalized parameter problem is linear and easily solvable. The idea of generalized parameter problems was introduced in [18].

The great advantage of the method proposed in this paper from a numerical point of view, besides its elegance, is that it handles the singularities at the corners of Ω in an essentially exact way. We expect that it would be very difficult to obtain comparable performance from finite-difference or finite-element methods. This disadvantage of our method is that it is very special, being unable to treat more general domains, boundary conditions, or differential equations.

Boundary-value problems for elliptic equations in non-smooth domains have been studied previously by a number of mathematicians from the viewpoint of the theory of partial differential equations. For surveys with many references, see Grisvard [5] and Kondrat'ev and Oleinik [11]. In particular, Grisvard has investigated Poisson's equation in polygonal domains with oblique derivative (and other) boundary conditions. Our own work was motivated originally by the paper by Harrison, Landau, and Shepp [7].

2. Application 1: Hall effect

The classical *Hall effect* is an effect that arises when electric current flows through a conductor or semiconductor in the presence of a transverse magnetic field [24]. The magnetic field induces a sideways force on each moving charge particle, and this results in an altered current distribution characterized by a compensating voltage difference between one side of the conductor and the other. The measurement of this voltage difference is of fundamental importance in semiconductor electronics. In particular, its sign reveals whether the material being tested carries current primarily be electrons (type n) or by 'holes' (type p).

To be more precise, let Ω be a planar polygonal domain as in Problem O, which we will think of as having been cut from a thin material of uniform electrical conductivity. See Fig. 5. On each boundary arc Γ_k with k in an index set $\Sigma \subseteq \{1, \ldots, n\}$, a fixed constant voltage $u = u_k \in \mathbb{R}$ will be applied, while the remainder of the boundary is insulated. We assume that Σ contains no adjacent pairs k, k + 1 except with $u_k = u_{k+1}$. Let $K \leq \frac{1}{2}n$, $K \geq 1$ be the number of disjoint components of $\bigcup_{k \in \Sigma} \overline{\Gamma}_k$, that is, the number of separate voltages applied on $\partial \Omega$.

Let a constant magnetic field **B** be applied perpendicular to Ω . The resulting steady-state voltage u(z), electric field E(z), and current density J(z) are related by

$$E = -\nabla u, \tag{2.1}$$

$$\boldsymbol{J} = \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B},\tag{2.2}$$

where we have set physical constants to 1 for convenience. If the real 2-vectors E and J are



Fig. 5. The Hall effect (Problem O_H) in a rectangle.

interpreted as complex scalars and B = |B|, (2.2) becomes

$$E = (1 - iB)J.$$
 (2.3)

Thus E and J form a constant angle $\alpha = \tan^{-1}B$ at every point $z \in \Omega$:

$$|J|/|E| = \cos \alpha, \quad \arg J - \arg E \equiv \alpha, \tag{2.4}$$

and α is called the *Hall angle*. Without loss of generality we may assume $|\alpha| < \frac{1}{2}\pi$.

It follows from (2.4) that the problem of determining u has the form of Problem O. On each side with $k \in \Sigma$, u is constant, which implies that it satisfies an oblique derivative boundary condition with $\theta_k = \frac{1}{2}\pi$. On each side with $k \notin \Sigma$, there can be no current through the boundary, i.e. J is tangent to Γ_k . By (2.1), (2.3) and (2.4), ∇u therefore makes an angle α clockwise from Γ_k in this case, which implies that the directional derivative boundary condition is indeed satisfied on every side of $\partial\Omega$. But in addition we have two further conditions: the values $u = u_k$ on Γ_k are specified for $k \in \Sigma$, and any physically meaningful solution must be continuous. We combine all of this into the following general Hall effect problem.

Problem O_H. Let Ω , Σ , u_k , and α be given as described above. Find a solution to Problem O subject to the oblique derivative boundary conditions (1.2) with

$$\theta_k = \frac{1}{2}\pi$$
 for $k \in \Sigma$, (2.5a)

$$\theta_k = \alpha \quad \text{for } k \notin \Sigma, \quad |\alpha| < \frac{1}{2}\pi,$$
(2.5b)

together with the additional conditions

$$u \in C(\overline{\Omega}), \tag{2.5c}$$

$$u(z) = u_k$$
 for all $z \in \Gamma_k$, $k \in \Sigma$. (2.5d)

In the last section we saw that in general Problem O has many solutions. However, the extra conditions (2.5c), (2.5d) of Problem O_H suffice to ensure uniqueness. In the following theorem let ϕ be a conformal map of Ω onto Im t > 0 as in Section 1. Let $\{q_k\}, 1 \le k \le K$, be the subset of $\{t_k\}$ at which the boundary changes from type $k \notin \Sigma$ to type $k \in \Sigma$, and let $\{r_k\}, 1 \le k \le K$, be the subset at which it reverts to $k \notin \Sigma$. In the product below, a factor should be omitted if the corresponding point q_k or r_k lies at ∞ .

Theorem 4. Problem O_H has a unique solution, which can be represented in the form $u = \text{Re}(\psi \circ \phi)$ with

$$\psi(t) = C_3 + e^{i\theta_{k_{\infty}-1}} \int_{T_{\psi}}^{t} p(t') \prod_{k=1}^{K} (t' - q_k)^{\alpha/\pi - 1/2} (t' - r_k)^{-\alpha/\pi - 1/2} dt', \qquad (2.6)$$

where p is a real polynomial of degree at most K-2.

Proof. First consider Problem O_H with condition (2.5d) omitted. By Theorem 3, the solutions are precisely the functions $u = \operatorname{Re}(\psi \circ \phi)$ with ψ given by (2.6), which constitute a vector space of dimension K. Thus what we have to show is that the addition of (2.5d) leads to existence and uniqueness. Since the number of equations in (2.5d) matches the dimension K of the vector space, it is enough to show just one of these.

We prove uniqueness. If u_1 and u_2 are two distinct solutions, then their difference $u = u_1 - u_2$ is harmonic is Ω , continuous on $\overline{\Omega}$, zero on Γ_k for $k \in \Sigma$, but not zero everywhere. By Theorem 3, the maximum and minimum of u on $\overline{\Omega}$ must be attained at vertices z_k satisfying (1.6). But by (2.5a-b), the only such vertices have $k - 1 \in \Sigma$, hence by continuity $u(z_k) = 0$. This is a contradiction.

An alternative proof of existence can be based on Theorems 1.4.3.1, 1.4.4.1, 4.4.2.1, and 5.1.3.1 of [5]. \Box

The idea of constructing solutions to problem O_H by Schwarz-Christoffel maps goes back to an impressive paper by Wick in 1954 [24] and has been pursued more recently by Versnel [21,22]. To begin with, consider the simple case in which Ω is a rectangle of height 1 and length L as in Fig. 5(a). Here K = 2, so p reduces to a constant in (2.6). To determine this constant and C_3 , one can simply set up a Schwarz-Christoffel integral (2.6), then shift and scale the resulting parallelogram so that $\operatorname{Re} f(\Gamma_1) = u_1$ and $\operatorname{Re} f(\Gamma_3) = u_3$. Once this is done, the height h of $f(\Omega)$ can be determined by evaluating an integral. This amounts to a determination of the total current I from Γ_3 to Γ_1 , for I is equal to the integral of the magnitude of the normal component of J along Γ_1 , or equivalently along $f(\Gamma_1)$, since current is invariant under conformal transformation. By (2.4), we have $|E| \equiv 1$ and $|J| \equiv \cos \alpha$ along $f(\Gamma_1)$, so the normal component of J has constant magnitude $\cos^2 \alpha$. Therefore

$$I = h \cos^2 \alpha. \tag{2.7}$$

Figure 6 shows the result of a such a calculation with L = 2, $\alpha = \frac{1}{6}\pi$, and $u_3 - u_1 = 1$. The numerically determined height and current are h = 0.615567 and I = 0.461675 (the domain and range are plotted on different scales). In each region a set of streamlines and equipotential lines is shown, meeting everywhere at an angle $\frac{1}{2}\pi - \alpha$. The grid in Fig. 6(a) is simply the conformal image in Ω computed by SCPACK of the grid of Fig. 6(b) in $f(\Omega)$. Table 1 records computed currents *I* for several other combinations of parameters.

If Ω is a more general polygon with $K \ge 3$ prescribed voltages, the picture changes qualitatively because now (2.6) contains a nontrivial polynomial p, and this introduces slits and/or branch points in $f(\Omega)$. For example, consider the 6-gon shown on the left in Fig. 7, whose vertices are -1 + i, -1, 0, $\frac{1}{2} - \frac{1}{2}i$, $2 - \frac{1}{2}i$, 2 + i. In this problem voltages $u_1 = 0$, $u_3 = U$, and $u_5 = 1$ are prescribed on sides Γ_1 , Γ_3 , and Γ_5 , while Γ_2 , Γ_4 , and Γ_6 are insulated. By Theorem 4 we know that the former three sides must be mapped to vertical lines and the latter to inclined ones,



Fig. 6. Numerical solution of the Hall effect problem of Fig. 5 with $\alpha = \frac{1}{6}\pi$. Streamlines and equipotential lines are plotted.

and that the polynomial p of degree 1 in (2.6) may introduce a slit in the boundary of the sort in Figs. 3(d) and 3(g). Let us write $p(t) = \text{const}(t - t_*)$, so that t_* marks the 'extra prevertex' corresponding to the end of the slit. For arbitrary U, t_* might lie anywhere on \mathbb{R} , but let us assume 0 < U < 1. Then there are exactly three possible topologies for $f(\Omega)$, and these are shown in Fig. 7. In each case one of the sides of $f(\Omega)$ protrudes beyond the vertex it might be expected to stop at, then doubles back, so that $f(\Omega)$ is not a 6-gon but a 7-gon, with vertices w_1, \ldots, w_6 and w_* . The three cases differ in the order in which the vertices and prevertices appear around the boundary:

- (a) $w_1, w_2, w_*, w_3, w_4, w_5, w_6,$
- (b) $w_1, w_2, w_3, w_*, w_4, w_5, w_6,$
- (c) $w_1, w_2, w_3, w_4, w_*, w_5, w_6$.

Table 1

Numerically determined currents $I = h \cos^2 \alpha$ as a function of Hall angle α and rectangle length L for the problem of Fig. 5

L	$\alpha = 0^{\circ}$	$\alpha = 15^{\circ}$	$\alpha = 30^{\circ}$	$\alpha = 45^{\circ}$	$\alpha = 60^{\circ}$	$\alpha = 75^{\circ}$	$\alpha = 90^{\circ}$
0.5	2.000000	1.901557	1.624521	1.220042	0.761746	0.330865	0
1.0	1.000000	0.965926	0.866025	0.707107	0.500000	0.258819	0
1.5	0.666667	0.650371	0.600951	0.516643	0.393868	0.225911	0
2.0	0.500000	0.490657	0.461675	0.409822	0.328193	0.202461	0
2.5	0.400000	0.393982	0.375031	0.340018	0.281797	0.183762	0
3.0	0.333333	0.329141	0.315803	0.290595	0.246974	0.168283	0
3.5	0.285714	0.282629	0.272736	0.253726	0.219825	0.155219	0
4.0	0.250000	0,247634	0.240006	0.225161	0.198055	0.144040	0



Fig. 7. Hall effect on a 6-gon—introduction of a slit of unknown length in $f(\Omega)$.

They also differ physically:

- (a) current flows into Γ_3 only,
- (b) current flows both into and out of Γ_3 ,
- (c) current flows out of Γ_3 only.

Of course it may happen that $f(\Omega)$ has no slit in it, but only when the problem by chance falls on the borderline between (a) and (b) or (b) and (c).

If U is specified in Fig. 7, we do not know a priori which of the case (a)–(c) will occur, nor do we know the correct slit length or any of the other vertical dimensions of $f(\Omega)$. On the other hand we do know its horizontal (i.e. oblique) dimensions. This is why the determination of ψ is a generalized parameter problem, in contrast to the usual Schwarz-Christoffel situation in which



Fig. 8. Numerical solution of the Hall effect problem of Fig. 7, $\alpha = \frac{1}{6}\pi$.



Fig. 9. Numerical solution of the Hall effect problem of Fig. 7, $\alpha = 0$.

all dimensions of the image polygon are known at the start. But since the generalized parameter problem is linear, it is easy to solve despite the complexity of its geometric interpretation. A related but nonlinear generalized parameter problem is described in the paper in this volume by Elcrat and Trefethen on free-streamline flows [3].

For a practical illustration with m > 2, we computed the numerical solution shown in Fig. 8 with $U = \frac{1}{2}$, $\alpha = \frac{1}{6}\pi$. One sees that in this case the solution has turned out to be of type (b). The current between each adjacent pair of streamlines is 0.05. The total currents are 0.449016 out of Γ_5 , 0.166622 into Γ_3 , 0.045959 out of Γ_3 , and 0.328354 into Γ_1 .

The numerical method we have described in this section is new even in the case $B = \alpha = 0$ in which the magnetic field is not present, so all that remains is a mixed Dirichlet-Neumann problem. Now u is the real part of a conformal map f onto a domain with horizontal and vertical sides and with sides and slits of unknown length. To illustrate, Fig. 9 shows a computation for the same problem as in Fig. 8 but with $\alpha = 0$. Again the solution turns out to be of type (b). Now the total currents are 0.452474 out of Γ_5 , 0.136224 into Γ_3 , 0.079972 out of Γ_3 , and 0.396222 into Γ_1 .

To show that cases (a) and (c) can indeed occur, Fig. 10 presents a 3×3 array of numerically determined boundary polygons $f(\partial \Omega)$ as in Figs. 8 and 9 for the problem of Fig. 7 with



Fig. 10. Image polygons $f(\partial \Omega)$ for the Hall effect problem of Fig. 7 for various α and U.

 $\alpha = -\frac{1}{6}\pi$, 0, $\frac{1}{6}\pi$ and U = 0.1, 0.5, 0.9. In this experiment we have evidently obtained case (a) with U = 0.1, (b) with U = 0.5, and (c) with U = 0.9.

Most of the computer time in any SCPACK computation is ultimately spent in the calculation of complex logarithms needed to evaluate the powers of $(t' - t_k)$ in (1.4) and (1.5). As a result, one can conveniently estimate the work done by SCPACK in a machine-independent way by counting logarithms. In practice the most time-consuming parts of the computation are the solution of nonlinear parameter problems, where the cost increases roughly in proportion to n^3 , and the construction of plots, where mapping of many points is required. Approximate logarithm counts for the computations of this section are 90 000 for Fig. 6, 80 000 for Table 1, 180 000 each for Figs. 8 and 9, and 20 000 for Fig. 10.

In their papers on the Hall effect, Wick and Versnel consider the situation in which two edges are held at fixed voltages while several others are passive conductors whose voltage is free to vary, but across which no net current may flow [21,22,24]. This leads to a somewhat different linear generalized parameter problem in which some of the equations enforce the condition that the two sides of a slit must have equal (but unknown) length. Wick and Versnel confine themselves to symmetrical domains for which the map ϕ can be obtained more or less analytically.

3. Application 2: Reflected Brownian motion

The mathematical concept of two-dimensional *Brownian motion* is a formalization of the idea of random motion of a particle in the plane [10]. Under suitable restrictions, Brownian motion confined to a domain can be described by a probability density p(z, t) that diffuses with taccording to the heat equation, and in the limit $t \to \infty$ converges to a steady-state probability density u(z) that satisfies Laplace's equation. Of course the nature of u will depend on boundary conditions, which describe the behavior of the particle when it hits the boundary. The connection of this topic with the present paper is that in several problems of practical interest, the appropriate boundary conditions for the steady-state density involve *oblique reflection*, which leads to instances of Problem O. In particular, in *queueing theory*, Brownian motion with oblique reflection at the boundary of a polygonal domain arises as a diffusion approximation to the queue-length process of a tandem queue under conditions of heavy traffic (approximate equality of average arrival and service rates) [6]. The same problem arises under similar conditions in *storage theory* in modeling the contents process of a two-buffer production or storage network [2,23].

To be more precise, let Ω and $\theta_1, \ldots, \theta_n$ be defined as in Problem O, with $|\theta_k| < \frac{1}{2}\pi$ for all k. The goal is to construct the steady-state density u of a strong Markov process Z with continuous sample paths that (a) behaves like Brownian motion in the interior of Ω and (b) is reflected instantaneously on each side Γ_k at the angle $-\theta_k$ from the normal. It is known that such a process can always be constructed with absorption at the vertices of Ω . However, if Ω is a bounded domain with external angle parameters β_k in the range (-1, 1), and if the reflection does not provide too great a 'push' towards any finite vertex, then it has also been shown that there is also a unique such process that spends zero time (in the sense of Lebesgue measure) at the vertices. A necessary and sufficient condition for this is

$$\beta_k < 1 - (\theta_k - \theta_{k-1})/2\pi \quad \text{for all } k. \tag{3.1}$$

The formal definition of the reflected Brownian motion process and the derivation of these results follow from the work of Varadhan and Williams [19] by localization, as explained by Harrison, Landau, and Shepp in [7].

For example, consider again Fig. 2(a). Here $\theta_2 = \theta_3 = 0$, $\theta_4 = -\theta_1 = \frac{1}{4}\pi$, and $\beta_k = \frac{1}{2}$ for k = 1, 2, 3, 4. The reflected Brownian motion for this geometry can be interpreted as the diffusion limit of a system of two finite-capacity queues Q_1 and Q_2 in parallel, where x and y positions in the rectangle represent the normalized numbers of customers waiting in Q_1 and Q_2 , respectively, and the queues are coupled as follows: if Q_1 becomes empty, it takes customers from Q_2 , while if Q_2 becomes full, it transfers customers to Q_1 . Since (3.1) is satisfied, there is a unique process Z that spends zero time at the vertices, behaves like Brownian motion inside the rectangle, and is reflected instantaneously in the normal direction on Γ_2 and Γ_3 and at angles $\frac{1}{4}\pi$ to the right on Γ_4 and $\frac{1}{4}\pi$ downwards on Γ_1 . Note that in keeping with the minus sign in condition (b) above, the directions of reflection on each side are not the same as the directions in which the derivative is zero, but are equal to these directions flipped about the respective normals.

Our purpose is to compute the steady-state density u for reflected Brownian motion on a bounded polygon Ω by taking the real part of a conformal map f of the kind described in Theorems 1 and 2. As usual, the statement of Problem O does not by itself include enough conditions to characterize u. The additional conditions needed turn out to be quite different from those that made sense for the Hall effect problem. First, since u is a probability density it must be nonnegative— $f(\Omega)$ must lie in the right half plane. Second, the total probability $\int_{\Omega} u$ must be finite and equal to 1—this determines the scale for $f(\Omega)$. Finally, there is a third restriction that is less obvious. This is a global 'corner condition' that amounts to a statement that particles may not disappear at one vertex and reappear at another.

Corner condition. If a and b are two points on any two sides Γ_j and Γ_k , respectively, with $\theta(a) = \theta_j$ and $\theta(b) = \theta_k$, and Λ is a differentiable curve in Ω that joins a to b, then

$$-\int_{\Lambda} \frac{\partial u}{\partial n}(z) d\sigma = u(b) \tan \theta(b) - u(a) \tan \theta(a), \qquad (3.2)$$

where $\partial/\partial n$ denotes the derivative in the normal direction to Λ that lies to the left as Λ is traversed from a to b.

Expressed in this form the corner condition may seem obscure, but there is a simple equivalent statement in terms of the analytic function f: it must be possible to choose f so that $f(\Omega)$ is a radial domain. By this we mean that all of the boundary arcs $f(\Gamma_k)$ are subsets of straight lines passing through the origin, hence in this particular application,

$$\arg f(z) \equiv \theta_k \pmod{\pi} \quad \text{for all } z \in \Gamma_k. \tag{3.3}$$

This observation was first made by Newell [15]. To see that (3.2) and (3.3) are equivalent, note that if f = u + iv, then by the Cauchy-Riemann conditions, $-\partial u/\partial n = \partial v/\partial \sigma$. Therefore the left side of (3.2) is equal to v(b) - v(a), so (3.2) amounts to the statement that $v(z) = u(z) \tan \theta(z) + C$ on each Γ_k , for some fixed constant C. By choosing v so that this constant is zero, (3.2) is reduced to (3.3).

The determination of conformal maps onto radial domains is easier than the general Schwarz-Christoffel problem, since arg f rather than just arg f' is known on each side Γ_k . By



Fig. 11. Reflected Brownian motion (Problem O_B) in a rectangle.

repeating the proof of Theorem 2 with appropriate changes, one can show that assuming u grows at most algebraically at each vertex, f can be written in the form $f = \psi \circ \phi$ with

$$\psi(t) = e^{i\theta_{k_{\infty}^{-1}}}p(t)\prod_{\substack{k=1\\k\neq k_{\infty}}}^{n} (t-t_{k})^{m_{k}-\beta_{k}'}$$
(3.4)

in analogy to (1.5). Thus no integration is required for ψ , which is of course a considerable help in numerical computations. We will see further in Theorem 5 below that if u is nonnegative, then p and $\{m_k\}$ in (3.4) can be omitted.

Returning to the example of Fig. 2, recall that many solutions to Problem O were depicted in Fig. 3. In the present situation all of these except Fig. 3(f) are excluded by the nonnegativity condition on u. Figure 3(f), in turn, is excluded by the condition that $f(\Omega)$ must be radial. To get a radial domain, the slit w_2 - w_3 - w_4 in Fig. 3(f) must be lowered to the real axis. Unless Ω is a square, w_3 will have to move away from the end of the slit to make this possible. Thus the proper solution to the reflected Brownian motion problem of Fig. 2 is given by the conformal map indicated in Fig. 11. The new symbols z_* and w_* denote the end of the slit and its preimage on $\partial\Omega$.

We are now ready to formulate the variant of Problem O motivated by the reflected Brownian motion problem. Note that in contrast to Problem O_H , here we do not require $u \in C(\overline{\Omega})$, and indeed often u will be infinite at a vertex.

Problem O_B. Let Ω and θ_k be given as usual with $|\theta_k| < \frac{1}{2}\pi$ for each k. Find a solution to Problem O subject to the additional conditions

- (i) $u(z) \ge 0$ for all $z \in \Omega \cup \bigcup_k \Gamma_k$;
- (ii) $u \in L^1(\Omega)$ with $\int_{\Omega} u = 1$;
- (iii) the corner condition (3.2) is satisfied.

Like Problem O_H , Problem O_B is well posed. As usual, in the following theorem ϕ is a fixed conformal map (1.4) of Ω onto Imt > 0.

Theorem 5. Problem O_{β} has a solution if and only if (3.1) holds for each k. The solution is unique,

and can be represented in the form $u = \operatorname{Re}(\psi \circ \phi)$ with

$$\psi(t) = U e^{i\theta_{k_{\infty}-1}} \prod_{\substack{k=1\\k\neq k_{\infty}}}^{n} (t-t_{k})^{-\beta_{k}'}, \qquad \beta_{k}' = \frac{1}{\pi} (\theta_{k} - \theta_{k-1}), \ U \in (0, \infty).$$
(3.5)

Proof. Suppose u in a solution to Problem O_B . By Theorem 1, u can be written as $u = \operatorname{Re} f = \operatorname{Re}(\psi \circ \phi)$ for some function ψ analytic in $\operatorname{Im} t \ge 0$ except possibly at the points $\{t_k\}$. By conditions (i) and (iii) (= (3.3)), ψ must map an interval $[t_k - \epsilon, t_k)$ into the ray $[0, \infty)e^{i\theta_{k-1}}$, an interval $(t_k, t_k + \epsilon]$ into the ray $[0, \infty)e^{i\theta_k}$, and the half-plane $\operatorname{Im} t > 0$ into the half-plane $|\arg w| \le \frac{1}{2}\pi$. Therefore the function

$$g_k(t) = \psi(t)(t - t_k)^{\beta_k}$$

maps both of these intervals into $[0, \infty)e^{i\theta_k}$. If $\theta_{k-1} \leq \theta_k$, the values assumed by g_k in $\operatorname{Im} t > 0$ satisfy $-\frac{1}{2}\pi \leq \arg w \leq \frac{1}{2}\pi + \theta_k - \theta_{k-1}$, while if $\theta_{k-1} \geq \theta_k$, they satisfy $-\frac{1}{2}\pi + \theta_k - \theta_{k-1} \leq \arg w$ $\leq \frac{1}{2}\pi$. Since $|\theta_{k-1}| < \frac{1}{2}\pi$ by assumption, i.e., $|\theta_{k-1}| = \frac{1}{2}\pi - \delta$ with $\delta > 0$, we have in either case

$$|\arg g_k(t) - (\theta_k \pm \pi)| \ge \delta > 0$$

for Imt > 0. In other words, g_k takes no values in the sector of half-angle $\delta > 0$ about the ray $-[0, \infty)e^{i\theta_k}$.

Now g_k can be extended to Im t < 0 by reflection in either $[t_k - \epsilon, t_k)$ or $(t, t_k + \epsilon]$. Since both of these intervals map into the same ray $[0, \infty)e^{i\theta_k}$, the extension is independent of which interval is chosen. Thus g_k is a single-valued analytic function in a punctured neighborhood of t_k , and must have an essential singularity, a pole, or a removable singularity at t_k . But by the geometry of the reflection process it is clear that the extended function g_k also takes no values in the sector described above. Therefore t_k can only be a removable singularity.

These arguments imply that

$$\psi(t)\prod_{\substack{k=1\\k\neq k_{\infty}}}^{n}(t-t_{k})^{\beta_{k}^{\prime}}$$

can be extended by reflection in \mathbb{R} to a function that is analytic for all $t \in \mathbb{C}$. Moreover, by reasoning as above it follows that it is also analytic at $t = \infty$. Therefore it is a constant, and since its argument is $e^{i\theta_{k_{\infty}}-1}$, this establishes the representation (3.5).

At this point we have shown that any solution u to Problem O_B has the form claimed. Conversely, given an instance of Problem O_B , let u be constructed as $\operatorname{Re}(\psi \circ \phi)$ from (3.5). It is easy to see that conditions (i) and (iii) are satisfied. Turning to (ii), it is clear that u will be integrable on Ω if and only if it is integrable near each z_k . If z_k is finite, f has a singularity of type

$$|z-z_k|^{-(\theta_k-\theta_{k-1})/\pi(1-\beta_k)}$$

there, and u is integrable if and only if the exponent is greater than -2, i.e., if and only if (3.1) holds. A similar consideration gives (3.1) also for $z_k = \infty$. This establishes the existence claim of the theorem. Since the parameter U must be chosen to ensure $\int_{\Omega} u = 1$, the solution is unique. \Box

By a slight modification of the results of [7] and [25], it follows that the solution u of Problem O_B given by Theorem 5 is the unique stationary density for the Markov process Z, provided that



Fig. 12. Image domain $f(\Omega)$ for the reflected Brownian motion problem of Fig. 11 with L = 2. The vertical lines represent probability densities u = 0.1, 0.2, ..., 2, and the arrow indicates w_3 .

 Ω is bounded with $\beta_k \in (-1, 1)$ for each k. As conjectured in [7] but not yet proved, the same result probably holds for unbounded polygons with $\beta_k \in (-1, \frac{3}{2})$. With $\beta_k \ge \frac{3}{2}$ there is no chance of satisfying (3.1).

We will now illustrate the Schwarz-Christoffel solution of Problem O_B by giving results computed with SCPACK for the problem of Fig. 11. (Actually this geometry is simple enough that it could have been treated semi-analytically via elliptic functions.) Let Ω have height 1 and length L. In $f(\Omega)$, any vertical line $\operatorname{Re} f = \operatorname{const}$ is a line of constant probability density. Figure 12 shows $f(\Omega)$ together with a system of such lines $u = 0.1, 0.2, \ldots, 2$ for the case L = 2. By mapping these lines back to Ω , we obtain curves of constant probability density in the problem domain which reveal quite a bit about the steady-state behavior of the reflected Brownian motion. Figure 13 shows results of this kind for L = 1, 2, 3. Beginning at the upper-left corner, we have u = 0 at z_1 , then the curves $u = 0.1, 0.2, \ldots, 2$ in succession. Note that u has poles at both z_2 and z_4 , but that as L increases, the former rapidly weakens in strength. The explanation for this is that the rightward motion imparted to the process by the upper side makes the probability density decay exponentially with distance from the right side. At L = 3 the right-wall behavior is already very close to what it would be in the limit $L = \infty$, a case investigated earlier by Harrison and Shepp in [9]. In this limit w_* will approach 0, and the lower half of $f(\Omega)$ will correspond to a small portion of Ω near z_2 .

Similar plots showing the behavior of u near the corners have been constructed previously by Newell by hand. See [15, Figs. IV-13–IV-15].

Table 2 lists computed values of

$$I_k = \int_{\Gamma_k} u, \quad k = 1, 2, 3, 4, \tag{3.6}$$

and

$$\langle x \rangle = \int_{\Omega} x u(z), \quad \langle y \rangle = \int_{\Omega} y u(z)$$
 (3.7)



Fig. 13. Numerical solution of the reflected Brownian motion problem of Fig. 11 with L = 1, 2, 3. The curves are lines of constant probability density u = 0.1, 0.2, ..., 2 obtained by applying f^{-1} to Fig. 12 or its analogs for L = 1, 3.

for various lengths L, together with corresponding quantities w_* and w_3 . Here the integrals along Γ_k are taken with respect to arc length and those over Ω are taken with respect to Lebesgue measure in \mathbb{R}^2 . The number I_k can be interpreted as a measure of the rate at which the Markov process Z hits side Γ_k in the steady state, or alternatively, as a measure of the average amount of 'pushing' that must be exerted on Γ_k to keep the Brownian motion in Ω . $\langle x \rangle$ and $\langle y \rangle$ represent the long-run average x and y positions. In practice these quantities are most conveniently computed as one-dimensional integrals by means of the identities

$$\langle x \rangle = \frac{1}{2} L^2 I_3 - \frac{1}{2} \int_{\Gamma_4} u x^2, \quad \langle y \rangle = \frac{1}{2} I_4 + \frac{1}{2} \int_{\Gamma_i} u y^2.$$
 (3.8)

L	<i>w</i> *	w ₃	I ₁	$I_2 = I_3$	I ₄	$\langle x \rangle$	<i>(y)</i>
0.5	1.693474	2.912616	1.871418	2.412890	0.541472	0.258585	0.380018
1.0	1.138859	1.138859	0.805295	1.610589	0.805295	0.551506	0.448494
1.5	0.686627	0.831998	0.446669	1.340225	0.893557	0.879534	0.471624
2.0	0.423368	0.728154	0.270736	1.206445	0.935709	1.239964	0.482830
2.5	0.268692	0.678353	0.171214	1.130587	0.959373	1.628342	0.489146
3.0	0.174154	0.649951	0.110891	1.084582	0.973691	2.040075	0.492970
3.5	0.114465	0.632412	0.072873	1.055585	0.982712	2.471022	0.495381
4.0	0.075922	0.621165	0.048334	1.036868	0.988534	2.917572	0.496936
×	0	0.599070	0	1	1	L-1.242454	$\frac{1}{2}$

Table 2 Numerically determined quantities w_* , w_3 , I_k , $\langle x \rangle$, $\langle y \rangle$ as a function of rectangle length L.

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These relations were obtained by substituting $g = x^3$ and $g = y^3$ in the following equation derived by applying Green's theorem to smooth domains approximating Ω and taking limits, using property (3.2) of u and the behavior of u near corners:

$$\int_{\Omega} u\Delta g + \sum_{k=1}^{7} \int_{\Gamma_{k}} uD_{k}g = 0 \quad \text{for all } g \in C^{2}(\overline{\Omega}),$$
(3.9)

where

$$D_k g = g_n - \tan \theta_k g_\sigma, \quad k = 1, \dots, 4.$$

Similarly, the magnitude of the constant U in (3.5) needed to accomplish the normalization (ii) can be determined by computing $\int_{\Omega} u$ as the one-dimensional integral

$$\int_{\Omega} u = I_4 + \int_{\Gamma_1} y u \tag{3.10}$$

derived by substituting $g = y^2$ in (3.9). For numerical evaluation of (3.8) and (3.10) a natural method is Gauss-Jacobi quadrature, since the integrands are analytic except for algebraic singularities of known type at the endpoints. This approach conforms nicely to the spirit of SCPACK, where all Schwarz-Christoffel integrals are already computed by compound Gauss-Jacobi quadrature using a well-known program by Golub and Welsch referenced in [16].

By substituting g = x and g = y in (3.9), one obtains the interesting balance relations

$$I_2 = I_4 + I_1 = I_3 \tag{3.11}$$

(compare [15, equation III.2.3f]). The left and right equalities here mirror the fact that in the steady state, there can be no force of the sides on the Brownian motion in the vertical and horizontal directions, respectively. These identities are verified numerically by the data in Table 2. The final line in the table lists the limiting value of each column as $L \to \infty$, where Ω becomes the half-strip considered in [9]. The limit for w_3 has been calculated previously in equation (1.2) of [9],

$$\lim_{L\to\infty} w_3 = \pi \sqrt{2\pi} / \Gamma\left(\frac{1}{4}\right)^2 \approx 0.599070.$$

The limit

 $\lim_{L \to \infty} L - \langle x \rangle \approx 1.242454$

was obtained numerically with SCPACK by treating the oblique derivative problem for a semi-infinite strip directly. Each column in Table 2 apparently approaches its limit geometrically, with an error asymptotically proportional to $\exp(-\frac{1}{4}\pi L)$.

As in the last section, we can measure the bulk of the effort required for these computations in a machine-independent way by counting complex logarithms. Roughly, Figs. 11 and 12 required 70 000 logs and Table 2 required 180 000.

Appendix: basic theorems on the oblique derivative problem

In this appendix we give three proofs that were omitted from Section 1 in the interest of readability. The first argument confirms that all solutions to Problem O can be obtained by conformal mapping.

Proof of Theorem 1. The proof of the 'if' part is straightforward, and was sketched in the discussion in Section 1. To prove the 'only if' part, let u be a solution to Problem O. Since $u \in C^2(\Omega) \cap C^1(\Omega \cup \bigcup_k \Gamma_k)$, the function

$$g(z) = u_x(z) - iu_y(z) \tag{A.1}$$

belongs to $C^1(\Omega) \cap C^0(\Omega \cup \bigcup_k \Gamma_k)$, and since *u* is harmonic in Ω , *g* satisfies the Cauchy-Riemann equations and is therefore analytic in Ω . Suppose Γ_k is inclined at an angle η_k counterclockwise from the real axis. By an elementary calculation based on (1.2) and (A.1), one can verify that

$$\arg g(z) \equiv \theta_k - \eta_k \pmod{\pi} \quad \text{for all } z \in \Gamma_k. \tag{A.2}$$

Therefore g maps each straight boundary segment Γ_k into a subset of the straight line in the complex plane passing at angle $\theta_k - \eta_k$ through the origin. By the reflection principle, it follows that g can be extended analytically across Γ_k . Thus g is analytic in $\Omega \cup \bigcup_k \Gamma_k$.

Let $z_* \in \Omega$ be arbitrary, and consider the new function

$$f(z) = u(z_{*}) + \int_{z_{*}}^{z} g(z') dz', \qquad (A.3)$$

where the path of integration is any rectifiable arc in $\Omega \cup \bigcup_k \Gamma_k$ that joins z_* to z. Since g is analytic and Ω is simply connected, f(z) is single-valued and analytic throughout $\Omega \cup \bigcup_k \Gamma_k$. Moreover for $z \in \Omega \cup \bigcup_k \Gamma_k$, we have

$$u(z) = u(z_{*}) + \int_{z_{*}}^{z} u_{x} dx + u_{y} dy$$

= $u(z_{*}) + \int_{z_{*}}^{z} \operatorname{Re}\{(u_{x} - iu_{y})(dx + i dy)\} = \operatorname{Re} f(z).$

Now for any two points $z_1, z_2 \in \Gamma_k$, (A.3) implies

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} g(z) dz.$$

Since dz has constant argument $\eta_k \pmod{\pi}$ along Γ_k , it follows from (A.2) that $f(z_2) - f(z_1)$ has argument $\theta_k \pmod{\pi}$. Therefore f maps Γ_k into a straight line oriented at angle θ_k counterclockwise from the real axis, as claimed. \Box

The next argument shows that f can be written in Schwarz-Christoffel form, provided that the condition of algebraic growth at the vertices is satisfied.

Proof of Theorem 2. Suppose first that ψ is a map of the form (1.5). By construction, the integrand of (1.5) (including the factor in front) has argument $\theta_{k_x-1} \pmod{\pi}$ at T_{ψ} and therefore on $\phi(\Gamma_{k_x-1})$. Following the changes in argument of each factor $(t'-t_k)^{-\beta'_k}$ as t' moves leftward along the real axis, we see that the argument is $\theta_k \pmod{\pi}$ on each interval $\phi(\Gamma_k)$. Therefore ψ is an analytic function in $\operatorname{Im} t \ge 0$ that maps each $\phi(\Gamma_k)$ into a subset of a straight line oriented at the angle θ_k from the real axis. By Theorem 1, it follows that $u = \operatorname{Re}(\psi \circ \phi)$ is a solution to Problem O.

Conversely, let u be a solution to Problem O satisfying the given bounds at each vertex. Assume without loss of generality that we have already transplanted the problem so that Ω is the upper half *t*-plane; since the Schwarz-Christoffel map ϕ has only algebraic singularities at the corners of Ω , this will not interfere with the growth condition. For simplicity assume further (a minor modification of (1.4)) that $t_k = \phi(z_k)$ is finite for each k. Let ψ be the analytic function with $u = \text{Re}\psi$ provided by Theorem 1. Our goal is to show that ψ can be written in the form (1.5).

The first task is to show that the algebraic growth bounds on u imply that ψ' satisfies similar bounds. To this end, assume $t_k = 0$ for convenience and suppose $|u(t)| \le \text{const} |t|^{-M}$, M > 0. For t = r + is near 0 with s > 0, we can compute ψ' as the integral

$$\psi'(t) = \frac{1}{\pi i} \int_{|t-\tau|=s/2} \frac{u(\tau)}{(\tau-t)^2} d\tau$$

obtained by differentiating the Schwarz formula [1, p. 167]. Therefore $|\psi'|$ is bounded by

$$|\psi'(t)| \leq \operatorname{const} |t|^{-M} s^{-1}. \tag{A.4}$$

In any sector bounded away from the real axis, say for $s \ge \frac{1}{2}|r|$, s^{-1} is equivalent to $|t|^{-1}$, and (A.4) implies

$$|\psi'(t)| \le \operatorname{const} |t|^{-M-1}. \tag{A.5}$$

On the other hand consider t with $s < \frac{1}{2}|r|$. Integrating ψ' up to t from some fixed point, we obtain from (A.4) the estimate

$$|\psi(t)| \leq \operatorname{const}|t|^{-M} |\log|s| - \log|t|| \tag{A.6}$$

for |t| sufficiently small. Note that $\log |s| - \log |t| > \log 2$ for $|s| < \frac{1}{2} |r|$. (The term $-\log |t|$ comes from the assumption M > 0, which ensures that the integral of ψ' up to the point r + i|t| has order $|t|^{-M}$, not $|t|^{-M} \log |t|$.) Now by Theorem 1, ψ maps an interval of the real line near t into a straight line. In particular, ψ can be extended by symmetry to a function satisfying (A.6) on and inside the circle $\{\tau: |\tau - t| = \frac{1}{2} |t|\}$, and ψ' can then be obtained by a Cauchy integral over this circle. If we estimate this integral from (A.6), the log factors drop out in the integration, and we are left with (A.5) for all t, as desired.

The remainder of the proof is a modification of the usual Schwarz-Christoffel argument based on the symmetry principle [4,14]. Let t_k be a prevertex. By Theorem 1, ψ' maps an interval $[t_k - \epsilon, t_k)$ into the line through the origin oriented at the angle θ_{k-1} from the real axis, and an interval $(t_k, t_k + \epsilon]$ into the line through the origin at angle θ_k . Therefore for any integer m_k , the function $\psi'(t)(t - t_k)^{\beta'_k - m_k}$ maps $[t_k - \epsilon, t_k)$ and $(t_k, t_k + \epsilon]$ into the single line $e^{i\theta_k}\mathbb{R}$. Since ψ' has at most algebraic growth at t_k , we can choose m_k so that this function is bounded and continuous near t_k . By the reflection principle, it follows that it has an analytic extension to a disk around t_k .

These observations and choices of m_k imply that the function

$$p(t) = \mathrm{e}^{-\mathrm{i}\theta_{k_{\infty}-1}}\psi'(t)\prod_{k=1}^{n}(t-t_{k})^{\beta_{k}'-m_{k}}$$

is analytic throughout $\text{Im} t \ge 0$ and maps \mathbb{R} into itself. Moreover at $t = \infty$, ψ is analytic by Theorem 1 and our choice of ϕ , so p has at most a pole there (since $\Sigma \beta'_k = 0$). By reflection across \mathbb{R} , p can be extended to an entire function. Thus p must be a polynomial with real coefficients. Equation (1.5) now follows by integration. \Box

The final argument shows that the bounded, continuous solutions to Problem O make up a vector space of finite dimension.

Proof of Theorem 3. By Theorem 2, any continuous (hence bounded) solution to Problem O can be written $u = \operatorname{Re}(\psi \circ \phi)$ with ψ given by (1.5). At any finite prevertex t_k , the integrand of (1.5) has a singularity of the form $(t - t_k)^{\gamma_k}$ with $\gamma_k = m_k - \beta'_k + n_k$, where $n_k \ge 0$ is the number of zeros of p at t_k , if any, and u will be continuous at t_k if and only if $\gamma_k > -1$. Therefore we can obtain all continuous solutions by picking m_k so that $m_k - \beta'_k \in (-1, 0]$, that is, by taking $m_k = -1$ for $\beta'_k < 0$ as in (1.6), and $m_k = 0$ otherwise. We have used the fact that $\beta'_k \in (-1, 1)$ here. For u to be continuous throughout $\overline{\Omega}$, it remains only to ensure that ψ is continuous at $t = \infty$. At $t = \infty$, the integrand of (1.5) has a singularity of type $t^{\gamma_{\infty}}$ with $\gamma_{\infty} = \deg p + \sum_{k \neq k_{\infty}} (m_k - \beta'_k) = \deg p + \beta'_{k_{\infty}} + \sum_{k \neq k_{\infty}} m_k$. For continuity we need $\gamma_{\infty} < -1$, i.e., $\deg p < -(1 + \beta'_{k_{\infty}}) - \sum_{k \neq k_{\infty}} m_k$. If k_{∞} is of type (1.6), then $-(1 + \beta'_{k_{\infty}}) \in (-1, 0)$ and $-\sum_{k \neq k_{\infty}} m_k = K - 1$, and this condition becomes $\deg p \leqslant K - 2$. On the other hand if k_{∞} is not of type (1.6), then $-(1 + \beta'_{k_{\infty}})$ $\in (-2, -1]$ and $-\sum_{k \neq k_{\infty}} m_k = K$, and again we have $\deg p \leqslant K - 2$. This completes the proof of Theorem 3 except for the assertion regarding maxima and minima.

We treat the case of a maximum, that of a minimum being analogous. Let $u \in C(\Omega)$ be a nonconstant solution to Problem O. By the maximum principle for harmonic functions, u must attain its maximum u_{\max} somewhere on $\partial\Omega$. Suppose first that this occurs at a point $\zeta \in \Gamma_{k_0}$ with $\theta_{k_0} = \frac{1}{2}\pi$, or at a vertex adjacent to such a segment Γ_{k_0} . Then $u = u_{\max}$ on Γ_{k_0} and on any successive adjacent boundary segments Γ_{k_0+1} , Γ_{k_0+2} , ... with $\theta_k = \frac{1}{2}\pi$. If the entire boundary had $\theta_k = \frac{1}{2}\pi$, u would be constant, which we have ruled out. Therefore there must be a final boundary segment $\Gamma_{k_0+\Delta k}$ of this type, and the vertex $z_{k_0+\Delta k+1}$ is then a point with $u = u_{\max}$ at which (1.6) holds, as asserted.

Next, suppose $u = u_{\max}$ at some point $\zeta \in \Gamma_{k_0}$ with $\theta_{k_0} < \frac{1}{2}\pi$. By the representation (1.5) just established, f maps a segment of Γ_{k_0} around ζ onto a line segment oriented at angle θ_{k_0} to the real axis. For u to have a maximum at ζ we must have $f'(\zeta) = 0$, i.e. $p(\zeta) = 0$. But in this case (1.5) implies that values $u > u_{\max}$ will be attained at points near ζ interior to Ω . This is a contradiction.

Finally, suppose u attains its maximum at a vertex z_k with θ_{k-1} , $\theta_k < \frac{1}{2}\pi$. Again we know that the behavior near z_k is given by (1.5), so that f maps an arc of $\partial\Omega$ around z_k onto two straight line segments with a corner between them. If u is to have a maximum at z_k , f(z) must move rightward at an angle θ_{k-1} to the real axis as z moves along Γ_{k-1} to z_k , then leftward at an angle $\theta_k + \pi$ to the real axis as z continues past z_k along Γ_k , and the image region bounded by these two line segments must lie in the half plane $\text{Re}w < \text{Re}f(z_k)$. This implies $\theta_k < \theta_{k-1}$, i.e. (1.6), as claimed. \Box

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