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CONSTRUCTIVE APPROXIMATION

The CF Table

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Abstract. Let f be a continuous function on the circle $|z|=1$. We present a theory of the (untruncated) "Carathéodory-Fejér (CF) table" of best supremum-norm approximants to f in the classes $\tilde{R}_{mn}$ of functions

$$\tilde{r}(z)=\sum_{k=-\infty}^{m} a_k z^k \Big/ \sum_{k=0}^{n} b_k z^k,$$

where the series converges in $1<|z|<\infty$. (The case $m=n$ is also associated with the names Adamjan, Arov, and Krein.) Our central result is an equioscillation-type characterization: $\tilde{r}\in\tilde{R}_{mn}$ is the unique CF approximant $\tilde{r}^*$ to f if and only if $f-\tilde{r}$ has constant modulus and winding number $\omega\geq m+n+1-\delta$ on $|z|=1$, where δ is the "defect" of $\tilde{r}$. If the Fourier series of f converges absolutely, then $\tilde{r}^*$ is continuous on $|z|=1$, and ω can be defined in the usual way. For general continuous f, $\tilde{r}^*$ may be discontinuous, and ω is defined by a radial limit. The characterization theorem implies that the CF table breaks into square blocks of repeated entries, just as in Chebyshev, Padé, and formal Chebyshev-Padé approximation. We state a generalization of these results for weighted CF approximation on a Jordan region, and also show that the CF operator $K: f\mapsto\tilde{r}^*$ is continuous at f if and only if (m, n) lies in the upper-right or lower-left corner of its square block.

Notation

S, D:	complex unit circle, disk
L^∞:	space of bounded measurable functions on S
$\|\cdot\|$:	essential supremum norm in L^∞
$H^\infty, \bar{H}^\infty$:	subsets of L^∞ of functions analytic inside S, outisde S
C:	space of continuous functions on S
$\mathscr{W}$:	space of functions on S with absolutely convergent Fourier series
f:	function in C to be approximated
m, n:	integers in the ranges $m\in[-\infty,\infty)$, $n\in[0,\infty)$
$R_{mn}, \tilde{R}_{mn}$:	spaces of rational, extended rational functions (equation (1.5))
$P_m, \tilde{P}_m$:	spaces R_{m0}, $\tilde{R}_{m0}$ of polynomials, extended polynomials
$r, \tilde{r}$:	functions in R_{mn}, $\tilde{R}_{mn}$

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(μ, ν): exact type of r or $\tilde{r}$
δ: defect $\min\{m-\mu, n-\nu\}$ of r or $\tilde{r}$
ω: winding number of the error curve $(f-r)(S)$ or $(f-\tilde{r})(S)$
Δ: excess winding number $\omega-(\mu+\nu+1)$
$r^*, \tilde{r}^*$: best approximants to f in R_{mn}, $\tilde{R}_{mn}$

Introduction

Recently there has arisen a great deal of interest in certain rational approximation problems involving analytic functions and associated Hankel operators. The mathematical foundations of this topic are due to Carathéodory and Fejér [4], Schur [33], Takagi [34], and Achieser in the first part of this century and more recently to Adamjan, Arov, and Krein (AAK) [1], [2] and Peller and Hruščev [29], among others. On the one hand, these results are of great theoretical beauty, and have attracted various functional analysts; see, e.g., [12], [31], and [32]. On the other hand, they are of considerable practical importance in digital signal processing and control theory [3], [19], numerical linear algebra [5], and approximation theory [36], [40].

The aim of this paper is to connect these ideas with more familiar topics in approximation theory by constructing a theory of the (untruncated) "CF table" of CF approximants to a function f continuous on the circle $|z|=1$. In real Chebyshev approximation on an interval, and also in real or complex Padé approximation, we seek an approximant r to a function f out of a space R_{mn} of rational functions indexed by integers m, $n \geq 0$. In both of these problems the desired approximant can be characterized by a certain kind of equioscillatory behavior of $f-r$, and from these characterizations it follows that repeated entries in the *Walsh* (=*Chebyshev*) *table* and in the *Padé table* always occupy precisely square blocks [38]. An analogous phenomenon occurs in formal Chebyshev–Padé approximation on an interval [42]. In the present problem we are looking for a best approximant $\tilde{r}^*$ in the supremum norm on $|z|=1$ out of a larger space $\tilde{R}_{mn}$ of *extended rational functions*, where now m may also be negative. The main result of this paper is that here too, best approximations can be characterized by an equioscillation condition, which asserts that $f-\tilde{r}^*$ maps $|z|=1$ onto a *circular error curve* of sufficiently high winding number. Moreover, as is already known, $\tilde{r}^*$ can be constructed by means of a singular value analysis of a Hankel matrix H of Fourier coefficients of f, and $\|f-\tilde{r}^*\|$ is equal to the nth singular value of H.

The idea of the CF table originated in [36] and in [16] and [17], but it is in the AAK-related literature where the most powerful results for general f have been obtained (under the assumption $m=n$) [1], [2], [29]. Some of our results and proofs overlap developments in that literature, but, unfortunately, to give a complete account of all the connections is beyond our ability. What sets this paper apart is its attention to unequal values of m and n and to the elementary principle that we believe lies at the heart of CF approximation: circular error curves. Much of the structure of the CF table can be derived from little more than circular error curves and Rouché's theorem, and in this paper we live up to this ideal as far as we are able. In particular, we avoid as far as possible not only functional analysis, but also algebraic arguments based on singular values and vectors. In this respect our derivations differ from those in the AAK literature and also from the elementary derivations based on linear algebra by Kung [26], Gutknecht [17], and Glover [13].

Here is an outline of the paper. Section 1 establishes notation and defines the CF approximation problem. Section 2 proves some basic theorems involving error curves: if $\tilde{r}$ has a circular error curve of sufficiently high winding number, then it is the unique CF approximant $\tilde{r}^*$ (Theorem 1), but otherwise it is not (Theorem 2); all together, it is the CF approximant, if anywhere, precisely in some square block in the CF table (Theorem 3). In Section 3 we assume the analytic part of f is a polynomial, and reproduce the well-known fact that such a function $\tilde{r}$ exists and can be constructed by singular value analysis (Theorem 4). Section 4 takes limits of such polynomials to obtain the same results for the case in which f belongs to the *Wiener algebra* $\mathscr{W}$ of functions with absolutely convergent Fourier series (Theorem 5). (Compare Section 4 of [1], which comes closest in the AAK literature to the spirit of this work.) Section 5 shows that the CF operator $K: f \mapsto \tilde{r}^*$ is continuous at f with respect to $\mathscr{W}$ if and only if $\tilde{r}^*$ lies in the upper-right or lower-left corner of its square block (Theorem 6); related results have been obtained for Chebyshev and Padé approximation by Werner and Wuytack [41], [43], [44] and more recently for CF approximation by Peller [28] and Helton and Schwartz [22]. In Section 6 we abandon elementary arguments and consider CF approximation for arbitrary continuous f. Here the AAK results imply that $f-\tilde{r}^*$ is still in the class $QC=(H^\infty+C)\cap(\overline{H^\infty+C})$ of *quasi-continuous* functions on $|z|=1$, and, because of this, it has a well-defined winding number obtained as a limit on $|z|=\rho$, $\rho\downarrow 1$. On the basis of this winding number the equioscillation theorems continue to hold (Theorems 7-9). Finally, Section 7 outlines the extension of these results to *Faber*-CF *approximation* on a Jordan region in the presence of a weight function (Theorems 10-12).

Some of our results have been announced without proof in [38].

The importance of the CF table derives from the fact that CF approximants are often extremely close to best Chebyshev approximants r^* in the space R_{mn} of rational functions. The "CF method" for near-best approximation in R_{mn} consists of truncating $\tilde{r}^*$, the exactly constructible best approximation in the large space $\tilde{R}_{mn}$, to obtain a "truncated" CF approximant r^{cf} in R_{mn}. Since $\tilde{r}^*$ has a circular error curve, r^{cf} has a nearly circular error curve, and this implies that it is near to r^*. For example, in approximation to $f(z)=\exp((z+2)^{-1})$ with $m=n=2$, we have $\|f-\tilde{r}^*\|\approx 1.11329096\times 10^{-5}$, $\|f-r^{\text{cf}}\|\approx 1.11329104\times 10^{-5}$. The best approximation error $\|f-r^*\|$ must lie somewhere between these numbers. Further examples can be found in [35]-[37] and [40], and general estimates for $\|r^{\text{cf}}-r^*\|$ are given in [13] and [36]. Matlab software for CF approximation is described in [39], and can be obtained from the second author.

1. Preliminaries

Let $\mathbf{C}$ denote the complex plane, D the unit disk $\{z\in\mathbf{C}: |z|<1\}$, S the unit circle $\{z\in\mathbf{C}: |z|=1\}$, and $\bar{D}$ the closed disk $\bar{D}=D\cup S$. Let L^2 and L^∞ be the spaces of square-integrable and essentially bounded Lebesgue measurable functions on S, respectively, and let $\|\cdot\|$ be the essential supremum norm on L^∞. A function $f \in L^2$ has a Fourier series defined by

$$f(z)=\sum_{k=-\infty}^{\infty} c_k z^k, \qquad c_k=\frac{1}{2\pi i}\int_S z^{-k-1} f(z)\,dz, \tag{1.1}$$

where the equality in the first formula signifies convergence in the L^2 norm of the partial sums $\sum_{-K}^{K} c_k z^k$. Let C be the set of complex continuous functions on S, and let $\mathscr{W}$ be the *Wiener algebra* of functions in L^∞ whose Fourier series converge absolutely. The following inclusions hold:

$$\mathscr{W} \subseteq C \subseteq L^\infty \subseteq L^2. \tag{1.2}$$

Let H^∞ and $\bar{H}^\infty$ be the subsets of L^∞ of functions whose Fourier Coefficients of negative and positive degree vanish, respectively. It is well known that if $f \in H^\infty$ is extended by a Cauchy integral or by the sum in (1.1) to an analytic function in D, then the resulting function is bounded in D with supremum norm equal to $\|f\|$, and has nontangential limits that agree with f almost everywhere on S [12], [23]. Similarly, a function $f \in \bar{H}^\infty$ is the nontangential limit a.e. of a bounded analytic function in $1<|z|\le\infty$. We will be careless about the distinction between functions defined on S and on some disk or annulus adjacent to S, relying on context to make the domains evident.

Let m and n be integers with $n \ge 0$. By a *stable rational function of type* (m, n) we mean a meromorphic function in $\mathbf{C} \cup \{\infty\}$ that can be written as a quotient

$$r(z)=\frac{p(z)}{q(z)}=\frac{\sum_{k=0}^{m} a_k z^k}{\sum_{k=0}^{n} b_k z^k} \tag{1.3}$$

with a_k, $b_k \in \mathbf{C}$ and $q(z) \neq 0$ for $z \in \bar{D}$. (If $m<0$, then $r \equiv 0$.) We define

$$R_{mn}=\{\text{stable rational functions of type } (m, n)\}$$

and

$$P_m = R_{m0}.$$

The quotient (1.3) is in *normal form* if p and q have no common zeros and $b_0=1$. We say that r has *exact type* (μ, ν) if p and q have exact degrees μ and ν in the normal form for r, respectively; throughout this paper, μ and ν always denote the exact degrees of whatever function is under discussion. (If $r \equiv 0$, then $p \equiv 0$, and we define $\mu=-\infty$; otherwise $\mu \ge 0$.) The *defect* of r with respect to R_{mn} is the integer

$$\delta = \min\{m-\mu, n-\nu\} \ge 0. \tag{1.4}$$

An equivalent definition of R_{mn} is that it is the set of meromorphic functions in $C \cup \{\infty\}$ that have no poles in $\bar{D}$ and ν poles in $1<|z|<\infty$ for some $\nu \le n$, and have order exactly $z^{\mu-\nu}$ at $z=\infty$ for some $\mu \le m$. Throughout this paper, zeros and poles are always counted with multiplicity.

Analogously, an *extended stable rational function of type* (m, n) is a meromorphic function in $1<|z|\le\infty$ that can be written as a quotient

$$(1.5)\qquad \tilde{r}(z)=\frac{\tilde{p}(z)}{q(z)}=\frac{\sum_{k=-\infty}^{m}}{\sum_{k=0}^{n} b_k z^k},$$

again with $q(z)\ne 0$ for $z\in\bar{D}$, where the series for $\tilde{p}$ converges in $1<|z|<\infty$ and is bounded there except possibly near ∞. We define

$$\tilde{R}_{mn}=\{\text{extended stable rational functions of type } (m,n)\}$$

and

$$\tilde{P}_m=\tilde{R}_{m0}.$$

If the exact degree of $\tilde{p}$ is understood to mean the index of its highest nonzero term, or $-\infty$ if $\tilde{p}\equiv 0$, then the *normal form, exact type,* and *defect* of $\tilde{r}\in\tilde{R}_{mn}$ are defined exactly as before. Now μ may be negative. An equivalent definition of $\tilde{R}_{mn}$ is that it is the set of meromorphic functions in $1<|z|\le\infty$ that have ν poles in $1<|z|<\infty$ for some $\nu\le n$ and order exactly $z^{\mu-\nu}$ at $z=\infty$ for some $\mu\le m$, and are bounded except near these points. Two further equivalent definitions are

$$(1.6)\qquad \tilde{R}_{mn}=z^{m-n}\tilde{R}_{nn}=z^{m-n}(R_{nn}+\bar{H}^{\infty})$$

and

$$(1.7)\qquad \tilde{R}_{mn}=z^{m-n}B^{(n)}\bar{H}^{\infty},$$

where $B^{(n)}$ is the set of finite Blaschke products with at most n zeros in D. Equation (1.6) makes the connection between our formulation for general m and n and the formulation of AAK and others, who consider approximation in $R_{nn}+\bar{H}^{\infty}$—or actually, in the space $\bar{R}_{nn}+H^{\infty}$ obtained from $R_{nn}+\bar{H}^{\infty}$ by the inversion $z\mapsto 1/z$.

The motivation for this work is the problem of complex Chebyshev approximation of an analytic function $f\in C\cap H^{\infty}$. However, nothing is lost by dropping the assumption of analyticity, although traces of it remain in the stability restriction in R_{mn}.

Chebyshev Approximation Problem. Given $f\in C$, find a Chebyshev approximant $r^*\in R_{mn}$ such that

$$(1.8)\qquad \|f-r^*\|\le\|f-r\|,\qquad \forall r\in R_{mn}.$$

The difficulty with Chebyshev approximation is that although r^* exists, it has no nontrivial characterization, cannot be readily constructed, and need not be unique [16], [20]. Therefore we turn to the

CF Approximation Problem. Given $f\in C$, find an (untruncated) CF approximant $\tilde{r}^*\in\tilde{R}_{mn}$ such that

$$(1.9)\qquad \|f-\tilde{r}^*\|\le\|f-\tilde{r}\|,\qquad \forall\tilde{r}\in\tilde{R}_{mn}.$$

In contrast, not only does $\tilde{r}^*$ exist, but as is already known, it can be constructed by singular value analysis, and it is always unique. The following sections reproduce these facts and show that $\tilde{r}^*$ can be characterized by a circular error curve of sufficiently high winding number.

2. Circular Error Curves

For arbitrary $f \in C$, it is shown in Remark 3.2 of [1] that $\tilde{r}^*$ need not belong to C. In Section 6 we consider this general case, but, in Sections 3-5, f is smooth enough to ensure $\tilde{r}^* \in C$. Given $f \in C$ and $\tilde{r} \in \tilde{R}_{mn} \cap C$, let the *error curve* for $\tilde{r}$ be the image $(f-\tilde{r})(S)$. Throughout this paper, the *winding number* $\omega = \omega(\tilde{r})$ of such a curve refers to its winding number in the positive sense about the origin, and is undefined if the curve passes through the origin. As already mentioned, the purpose of this paper is to prove theorems to the following effect: $\tilde{r} \in \tilde{R}_{mn}$ is the CF approximant to f if and only if its error curve is a circle about the origin of winding number at least $m+n+1-\delta$, where δ is the defect of $\tilde{r}$ with respect to $\tilde{R}_{mn}$. We are now prepared to establish the two halves of this claim in their most basic form.

The following result is a variation on the theme of Rouché's theorem. Compare Lemma 2.3 of [36].

Theorem 1 ("Circular Implies Best," Continuous Case). *Assume that $f \in C$, $\tilde{r} \in \tilde{R}_{mn} \cap C$, and the error curve of $\tilde{r}$ is a circle about the origin of winding number $\omega \geq m+n+1-\delta$. Then $\tilde{r}$ is a best approximant $\tilde{r}^*$ to f in $\tilde{R}_{mn}$. Moreover, it is the unique such best approximant.*

Proof. Assume to the contrary that $\|f-\tilde{s}\| < \|f-\tilde{r}\|$ for some $\tilde{s} \in \tilde{R}_{mn}$. Then $\tilde{r}-\tilde{s}$ belongs to $\tilde{R}_{m+n-\delta,2n-\delta}$, and thus is meromorphic in $1<|z|\leq\infty$ with $\nu_{\tilde{s}} \leq 2n-\delta$ poles in $1<|z|<\infty$ and at most $m+n-\delta-\nu_{\tilde{s}}$ poles at $z=\infty$, i.e., at most $m+n-\delta$ poles in $1<|z|\leq\infty$ all told. (If $m+n-\delta-\nu_{\tilde{s}}<0$, we are counting zeros at $z=\infty$ as poles of negative multiplicity.)

Assume first $\tilde{s} \in \tilde{R}_{mn} \cap C$. Then by the assumption on $\tilde{r}$, $(\tilde{r}-\tilde{s})(S)$ is a continuous curve with winding number $\omega \geq m+n+1-\delta$. But ω must equal the number of poles minus the number of zeros of $\tilde{r}-\tilde{s}$ in $1<|z|\leq\infty$, so this contradicts the count above.

On the other hand, assume $\tilde{s} \in \tilde{R}_{mn}$ is arbitrary. Then for small $\rho>1$, the function $\tilde{s}_\rho(z) = \tilde{s}(\rho z)$ belongs to $\tilde{R}_{mn} \cap C$, and we claim that

$$\lim_{\rho\downarrow 1} \|f-\tilde{s}_\rho\| = \|f-\tilde{s}\|. \tag{2.1}$$

This implies that for small enough ρ, $\tilde{s}_\rho$ is a continuous approximant to f with $\|f-\tilde{s}_\rho\| < \|f-\tilde{r}\|$, so that the winding number argument can be applied.

To establish (2.1), let us generalize the notation above by defining $F_\rho(z) = F(\rho z)$ for $z \in S$, where F is any function defined in $\{1 \leq |z| \leq 1+\rho\}$. Also, for any $F \in L^1(S)$, let $\hat{F}$ denote its Poisson extension to a (complex) harmonic function in $\{1<|z|\leq\infty\}$. For $z \in S$, let $f-\tilde{s}_\rho$ be decomposed into three parts:

$$(f-\tilde{s}_\rho) = (f-\hat{f}_\rho)+(\hat{f}_\rho-\hat{\tilde{s}}_\rho)+(\hat{\tilde{s}}_\rho-\tilde{s}_\rho). \tag{2.2}$$

Since f is continuous on S, $\|f-\hat{f}_\rho\| \to 0$ as $\rho \downarrow 1$. It is also true that $\|\hat{\tilde{s}}_\rho - \tilde{s}_\rho\| \to 0$ as $\rho \downarrow 1$. To see this, write $\tilde{s} = q + h$, where $h \in \bar{H}^\infty$ and q is a stable rational function; then $\hat{h}_\rho = h_\rho$ and $\hat{\tilde{s}} - \tilde{s} = \hat{q} - q$, and so $\|\hat{\tilde{s}}_\rho - \tilde{s}_\rho\| = \|\hat{q}_\rho - q_\rho\|$, which approaches 0 as $\rho \downarrow 1$ since q is continuous on S. Applying the triangle inequality to (2.2), we conclude that

$$\lim_{\rho \downarrow 1} \|f - \tilde{s}_\rho\| = \lim_{\rho \downarrow 1} \|\hat{f}_\rho - \hat{\tilde{s}}_\rho\|.$$

The right-hand side of this identity is equal to $\|f-\tilde{s}\|$—by the maximum modulus principle for Poisson extensions, which follows from the Poisson integral itself—and this proves (2.1).

To prove the uniqueness of $\tilde{r}^*$, we regret that we have not found an elementary argument. Suppose that $\tilde{r}_1^*$ and $\tilde{r}_2^*$ are best approximations to f from $\tilde{R}_{mn}$ with exact types (μ_1, ν_1) and (μ_2, ν_2) and defects δ_1 and δ_2, respectively, and that $\tilde{r}_1^* \in C$ with winding number $\omega \geq m+n+1-\delta_1$. We may then write

$$\tilde{r}_j^* = z^{\mu_j - \nu_j} \bar{B}_j \bar{h}_j \qquad (j=1,2),$$

where B_j is a Blaschke product of exact order ν_j, $h_j \in H^\infty$, $h_j(0) \neq 0$, $B_j(0) \neq 0$, and B_j and h_j have no common zeros in $|z|<1$; see (1.7). Let $\lambda = \max(\mu_1 - \nu_1, \mu_2 - \nu_2)$. Then $\overline{B_1 B_2} \tilde{r}_1^* = \bar{B}_2 \bar{h}_1 z^{\mu_1 - \nu_1}$ and $\overline{B_1 B_2} \tilde{r}_2^* = \bar{B}_1 \bar{h}_2 z^{\mu_2 - \nu_2}$, so both functions are in $\tilde{R}_{\lambda 0} = z^\lambda \bar{H}^\infty$. We will show that $\overline{B_1 B_2} \tilde{r}_1^*$ is a best approximation to $\overline{B_1 B_2} f$ from $\tilde{R}_{\lambda 0}$, where uniqueness is easily established by a standard duality argument (see Theorem 1.7 of Chapter IV of [12]). Since $\|\overline{B_1 B_2} f - \overline{B_1 B_2} \tilde{r}_1^*\| = \|\overline{B_1 B_2} f - \overline{B_1 B_2} \tilde{r}_2^*\|$, this will then imply $\tilde{r}_1^* = \tilde{r}_2^*$.

There are two cases to consider:

Case 1: $\mu_1 - \nu_1 \geq \mu_2 - \nu_2$. Let $\omega(G)$ denote the winding number of the curve $G(S)$. Since $\omega(f - \tilde{r}_1^*) \geq m+n+1-\delta_1$, we have

$$\omega(\overline{B_1 B_2} f - \overline{B_1 B_2} \tilde{r}_1^*) \geq m+n+1-\delta_1-\nu_1-\nu_2.$$

Since $\overline{B_1 B_2} \tilde{r}_1^*$ has defect 0 relative to $\tilde{R}_{\lambda 0}$, it suffices to show that $m+n+1-\delta_1-\nu_1-\nu_2 \geq \mu_1 - \nu_1 + 1$, i.e., that $m+n-\delta_1-\nu_2 \geq \mu_1$. This inequality holds since $m - \delta_1 \geq \mu_1$ and $n - \nu_2 \geq 0$.

Case 2: $\mu_1 - \nu_1 \leq \mu_2 - \nu_2$. Here, again, the defect of $\overline{B_1 B_2} \tilde{r}_1^*$ in $\tilde{R}_{\lambda 0}$ is 0, so it suffices to show that $m+n+1-\delta_1-\nu_1-\nu_2 \geq \mu_2 - \nu_2 + 1$, i.e., that $m+n-\delta_1-\nu_1 \geq \mu_2$. Since $m \geq \mu_2$ and $n - \delta_1 - \nu_1 \geq 0$, we are done. ■

The above proof can be adapted to the more general case where $f \in C$ but neither $\tilde{r}_1^*$ nor $\tilde{r}_2^*$ is continuous. We need, however, the generalized notion of winding number presented in Section 6.

For an algebraic proof of uniqueness in the (n, n) case, see [2] or [29].

Two further remarks should be made regarding Theorem 1. First, the same argument shows that a function $\tilde{r}$ with a nearly circular error curve of winding number $\omega \geq m+n+1-\delta$ is a nearly best approximation to f in $\tilde{R}_{mn}$. We get the following estimate, an analogue of the de La Vallée Poussin lower bound in real Chebyshev approximation on an interval [27]:

$$\min_{z\in S}|(f-\tilde{r})(z)| \le \|f-\tilde{r}^*\| \le \|f-\tilde{r}\|. \tag{2.3}$$

(Since we have not yet shown the existence of $\tilde{r}^*$, $\|f-\tilde{r}^*\|$ can be interpreted here as $\inf_{\tilde{s}\in\tilde{R}_{mn}}\|f-\tilde{s}\|$.) Second, "circular implies best" holds also for standard rational approximation: Theorem 1 and (2.3) remain valid if all tildes are removed. The difference is that in R_{mn}, approximants with circular error curves rarely exist; as a minimum, f must be rational for this to happen [36, Proposition 2.1].

The next theorem gives a partial converse to Theorem 1. The proof is an elaboration of arguments given by Poreda and by Gamelin *et al.* in their discussion of "badly approximable functions" [11], [30].

Theorem 2 ("Best Implies Circular," Continuous Case). *Assume that $f\in C$, $\tilde{r}\in\tilde{R}_{mn}\cap C$ with $\tilde{r}\not\equiv f$, and the error curve of $\tilde{r}$ either is not a circle about the origin, or does not have winding number $\beta\ge m+n+1-\delta$. Then $\tilde{r}$ is not a best approximant to f in $\tilde{R}_{mn}$.*

Proof. Let $\tilde{r}$ be written $\tilde{p}/q$ in the normal form (1.5). We will show that there exist $\Delta\tilde{p}\in\tilde{P}_m\cap C$ and $\Delta q\in P_n$ such that

$$\left\|f-\frac{\tilde{p}+\varepsilon\Delta\tilde{p}}{q-\varepsilon\Delta q}\right\| < \|f-\tilde{r}\|$$

for all sufficiently small $\varepsilon>0$. Let E be the extremal set

$$E=\{z\in S\colon |(f-\tilde{r})(z)|=\|f-\tilde{r}\|\}.$$

Since all functions in question are continuous on S, it is sufficient to find $\Delta\tilde{p}$ and Δq such that

$$\frac{\tilde{p}+\varepsilon\Delta\tilde{p}}{q-\varepsilon\Delta q}-\frac{\tilde{p}}{q}=\varepsilon\frac{q\Delta\tilde{p}+\tilde{p}\Delta q}{q(q-\varepsilon\Delta q)}$$

is nonzero on E and has argument on E differing from $\arg(f-\tilde{r})$ by at most $\alpha<\pi/2$, for all sufficiently small ε. For this it is in turn sufficient to find $\Delta\tilde{p}$ and Δq such that

$$|\arg[q\Delta\tilde{p}+\tilde{p}\Delta q]-\arg[q^2(f-\tilde{r})]|\le\beta \qquad \text{on } E \tag{2.4}$$

for some $\beta<\alpha$. Now if $(f-\tilde{r})(S)$ does not have constant modulus, then E is a closed proper subset of S and therefore omits some arc along S, so by putting a jump in argument in this interval, we can define a continuous argument for $f-\tilde{r}$ on E. On the other hand, if $(f-\tilde{r})(S)$ has constant modulus but winding number $\omega\le m+n-\delta$, then $E=S$ but $z^{-\omega}(f-\tilde{r})$ can be given a continuous argument on E. Using the fact that all zeros of q lie in $|z|>1$, so that q has winding number 0 on S, we conclude in either case that for some $k\le m+n-\delta$, $z^{-k}q^2(z)(f-\tilde{r})(z)$ has a continuous argument $\varphi(z)$ on E. Let v be a continuously differentiable real function defined on all of S such that

$$|v(z)-\varphi(z)|\le\beta \qquad \text{on } E.$$

Let $-u(z)$ be the harmonic conjugate of v in $1 \le |z| \le \infty$ with $u(\infty) = 0$. Since v is smooth, u is continuous in $1 \le |z| \le \infty$ and hence also bounded. Therefore the function $g(z) = \exp(u(z) + iv(z))$ is analytic in $1 < |z| \le \infty$, continuous in $1 \le |z| \le \infty$, and bounded both above and below in modulus. Moreover,

$$(2.5) \qquad |\arg[z^k g(z)] - \arg[q^2(z)(f - \tilde{r})(z)]| \le \beta \qquad \text{on } E.$$

By (2.4) and (2.5), we are done if we can show that $z^k g(z)$, or more generally any $h \in \tilde{P}_{m+n-\delta}$, can be written in the form

$$(2.6) \qquad h = q\Delta\tilde{p} + \tilde{p}\Delta q$$

for some $\Delta\tilde{p} \in \tilde{P}_m$ and $\Delta q \in P_n$.

If $\tilde{r}$ has exact type (μ, ν), then by definition of δ we have $\mu \le m - \delta$ and $\nu \le n - \delta$, and at least one of these inequalities is an equality. First, assume $\nu = n - \delta$. Let Δq be the (unique) polynomial in $P_{\nu-1}$ that interpolates $h/\tilde{p}$ at the ν roots of q. Since $\tilde{p}$ is nonzero at these points by the normality assumption, it follows that $(h - \tilde{p}\Delta q)/q$ is analytic in $1 \le |z| < \infty$, and at $z = \infty$ has order at most $\max\{m + n - \delta, \mu + \nu - 1\} - \nu = m + n - \delta - \nu = m$. Thus we can indeed take $\Delta\tilde{p} \in \tilde{P}_m$ in (2.6), and we are done.

On the other hand, suppose $\mu = m - \delta$. Now let Δq be the polynomial in P_n that not only interpolates $h/\tilde{p}$ at the roots of q, but in addition uses its remaining $n + 1 - \nu$ coefficients to match the behavior of $h/\tilde{p}$ at $z = \infty$ of orders $\nu, \dots, n$. Since $h/\tilde{p}$ has order at most $m + n - \delta - \mu = n$ at $z = \infty$, this construction yields a difference $h/\tilde{p} - \Delta q$ of order at most $\nu - 1$. Therefore $(h - \tilde{p}\Delta q)/q$ has order at most $\nu - 1 + \mu - \nu \le m - 1$, so again it belongs to $\tilde{P}_m$, as required. ■

Theorems 1 and 2 provide all that is needed to establish a basic theorem on square blocks in the CF table:

Theorem 3 (Square Blocks, Continuous Case). *Assume that $f \in C$, $\tilde{r} \in \tilde{R}_{\mu\nu} \cap C$, $\tilde{r}$ has exact type (μ, ν), and the error curve of $\tilde{r}$ is a circle about the origin of winding number $\omega = \mu + \nu + 1 + \Delta$, $\Delta \ge 0$. Then $\tilde{r}$ is a best approximant to f in $\tilde{R}_{mn}$ if and only if (m, n) lies in the $(\Delta + 1) \times (\Delta + 1)$ square block*

$$(2.7) \qquad \mu \le m \le \mu + \Delta, \qquad \nu \le n \le \nu + \Delta.$$

In the special cases $\tilde{r} \equiv 0$ ($\mu = -\infty$) and $\tilde{r} \equiv f$ ("$\omega = \infty$"), $\tilde{r}$ is a best approximant if and only if (m, n) lies in the infinite square blocks

$$(2.8) \qquad -\infty < m \le \omega - 1, \qquad 0 \le n < \infty \qquad (\textit{case } \tilde{r} \equiv 0)$$

and

$$(2.9) \qquad \mu \le m < \infty, \qquad \nu \le n < \infty \qquad (\textit{case } \tilde{r} \equiv f).$$

Proof. By Theorems 1 and 2, $\tilde{r}$ is the best approximation to f in $\tilde{R}_{mn}$ if and only if $\omega \ge m + n + 1 - \delta$, i.e., $\mu + \nu + \Delta \ge m + n - \delta$, or, by (1.4),

$$\mu + \nu + \Delta \ge m + n - \min\{m - \mu, n - \nu\}.$$

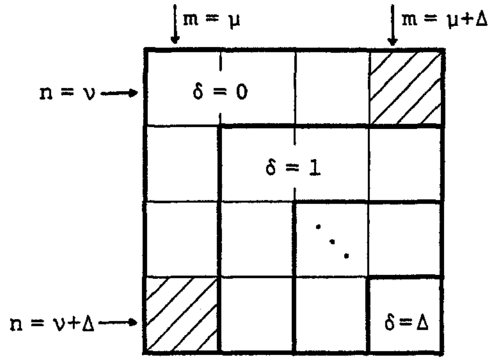

Fig. 1. Anatomy of a square block in the CF table for a given function f and a given extended stable rational function $\tilde{r}$ of exact type (μ, ν). The winding number of $(f-\tilde{r})(|z|=1)$ is $\mu+\nu+1+\Delta$, and δ is the defect of $\tilde{r}$ with respect to the space $\tilde{R}_{mn}$ $(m\ge\mu, n\ge\nu)$. The shaded corners mark the positions where the CF operator $K: f\mapsto\tilde{r}^*$ is continuous, as discussed below in Section 5.

If $m-\mu\le n-\nu$, this becomes $\nu+\Delta\ge n$, which gives the lower-left half of the square block (2.7). The alternative $n-\nu\le\mu$ leads to $\mu+\Delta\ge m$, which gives the upper-right half of the block. The special case $\tilde{r}\equiv 0$ has $\delta=n$, from which (2.8) follows, and (2.9) is trivial. ■

Figure 1 illustrates a square block of the kind described in Theorem 3, showing how μ, ν, and δ relate to position in the block.

An important example of block structure occurs if $f(z)$ is even, in which case the CF table breaks into 2×2 blocks (or larger) whose upper-left corners lie in positions with $(m, n)=(\text{even}, \text{even})$. If $f(z)$ is odd, the table breaks into 2×2 blocks with upper-left corners in positions $(m, n)=(\text{odd}, \text{even})$.

To summarize what has been accomplished so far: without introducing any singular values, we have shown that when continuous CF approximants $\tilde{r}^*$ exist, they are characterized by circular error curves and lie in square blocks in the CF table. It is now time to turn to the singular values to reproduce the known fact that they do indeed exist.

3. The CF Table: Polynomial f

Throughout this section let f be a function in $\tilde{P}_K\cap C$ for some K. That is, f is a continuous function whose analytic part reduces to a polynomial of degree at most K, and it has the Fourier series

$$f(z)=\sum_{k=-\infty}^{K} c_k z^k. \tag{3.1}$$

Let $d\le K$ be an integer, and let H be the finite *Hankel matrix*

$$H=\begin{bmatrix} c_d & c_{d+1} & \cdots & c_K \\ c_{d+1} & & \cdot^{\cdot^{\cdot}} & \\ \vdots & \cdot^{\cdot^{\cdot}} & 0 & \\ c_K & & & \end{bmatrix}. \tag{3.2}$$

Let $\sigma \in [0, \infty)$ and let $U = (u_0, \ldots, u_{K-d})^T \in \mathbf{C}^{K-d+1}$ be a *singular value* and corresponding *left singular vector* for H, that is, a scalar and vector satisfying

$$H\bar{U} = \sigma U. \tag{3.3}$$

(To get this form we have used the fact that H is complex symmetric [34].) All together, H has $K-d+1$ linearly independent singular vectors and $K-d+1$ corresponding singular values, counted with multiplicity [15].

Let u and $\bar{u}$ denote the polynomials

$$u(z) = \sum_{k=0}^{K-d} u_k z^k, \qquad \bar{u}(z) = \sum_{k=0}^{K-d} \bar{u}_k z^{-k}. \tag{3.4}$$

From (3.1)-(3.3) it follows that we have

$$f(z)\bar{u}(z) - \sigma z^d u(z) \in \tilde{P}_{d-1}. \tag{3.5}$$

Conversely, the coefficients of any polynomial $u \in P_{K-d}$ that satisfies (3.5) for some $\sigma \in [0, \infty)$ define a singular vector of H. Now define

N = number of zeros of u in $|z| < 1$,
γ = number of zeros of u at $z = 0$,

and consider the meromorphic function $\tilde{r}$ in $1 < |z| \le \infty$ defined by

$$f(z) - \tilde{r}(z) = \sigma z^d \frac{u(z)}{\bar{u}(z)}. \tag{3.6}$$

Equation (3.5) asserts that $\tilde{r}\bar{u} \in \tilde{P}_{d-1}$ and, therefore, $\tilde{r}(z) = O(z^{d-1+\gamma})$ as $z \to \infty$. On the other hand, by (3.6), $\tilde{r}$ has at most $N - \gamma$ poles in $1 < |z| < \infty$. (The reason it might be fewer than $N - \gamma$ is that some zeros of $\bar{u}$ might be canceled by zeros of u.) Therefore if we define

$$M = N - 1 + d,$$

then $\tilde{r} \in \tilde{R}_{M,N-\gamma}$ and, *a fortiori*,

$$\tilde{r} \in \tilde{R}_{MN}. \tag{3.7}$$

From (3.6) it is evident that $(f - \tilde{r})(S)$ is a circle about the origin of radius σ and winding number $\omega = 2N + d$ if u has no zeros on S, or $\omega > 2N + d$ if it does have zeros there. That is,

$$\omega(\tilde{r}) \ge M + N + 1. \tag{3.8}$$

Therefore, by Theorem 1, *$\tilde{r}$ is the unique CF approximant $\tilde{r}^*$ to f in $\tilde{R}_{MN}$, and the corresponding approximation error is $\|f - \tilde{r}^*\| = \sigma$.*

Let the singular values of H be writtten $\sigma_0 \ge \sigma_1 \ge \cdots \ge \sigma_{K-d} \ge 0$, and assume for a moment that they are all distinct. We have just shown that each σ_n is the error $\|f - \tilde{r}^*\|$ for a CF approximant in some class $\tilde{R}_{N-1+d,N}$ with $0 \le N \le K - d$. Since $\|f - \tilde{r}^*\|$ must decrease or remain constant as N increases, this is possible only if $N(n) = n$, i.e.,

$$\|f - \tilde{r}^*_{n-1+d,n}\| = \sigma_n, \qquad 0 \le n \le K - d. \tag{3.9}$$

But if this formula holds when the singular values are simple, it must also hold when they are multiple: for if f and $\varepsilon > 0$ are given, consider a perturbation f_ε

with $\|f_\varepsilon - f\| \le \varepsilon$ whose Hankel matrix H_ε has simple singular values and nth singular value σ_ε with $|\sigma_\varepsilon - \sigma| \le \varepsilon$. By the above arguments $(f_\varepsilon - \tilde{r}_\varepsilon^*)(S)$ is a circle of radius σ_ε, so $(f - \tilde{r}_\varepsilon^*)(S)$ is a near-circle of radius varying in the range $[\sigma_\varepsilon - \varepsilon, \sigma_\varepsilon + \varepsilon] \subseteq [\sigma - 2\varepsilon, \sigma + 2\varepsilon]$. Therefore, by (2.3), we have $\|f - \tilde{r}^*\| \in [\sigma - 2\varepsilon, \sigma + 2\varepsilon]$, and taking $\varepsilon \to 0$ gives (3.9).

We have proved the following theorem. Except for the winding number characterization, these results go back to Carathéodory and Fejér [4], Schur [33], Takagi [34], and Gutknecht [17], and in any case the theorem is a special case of Theorems 5 and 9 below, but we present it separately to make our treatment of polynomial CF approximation self-contained. As always, δ denotes the defect of $\tilde{r}$ with respect to $\tilde{R}_{mn}$.

Theorem 4 (CF Table for Polynomial f). *Any $f \in \tilde{P}_K \cap C$ has a unique best approximant $\tilde{r}^*$ in $\tilde{R}_{mn}$, and $\tilde{r}^*$ is characterized as the unique continuous function in $\tilde{R}_{mn}$ whose error curve is a circle about the origin of winding number $\omega \ge m + n + 1 - \delta$. Moreover, $\tilde{r}^*$ is given by*

$$(f - \tilde{r}^*)(z) = \sigma_n z^{m-n+1} \frac{u(z)}{\bar{u}(z)} \tag{3.10}$$

and satisfies

$$\|f - \tilde{r}^*\| = \sigma_n, \tag{3.11}$$

where σ_n, u, etc., are defined above. The CF table for f breaks into square blocks of identical entries, as described in Theorem 3.

Note that in this theorem u is permitted to be the polynomial constructed from any singular vector corresponding to σ_n; if σ_n is not simple, there will be several linearly independent possibilities. By the uniqueness statement of Theorem 1, these all lead to the same quotient $u/\bar{u}$, which implies that they differ only by self-reciprocal polynomial factors.

Theorem 4 is summarized in Fig. 2, which suggests the somewhat arbitrary pattern of square blocks that may occur in the CF table for $f \in \tilde{P}_K$. (It is not

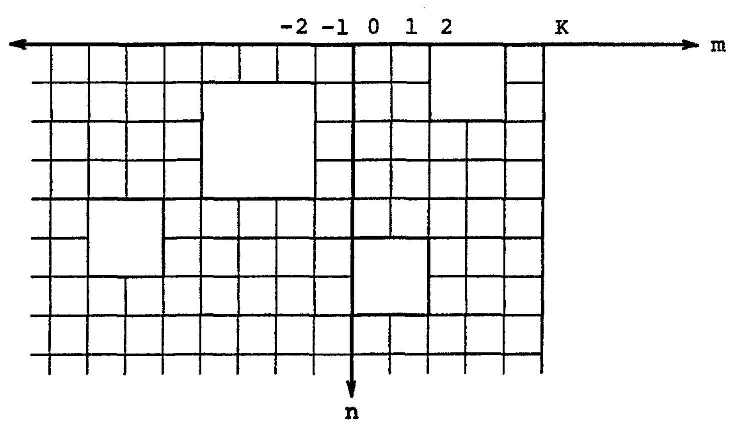

Fig. 2. The CF table for an extended polynomial f of degree K.

completely arbitrary [38].) All entries along any diagonal $m-n=d-1=\text{const.}$ correspond to a single Hankel matrix (3.2), and the singular values of this matrix are the approximation errors as we descend along this diagonal. The length of the intersection of the diagonal with a square block equals the multiplicity of the corresponding singular value.

It is possible to generalize the above proof to a construction of the CF table for the case in which f is rational rather than polynomial [13], [17], [26], but we omit this possibility and pass directly from polynomials to continuous functions in the Wiener algebra.

4. The CF Table: f in the Wiener Algebra

Now let f be any function in the Wiener algebra $\mathscr{W}$, that is,

$$f(z)=\sum_{k=-\infty}^{\infty} c_k z^k, \qquad \sum_{k=-\infty}^{\infty} |c_k|<\infty, \tag{4.1}$$

and, for any K, let $f^{(K)} \in \tilde{P}_K \cap \mathscr{W}$ denote the truncation

$$f^{(K)}(z)=\sum_{k=-\infty}^{K} c_k z^k. \tag{4.2}$$

Let ℓ^2 and ℓ^1 be the usual spaces of square-summable and summable sequences, respectively, with norms $\|\cdot\|_2$ and $\|\cdot\|_1$; we abuse notation by using the same symbols for both the semi-infinite and the bi-infinite cases. There is a one-to-one correspondence given by (1.1) between functions in L^2 or $\mathscr{W}$ and Fourier series in ℓ^2 or ℓ^1, respectively. By Parseval's formula we have $\|\phi\|_2=\|\hat{\phi}\|_2$ in the former case, and let us define a norm $\|\cdot\|_{\mathscr{W}}$ by

$$\|\phi\|_{\mathscr{W}}=\|\hat{\phi}\|_1, \qquad \phi\in\mathscr{W},$$

for the latter. Obviously any $\phi\in\mathscr{W}$ satisfies

$$\|\phi\|\le\|\phi\|_{\mathscr{W}}. \tag{4.3}$$

A great deal is known about the space $\mathscr{W}$. In particular, although functions in $\mathscr{W}$ cannot be characterized by smoothness properties, various necessary or sufficient conditions have been established. For example, if f is in $\text{Lip}_\alpha(S)$ for some $\alpha>\frac{1}{2}$ (Bernstein), or of bounded variation and in $\text{Lip}_\alpha(S)$ for some $\alpha>0$ (Zygmund), or of bounded variation and equal to the boundary values of an analytic function in D (Hardy-Littlewood), then $f\in\mathscr{W}$. See Sections I.6 of [24] and Chapter 5 of [23].

Let H be the infinite Hankel matrix

$$H=\begin{bmatrix} c_d & c_{d+1} & \cdots \\ c_{d+1} & \ddots & \\ \vdots & & \end{bmatrix} \tag{4.4}$$

for some integer d, and let $H^{(K)}$ be the corresponding infinite Hankel matrix for $f^{(K)}$, which has finite rank at most $K-d+1$. The matrix $H^{(K)}-H^{(K-1)}$ is nonzero only on one cross-diagonal, where it has the constant value c_K, so $H^{(K)}-H^{(K-1)}$ has norm exactly $|c_K|$ as an operator on either ℓ^2 or ℓ^1. From

$$H=\sum_{K=d}^{\infty}(H^{(K)}-H^{(K-1)}),$$

we conclude that H is a compact operator on both ℓ^2 and ℓ^1. The ℓ^2-compactness of H is a special case of Hartman's theorem [31], [32].

The *singular values* (=*s-numbers*) of H are defined by

$$\sigma_n=\inf\|H-C_n\|_2, \qquad n\geq 0, \tag{4.5}$$

where C_n ranges over all ℓ^2-bounded infinite matrices of rank n [14]. Since H is compact, $\sigma_n\to 0$ as $n\to\infty$. Repeating the argument following (3.9) in the last section, we can show as follows that the errors in CF approximation to f are exactly

$$\inf_{\tilde{r}\in\tilde{R}_{mn}}\|f-\tilde{r}\|=\sigma_n, \tag{4.6}$$

where $m=n-1+d$ as usual. (This result is essentially due to AAK [2].) Let $\varepsilon>0$ be arbitrary, and pick $K=K(\varepsilon)$ large enough so that $\|f-f^{(K)}\|_{\mathscr{W}}\leq\varepsilon$, hence by (4.3) also $\|f-f^{(K)}\|\leq\varepsilon$. Let $\sigma_n^{(K)}$ and $\tilde{r}_K^*$ be the corresponding nth singular value of $H^{(K)}$ and best approximant to $f^{(K)}$ in $\tilde{R}_{mn}$. By Theorem 4, $(f^{(K)}-\tilde{r}_K^*)(S)$ is a circle of radius $\sigma_n^{(K)}$, so $(f-\tilde{r}_K^*)(S)$ is a near-circle of radius varying in the range $[\sigma_n^{(K)}-\varepsilon,\sigma_n^{(K)}+\varepsilon]$. On the other hand, since $\|H-H^{(K)}\|_2\leq\sum_{k=K+1}^{\infty}|c_k|<\varepsilon$, (4.5) implies $|\sigma_n^{(K)}-\sigma_n|\leq\varepsilon$. Therefore by (2.3) we have $\inf_{\tilde{r}}\|f-\tilde{r}\|\in[\sigma_n-2\varepsilon,\sigma_n+2\varepsilon]$, and taking $\varepsilon\to 0$ gives (4.6).

Given a singular value σ, there is a space of (*left*) *singular vectors* (=*Schmidt vectors*) $U\in\ell^2$ satisfying $H\bar{U}=\sigma U$, of dimension equal to the multiplicity of σ. (Again we have used the symmetry of H [14], [34].) Analogously, let $U^{(K)}$ denote a singular vector of $H^{(K)}$ with singular value $\sigma^{(K)}$. Obviously $U^{(K)}$ has only finitely many nonzero entries if $\sigma>0$, so it belongs to ℓ^1 as well as to ℓ^2. Following (3.4), let us define functions u, $u^{(K)}\in L^2$ by

$$u(z)=\sum_{k=0}^{\infty}u_kz^k, \qquad u^{(K)}(z)=\sum_{k=0}^{\infty}u_k^{(K)}z^k, \tag{4.7}$$

and $\bar{u}$ and $\bar{u}^{(K)}$ similarly.

The following lemma is the key to the construction of the CF table for $f\in\mathscr{W}$.

Lemma 1. *Given $f\in\mathscr{W}$, let σ be a nonzero singular value of H with multiplicity M. Then there is a corresponding singular vector U which is a limit in ℓ^1 of singular vectors $U^{(K)}$ of the finite-rank approximations $H^{(K)}$ of H. Moreover, there exist numbers $a, a_1,\ldots,a_k\in S$ $(k\leq M-1)$ such that U corresponds to a function $u\in\mathscr{W}$ of the form*

$$u(z)=av(z)\prod_{j=1}^{k}(1-\bar{a}_jz), \qquad v(z)\neq 0 \quad \text{on } S, \qquad v\in\mathscr{W}, \tag{4.8}$$

where v corresponds to a singular vector of the Hankel matrix $H_{(k)}$ obtained by deleting the first k columns of H, with the same singular value σ.

Proof. We noted above that H is a compact operator on both ℓ^1 and ℓ^2. Moreover, $\ell^1 \subseteq \ell^2 \subseteq \ell^\infty = (\ell^1)^*$, and H is symmetric. Thus the Riesz-Schauder theory of compact operators asserts that H has the same nonzero singular values and multiplicities (all of which are finite) whether viewed as an operator on ℓ^1, ℓ^2, or ℓ^∞ [45]. The inclusion $\ell^1 \subseteq \ell^\infty$ and dimensionality constraints imply that every ℓ^∞ singular vector is in ℓ^1 as well. Now for each K, let $\sigma^{(K)}$ and $U^{(K)}$ denote a singular value and vector of $H^{(K)}$, selected so that $\sigma^{(K)} \to \sigma$. Normalize $U^{(K)}$ to have unit norm in ℓ^1. Then the set $\{U^{(K)}\} = \{H\bar{U}^{(K)}/\sigma^{(K)}\}$ is precompact in ℓ^1 and, therefore, passing to a subsequence if necessary, $U^{(K)}$ converges in ℓ^1 to some vector U. It is easily checked that $\|U\|_1 = 1$ and that $H\bar{U} = \sigma U$.

To establish (4.8), let $U \in \ell^2$ be any singular vector for H with singular value σ, and let $u(z)$ denote the corresponding function. By the above paragraph, $u \in \mathscr{W}$. If $u(z) \neq 0$ on S, we are done. On the other hand, assume that $u(\lambda) = 0$ for $\lambda = e^{i\theta} \in S$, and consider the vectors

$$W_1 = (0, u_0, \bar{\lambda}u_0 + u_1, \bar{\lambda}^2 u_0 + \bar{\lambda}u_1 + u_2, \dots)^{\mathrm{T}}$$

and

$$W_2 = (u_0, \bar{\lambda}u_0 + u_1, \bar{\lambda}^2 u_0 + \bar{\lambda}u_1 + u_2, \dots)^{\mathrm{T}},$$

which correspond to the functions

$$w_1(z) = \frac{zu(z)}{1-\bar{\lambda}z} \quad \text{and} \quad w_2(z) = \frac{u(z)}{1-\bar{\lambda}z},$$

respectively. Since $\sum_{k=0}^{\infty} |u_k| < \infty$, W_1 and W_2 are in ℓ^∞, and, by a computation based on the assumption $\sum_{k=0}^{\infty} u_k \lambda^k = 0$, it is easily seen that

$$H\bar{W}_1 = -\sigma\bar{\lambda}\, W_2 \quad \text{and} \quad H\bar{W}_2 = -\sigma\bar{\lambda}\, W_1.$$

Therefore $W := i\,e^{-i\theta/2}(W_1 + W_2)$, which corresponds to the function

$$w(z) = i\,e^{-i\theta/2} u(z) \frac{1+z}{1-\bar{\lambda}z},$$

is an ℓ^∞ singular vector for H, which implies $W \in \ell^1$ also. Moreover, the vector

$$U_1 := i\,e^{-i\theta/2}\, W_2 \in \ell^\infty$$

is a singular vector for the matrix $H_{(1)}$ obtained by deleting the first column of H, with the same singular value σ. Hence U_1 corresponds to a function $u_1 \in \mathscr{W}$, and we have

$$u(z) = -i\,e^{i\theta/2}(1-\bar{\lambda}z)u_1(z).$$

If u_1 has any zeros on S, we may repeat the argument on $H_{(1)}$, u_1, σ. Since each additional zero of u on S leads to a new linearly independent singular vector for H (there is an additional zero at -1), the process must terminate, resulting in the factorization (4.8). ∎

We are now ready to prove the Wiener algebra analogue of Theorem 4. Part of this theorem is a special case of Theorem 9, below, but once again we formulate it separately to keep the development self-contained. What is not included in Theorem 9 is the regularity assertion that if f is in the Wiener algebra, then the same is true of $\tilde{r}^*$. Such "hereditary" properties of CF/AAK approximants have been considered by Peller and others [29], [31] in the case $m = n$, especially for $m = n = 0$.

Theorem 5 (CF Table for f in the Wiener Algebra). *Any $f \in \mathcal{W}$ has a unique best approximant $\tilde{r}^*$ in $\tilde{R}_{mn}$, and $\tilde{r}^*$ belongs to $\mathcal{W}$ and is characterized as the unique continuous function in $\tilde{R}_{mn}$ whose error curve is a circle about the origin of winding number $\omega \geq m + n + 1 - \delta$. Moreover, $\tilde{r}^*$ is given by*

$$(f - \tilde{r}^*)(z) = \sigma_n z^{m-n+1} \frac{u(z)}{\bar{u}(z)} \tag{4.9}$$

and satisfies

$$\|f - \tilde{r}^*\| = \sigma_n, \tag{4.10}$$

where σ_n, u, etc., are as defined above. The CF table for f breaks into square blocks of identical entries, as described in Theorem 3.

Proof. Let U be a singular vector of H associated with σ_n, and let $\{U^{(K)}\}$ be a sequence of singular vectors of $\{H^{(K)}\}$ with $U^{(K)} \to U$ in ℓ^1, as provided by Lemma 1. Letting u denote the function associated with U, we have $u(z) = av(z) \prod_{j=1}^{k} (1 - \bar{a}_j z)$ by (4.8). Since v belongs to $\mathcal{W}$ and is nonzero on S, Wiener's theorem implies that $1/\bar{v} \in \mathcal{W}$. Also, for $z \in S$, $(1 - \bar{a}_j z)/(1 - \alpha_j \bar{z}) = -\bar{a}_j z$, so

$$\frac{u(z)}{\bar{u}(z)} = z^k a^2 \frac{v(z)}{\bar{v}(z)} \prod_{j=1}^{k} (-\bar{a}_j) \in \mathcal{W}.$$

Thus $\tilde{r}^*$ defined by (4.9) is in $\mathcal{W}$. Moreover, since $H\bar{U} = \sigma_n U$, we have $\tilde{r}^* \bar{u} \in \tilde{P}_{d-1} \cap \mathcal{W}$, so $\tilde{r}^* \bar{v} = \tilde{r}^* \bar{u} / \bar{a} \prod_{j=1}^{k} (1 - a_j \bar{z}) \in \tilde{P}_{d-1} \cap \mathcal{W}$ as well. Now each $u^{(K)}$ has at most n zeros in D, so Hurwitz's theorem implies that u, and hence v, has at most n zeros in D. Thus $\bar{v}$ has at most n zeros in $1 < |z| \leq \infty$. Therefore $\tilde{r}^* =$

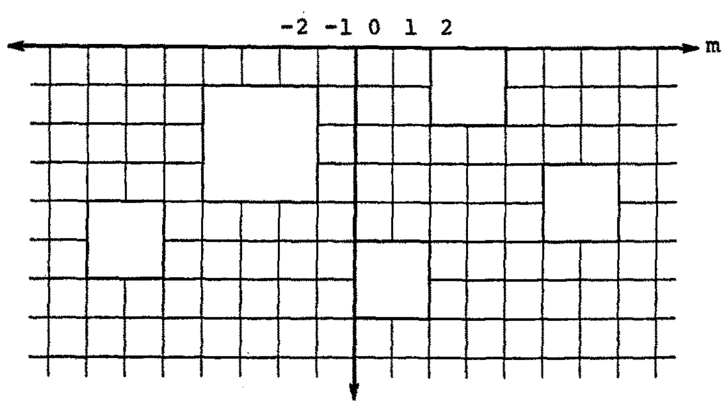

Fig. 3. The CF table for general continuous f.

$\tilde{r}^*\bar{v}/\bar{v} \in \tilde{R}_{d-1+n,n} \cap \mathcal{W} = \tilde{R}_{mn} \cap \mathcal{W}$, so, by (4.6), $\tilde{r}^*$ is a best approximant to f in $\tilde{R}_{mn}$. Theorem 2 now implies that the error curve for $\tilde{r}^*$ has winding number $\omega \geq m+n+1-\delta$, as claimed, and Theorem 1 establishes uniqueness. ■

We remark that, once again, the ratio $u/\bar{u}$ of singular vectors of H depends only on the singular values (see [29] or Section 6). Thus (4.6) is satisfied by *any* singular vector u with singular value σ_n.

The CF table for $f \in \mathcal{W}$ looks the same as in Fig. 2, except that in general there is no infinite block with $\tilde{r}^* = f$ for large m. See Fig. 3.

5. Continuity of the CF Operator

For any fixed m and n, let K denote the *CF operator*

$$K: f \mapsto \tilde{r}^*,$$

which according to Theorem 5 is well defined as a map from $\mathcal{W}$ into $\mathcal{W}$; in the next section it is shown that K extends to a map of C into $QC \subseteq L^\infty$. It is natural to ask under what circumstances and in what norms K is continuous. Analogous questions for the Padé operator P and the real Chebyshev approximation operator T have been investigated previously by Werner and Wuytack [43], [44] with the following outcome: P or T is continuous at a point f if and only if the corresponding approximant Pf or Tf has $\delta = 0$, that is, if and only if (m, n) lies in the first column or first row of its square block (Fig. 1). See [38] and [41] for discussions and generalizations of these results.

We will now prove a similar result for K viewed as a map from $\mathcal{W}$ to $\mathcal{W}$. But here the condition for continuity is stricter: (m, n) must lie in the upper-right or lower-left corner of its square block. See [38] for an explanation of the difference in terms of fracturing of square blocks. Related results for approximants with $n = 0$ have recently been obtained by Peller [28] and by Helton and Schwartz [22]. In the present theorem, continuity is defined with respect to the norm $\|\cdot\|_{\mathcal{W}}$, not $\|\cdot\|_\infty$.

Theorem 6 (Continuity of the CF Operator). *Consider $f \in \mathcal{W}$ with $K(f) \neq f$. The CF operator $K: \mathcal{W} \to \mathcal{W}$ is continuous at f if and only if the singular value σ of Theorem 5 is simple. That is, it is continuous if and only if (m, n) lies in the lower-left or upper-right corner of the square block for $\tilde{r}^*$.*

Proof. To show continuity in the corner positions, assume m, n, and f are chosen so that σ is simple. Let $\{f_\varepsilon\}$ be a sequence of functions in $\mathcal{W}$ with $\lim_{\varepsilon\to 0}\|f_\varepsilon - f\|_{\mathcal{W}} = 0$, and define u_ε, $\tilde{r}^*_\varepsilon$, and σ_ε correspondingly. By the argument of Lemma 1, u is unambiguously defined, $u(z) \neq 0$ on S, and $\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{\mathcal{W}} = 0$. By Wiener's theorem, we also have $1/u \in \mathcal{W}$ and $\lim_{\varepsilon\to 0}\|1/u_\varepsilon - 1/u\|_{\mathcal{W}} = 0$, which implies

$$\begin{aligned}\|\tilde{r}^*_\varepsilon - \tilde{r}^*\|_{\mathcal{W}} &= \|f_\varepsilon - \sigma_\varepsilon z^d u_\varepsilon/\bar{u}_\varepsilon - f + \sigma z^d u/\bar{u}\|_{\mathcal{W}} \\ &\leq \|f_\varepsilon - f\|_{\mathcal{W}} + \|\sigma_\varepsilon u_\varepsilon/\bar{u}_\varepsilon - \sigma u/\bar{u}\|_{\mathcal{W}} = o(1) \qquad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Thus K is continuous at f.

For the converse, we can use a simple argument based on the winding number characterization. The idea is that except in the corner positions of the square block, arbitrarily small perturbations of f can change the winding number of the error curve of its CF approximant $\tilde{r}^*$, and since $\|f-\tilde{r}^*\|>0$ by the hypothesis $K(f)\neq f$, this implies that K is discontinuous.

Let f, m, and n be chosen so that $(f-\tilde{r}^*)(S)$ is a circle about the origin of winding number $\mu+\nu+1+\Delta$, and (m,n) does not lie in the upper-right or lower-left corner of its square block in the CF table for f, defined by (2.7) as

$$\mu\leq m\leq u+\Delta, \qquad \nu\leq n\leq \nu+\Delta.$$

For any $\varepsilon<0$, define

$$b(z)=\frac{(1-\varepsilon)-z}{1-(1-\varepsilon)z}=1-\varepsilon\frac{1+z}{1-(1-\varepsilon)z},$$

a Möbius transformation with winding number 1 on S. Then the error curve $(fb-\tilde{r}^*b)(S)$ has winding number $\mu+\nu+2+\Delta$, and therefore, by Theorem 3, $\tilde{r}^*b$ is the CF approximant to the function fb precisely in the lower-right subblock

$$\mu+1\leq m\leq \mu+\Delta, \qquad \nu+1\leq n\leq \nu+\Delta. \tag{5.1}$$

Now if by chance $f(1)$ is 0, then we have $\lim_{\varepsilon\to 0} fb=f$ but $\lim_{\varepsilon\to 0}\tilde{r}^*b\neq\tilde{r}^*$, since necessarily $b(1)\neq 0$, and we are done. But since $f(1)\neq 0$ in general, define a new function f' (not the derivative of f) by

$$f'(z)=f(z)b(z)-f(1)[b(z)-1]z^{\mu-\nu}=f(z)+[f(z)-f(1)z^{\mu-\nu}][b(z)-1],$$

and define also

$$\tilde{r}^{*\prime}(z)=\tilde{r}^*(z)b(z)-f(1)[b(z)-1]z^{\mu-\nu}.$$

We claim that $\tilde{r}^{*\prime}$ is the CF approximant to f' precisely in the subblock (5.1). Since $f'-\tilde{r}^{*\prime}=b(f-\tilde{r}^*)$ maps S onto a circle of winding number $\mu+\nu+2+\Delta$, this will follow from Theorem 5 provided $\tilde{r}^{*\prime}$ has exact type $(\mu+1,\nu+1)$. Now $\tilde{r}^*b$ obviously has exact type $(\mu+1,\nu+1)$, so we need to show that subtracting $f(1)[b(z)-1]z^{\mu-\nu}$ from it will not change this. First, since the pole of b is also a pole of $\tilde{r}^*b$, but $\tilde{r}^*(1)\neq f(1)$, the denominator index $\nu+1$ is unchanged for sufficiently small ε. Second, since $f(1)[b(z)-1]z^{\mu-\nu}=O(z^{\mu-\nu})$ as $z\to\infty$, with a coefficient of order $\mu-\nu$ of magnitude $O(\varepsilon)$ and thus smaller than that of $\tilde{r}^*b$, the numerator index $\mu+1$ is also unchanged for sufficiently small ε.

Now the function f' just constructed converges in $\mathscr{W}$ to f as $\varepsilon\to 0$. This can be proved by noting that $f'\to f$ uniformly on S and $\|f'-f\|_{\mathscr{W}}$ is uniformly bounded as $\varepsilon\to 0$. Since the Fourier coefficients of $(f'-f)$ each tend to 0 as $\varepsilon\to 0$, we have $\|f'-f\|\mathscr{W}\to 0$ by the Lebesgue Dominated Convergence Theorem. On the other hand the CF approximant $\tilde{r}^{*\prime}$ to f' cannot converge to $\tilde{r}^*$ as $\varepsilon\to 0$, because the winding numbers of their error curves are different. If $m\geq\mu+1$ and $n\geq\nu+1$, $\tilde{r}^{*\prime}$ is the function constructed in the last paragraph, with winding number greater than that of $\tilde{r}^*$. If $m=\mu$ or $n=\nu$, then we have not constructed $\tilde{r}^{*\prime}$, but we know by the block structure statement of Theorem 5 that it is the CF approximant only in a 1×1 block of the CF table, and hence by the winding number characterization of Theorem 5, it must have winding number smaller than that of $\tilde{r}^*$. ■

6. The CF Table: Continuous f

As was remarked earlier, for general $f \in C$, the best approximation in $\tilde{R}_{mn}$ may be discontinuous (see [1] and, for example, Chapter 9 of [32]). However, it is still possible to define a winding number, or *index*, for the function $f - \tilde{r}^*_{mn}$. Although we do not prove it here, this index is a radial limit of winding numbers obtained by extending $f - \tilde{r}^*$ harmonically into $|z| > 1$.

The development of the CF table proceeds along the lines established in Sections 1-4. However, for technical reasons, we first prove the existence of best approximations in $\tilde{R}_{mn}$ directly. The following lemma is equivalent to Theorem 0.2 of [2].

Lemma 2. *Let $f \in L^\infty$ and (m, n) be fixed. Then there exists a function $\tilde{r}^* \in \tilde{R}_{mn}$ such that $\|f - \tilde{r}^*\| \le \|f - \tilde{s}\|$ for all $\tilde{s} \in \tilde{R}_{mn}$.*

Proof. Let $B^{(n)}$ again denote the set of Blaschke products of order at most n. By (1.7), $\tilde{R}_{mn} = \{z^{m-n} B\bar{g} : B \in B^{(n)}, g \in H^\infty\}$, so it suffices to consider the case $m = n$. Choose $\tilde{r}_j = B_j \bar{g}_j \in \tilde{R}_{nn}$ with $\|f - \tilde{r}_j\| \to \rho := \inf_{\tilde{s} \in \tilde{R}_{nn}} \|f - \tilde{s}\|$. Passing to a subsequence, we may assume without loss of generality that each B_j has exact order ν for some fixed $\nu \le n$. Let

$$B_j(z) = \tau_j \prod_{k=1}^{\nu} \frac{z - a_j k}{1 - \bar{a}_{jk} z} \qquad \text{with} \quad |a_{jk}| < 1, \quad |\tau_j| = 1.$$

Also, we may assume that $\lim_{j\to\infty} a_{jk} = \alpha_k$ $(k = 1, \ldots, \nu)$ and that $\lim_{j\to\infty} \tau_j = \tau$, with $|\tau| = 1$ and each $|\alpha_k| \le 1$. By a normal family argument, we may further assume that $g_j \to g$ uniformly on compact subsets of D. let

$$B(z) = \tau \left(\prod_{|\alpha_k| < 1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \right) \left(\prod_{|\alpha_k| = 1} (-\alpha_k) \right).$$

Then $B_j(z) \to B(z)$ uniformly on compact subsets of $\bar{D} \backslash \{\alpha_k : |\alpha_k| = 1\}$. Therefore for any fixed $h \in L^\infty$, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
(Bh)(z) &:= \frac{1}{2\pi} \int_0^{2\pi} B(e^{i\theta}) h(e^{i\theta}) P_z(e^{i\theta})\, d\theta \\
&= \lim_{j\to\infty} \frac{1}{2\pi} \int_0^{2\pi} B_j(e^{i\theta}) h(e^{i\theta}) P_z(e^{i\theta})\, d\theta \\
&= \lim_{j\to\infty} (B_j h)(z)
\end{aligned}$$

for $z \in D$, where P_z is the Poisson kernel. This implies that for the Poisson extension of $(B\bar{f})(e^{i\theta})$ into D,

$$\begin{aligned}\rho \le \|f - B\bar{g}\| &= \|B\bar{f} - g\| \\ &= \sup_{|z|<1} \{|(B\bar{f})(z) - g(z)|\} \\ &= \sup_{|z|<1} \{\lim_{j\to\infty} |(B_j\bar{f})(z) - g_j(z)|\} \\ &\le \lim_{j\to\infty} \|B_j\bar{f} - g_i\| \\ &= \lim_{j\to\infty} \|f - \tilde{r}_j\| = \rho.\end{aligned}$$

Since $B\bar{g} \in \tilde{R}_{nn}$, $\inf\{\|f - \tilde{s}\|: \tilde{s} \in \tilde{R}_{nn}\}$ is attained, and the lemma is proved. ■

Now assume that $\tilde{r}^*$ is a best approximation to f in $\tilde{R}_{mn}$, and write $\tilde{r}^* = z^{m-n}B\bar{h}$. Then $\|f - \tilde{r}^*\| = \|z^{n-m}\bar{B}f - \bar{h}\| = \|z^{m-n}B\bar{f} - h\|$, and thus h is evidently the best approximation to $z^{m-n}B\bar{f}$ in H^∞. By a standard duality argument [7], [12], [32], we have that $|z^{m-n}B\bar{f} - h| = |f - z^{m-n}B\bar{h}|$ is equal to the constant value $\|f - \tilde{r}^*\|$ a.e. on S. Thus the "error curves" are "circular."

We are now prepared to define the generalized winding number for invertible functions in L^∞. The following definition is essentially the same as has been applied in the Fredholm theory of Toeplitz operators; see p. 189 of [6] and p. 81 of [31].

Definition of generalized Winding Number Ind(u). Assume u, $1/u \in L^\infty$ and

$$u = A\bar{B}\, e^{w + i(\varphi + \psi^*)}, \tag{6.1}$$

where A and B are finite Blaschke products, ω, φ, $\psi \in L^\infty_R$ with $\|\varphi\| < \pi/2$, and ψ^* denotes the harmonic conjugate of ψ on D. Then Ind(u) is defined to be the winding number of $A\bar{B}$ about 0, that is, Ind(u) = (order of A) − (order of B).

Of course, we need to show that the index is well defined whenever it exists. To this end, assume that

$$u = A_1\bar{B}_1\, e^{w_1 + i(\varphi_1 + \psi_1^*)} = A_2\bar{B}_2\, e^{w_2 + i(\varphi_2 + \psi_2^*)}.$$

Let k and l be the winding numbers of $A_1\bar{B}_1$ and $A_2\bar{B}_2$, respectively, with $k \ge l$. By a harmless alteration of ψ_1 and ψ_2, we can write

$$u = z^k\, e^{w_1 + i(\varphi_1 + \psi_1^*)} = z^l\, e^{w_2 + i(\varphi_2 + \psi_2^*)}.$$

Thus, $w_1 = w_2$ a.e., and

$$e^{-w_2 - w_1 + \varphi - \psi^*}\, u\bar{u} = z^{k-l}\, e^{\varphi - \psi^* + i(\varphi + \psi^*)} > 0 \qquad \text{a.e. on } S,$$

where $\varphi := \varphi_1 - \varphi_2$ and $\psi := \psi_1 - \psi_2$. Since $\|\phi\| < \pi$ and $\psi \in L^\infty$, we have that $F = \exp[\varphi - \psi^* + i(\varphi + \psi^*)]$ is an outer function in $H^{1/2}$ (see [12]). By a result in [21], we can continue $z^{k-l}F(z)$ across S by reflection. Therefore $z^{k-l}F(z)$ may be regarded as a bounded entire function, and hence is constant. Therefore $k = l$, and Ind(u) is well defined.

Our next aim is to show that the index exists for the functions $f=\tilde{r}^*$. Suppose that $z^{m-n}B\bar{g}$ is the best approximation to f in $\tilde{R}_{mn}$ (we will always assume $f\notin \tilde{R}_{mn}$). Then, setting $\varphi=z^{n-m}B\bar{f}-g$, we have $|\varphi|=\|f-\tilde{r}^*\|=\text{const.}$ a.e. on S and, evidently, $\inf_{h\in H^\infty}\{\|\varphi-h\|\}=\|\varphi\|=\|f-\tilde{r}^*\|$. Since $\lim_{k\to\infty}\inf_{h\in H^\infty}\|z^k\varphi-h\|=0$, the existence of $\operatorname{Ind}(\varphi)$ reduces to the following result, which was first proved by Sarason.

Lemma 3. *Let $|\varphi|=1$ a.e. on S, and assume that $\|\varphi-h\|\geq 1$ for every $h\in H^\infty$, while $\|B\varphi-g_0\|<1$ for some $g_0\in H^\infty$ and finite Blaschke product B. Then* $\operatorname{Ind}(\varphi)\leq -1$.

Proof. Assume φ is as stated with $\|B\varphi-g_0\|<1$. Then $|g_0|$, $1/|g_0|\in L^\infty$, so we have $g_0=b\,e^{w+iw^*}$ for b inner and $w\in L^\infty_R$. Then $\varphi=\bar{B}b\,e^{w+iw^*}\,e^{t+iv}$, where $t, v\in L^\infty_R$ and $\|v\|<\pi/2$. To establish the desired result, we need to rule out the possibilities that b might have a nontrivial singular part and that b might have at least as many zeros as does B. If b has a nontrivial singular part, then it may be replaced by $[b-b(a)]/[1-\bar{b}(a)b]$, where $a\in D$ is chosen to make $b(a)$ suitably small (t, v, w must also be modified). Repeating this adjustment as many times as required, we can assume that b has arbitrarily many zeros. Thus if the conclusion of Lemma 3 fails, we can write

$$\varphi=b_1b_2\bar{B}\,e^{w+iw^*}\,e^{t+iv},$$

where $b_2\bar{B}$ has index zero and b_1 is inner. Since $|\varphi|=1$ a.e., $w+t=0$ a.e., so $\varphi=b_1\exp[i(u+v+s^*)]$, with $u\in C^\infty(S)$, $\|v\|<\pi/2$, and $s\in L^\infty$. Now let $g_\varepsilon=\varepsilon b_1\exp[(s-u^*)+i(s^*+u)]$. Since $u\in C^\infty$, $\|g_\varepsilon\|$ is small, $1/g_\varepsilon\in L^\infty$, and $\|\arg g_\varepsilon-\arg\varphi\|<\pi/2$, we have $\|\varphi-g_\varepsilon\|<1$ for small ε. This contradicts the assumption that $\|\varphi-h\|\geq 1$ for all $h\in H^\infty$. ∎

Assume that a unimodular function φ belongs to $\bar{H}^\infty+C$ and that $\varphi=k/\bar{k}$, where k is an outer function in H^2. Then $\bar{z}\bar{\varphi}=\bar{z}\bar{k}/k$, so by Nehari's Theorem (see Chapter 9 of [32]), $\|\bar{z}\bar{\varphi}-h\|\geq 1$ for every $h\in H^\infty$. By Lemma 3, we have $\operatorname{Ind}(\bar{z}\bar{\varphi})\leq -1$, hence $\operatorname{Ind}(\varphi)\geq 0$.

One more fact we will need is that for functions in $(H^\infty+C)\cap(\bar{H}^\infty+C)$ with finite index, the index of a product is the sum of the indices. Assume that $|u|=1$ a.e. on S and $u\in H^\infty+C$. Then $\lim_{n\to\infty}\inf_{g\in H^\infty}\|z^nu-g\|=0$, so if $\varepsilon>0$ is given, it is possible to find an interger n and $g\in H^\infty$ with $g=B\,e^{v+iv^*}$ such that $\|1-\bar{z}^n\bar{u}B\,e^{v+iv^*}\|$ is small, i.e.,

$$\bar{z}^n\bar{u}B\,e^{v+iv^*}=e^{\alpha+i\beta},$$

where $\alpha, \beta\in L^\infty_R$ are so small that $u=\bar{z}^nB\,e^{i(v^*-\beta)}$, where $\|\beta\|_e<\varepsilon$. Thus, for u_1 and u_2 in $(H^\infty+\bar{H}^\infty)+C$ with indices m and k, respectively, we have $u_j=A_j\bar{B}_j\,e^{\omega_j}\,e^{i(\varphi_j+\psi_j^*)}$ for $j=1, 2$ and $\|\varphi_j\|<\varepsilon<\pi/r$. Then

$$u_1u_2=A_1A_2\bar{B}_1\bar{B}_2\,e^{w_1+w_2}\,e^{i(\varphi_1+\varphi_2)+i(\psi_1^*+\psi_2^*)},$$

so this product has index equal to $m+k$.

We are now ready to state and prove the generalizations of Theorems 1 and 2.

Theorem 7 ("Circular Implies Best," General Case). *Assume that $f \in C$, $\tilde{r} \in \tilde{R}_{mn}$, and $(f-\tilde{r})(S)$ has constant modulus a.e. with* $\operatorname{Ind}(f-\tilde{r}) \geq m+n+1-\delta$. *Then $\tilde{r}$ is a best approximant $\tilde{r}^*$ to f in $\tilde{R}_{mn}$. Moreover, it is the unique such best approximant.*

Proof. Repeat the first paragraph of the proof of Theorem 1. Assume, without loss of generality, that $|f-\tilde{r}|=1$ a.e. on S. Then if $\tilde{s} \in \tilde{R}_{mn}$ and $\|f-\tilde{s}\|<1$, we have

$$\tilde{s}-\tilde{r}=(f-\tilde{r})-(f-\tilde{s})=(f-\tilde{r})(1-e^{a+ib}),$$

where b is a real function and $a \in L_R^\infty$ is negative but bounded away from 0 on S. Hence $1-e^{a+ib}$ has index 0. Because the unimodular function $f-\tilde{r}$ is in $\bar{H}^\infty + C$, its representation in the form (6.1) can be chosen with $\|\varphi\|$ as small as we like, so $\tilde{s}-\tilde{r}$ must have index at least $m+n+1-\delta$, yet lie in $R_{m+n-\delta,2n-\delta}$. This implies

$$\tilde{s}-\tilde{r}=z^{m-n}b_1\bar{b}_2\, e^{c-ic^*},$$

where b_1 and b_2 are Blaschke products and where b_1 has at most $2n-\delta$ zeros, i.e., the index of $\tilde{s}-\tilde{r}$ is at most $m-n+2n-\delta$. This is a contradiction, so $\|f-\tilde{s}\|<1$ is not possible. To prove uniqueness, follow the same argument as was used in the proof of Theorem 1. ∎

Theorem 8 ("Best Implies Circular," General Case). *Assume that $f \in C$ and $\tilde{r}^*$ is a best approximant to f in $\tilde{R}_{mn}$ with $\tilde{r}^* \not\equiv f$. Then $|f-\tilde{r}^*|=\|f-\tilde{r}^*\|$ a.e. on S, and $f-\tilde{r}^*$ has index* $\operatorname{Ind}(f-\tilde{r}^*) \geq m+n+1-\delta$.

Proof. The unimodularity has already been established. We follow the proof of Theorem 2 to show that $\omega=\operatorname{Ind}(f-\tilde{r}^*)$ is at least $m+n+1-\delta$. Assume that $\tilde{r}^*$ has exact type (μ, ν). Let $\tilde{r}^*=\tilde{p}/q$ in reduced form with $\tilde{p} \in \tilde{P}_\mu$, $q \in P_\nu$. Since q has no zeros in $|z| \leq 1$, we have

$$\operatorname{Arg}[\bar{z}^\omega q^2(f-\tilde{r}^*)]=\varphi+\psi^*,$$

where φ, $\psi \in L_R^\infty$ and $\|\varphi\|<\pi/2$ (in fact we can assume $\|\varphi\|$ is as small as we like). Let $g=e^{-\psi+i\psi^*}$. Then g, $1/g \in \bar{H}^\infty$ and

$$\|\operatorname{Arg} z^\omega g-\operatorname{Arg}[q^2(f-\tilde{r}^*)]\|<\pi/2.$$

If $\omega \leq m+n-\delta$, then as in the proof of Theorem 2 (see the discussion following (2.5)), it is possible to find $\Delta\tilde{p} \in \tilde{P}_\mu$ and $\Delta q \in P_n$ such that $(\tilde{p}+\varepsilon\Delta\tilde{p})/(q-\varepsilon\Delta q)$ is a better approximation than $\tilde{r}^*$ in $\tilde{R}_{mn}$, which is, by assumption, impossible. ∎

Theorems 7 and 8 show that the CF table for general $f \in C$ must break into square blocks. We now show that the entries in the table may still be computed from singular vectors of the appropriate Hankel matrices, as in the Wiener case. First let (m, n) be given and set $d=m-n+1$. Form the Hankel matrix H of (4.4) from the Fourier coefficients c_k of f. By Hartman's theorem, H is a compact operator on ℓ^2 [32]. If $\tilde{r}^*$ is the best approximation to f in $\tilde{R}_{mn}$, we must then have $|f-\tilde{r}^*|=\sigma:=\|f-\tilde{r}^*\|$ a.e. Suppose that (μ, ν) is the exact type of $\tilde{r}^*$. Then $\tilde{r}^*=z^{\mu-\nu}b\bar{h}$, with b a Blaschke product of order ν and $h \in H^\infty$. We then have as in (6.1),

$$\theta:=f-\tilde{r}^*=\sigma A\bar{B}\, e^{i(u+v^*)},$$

with the index of $A\bar{B}$ equal to $\omega \geq m+n+1-\delta$. Since $\operatorname{Arg}(b^2 z^{-2\nu})$ is smooth on S, we can write

$$\theta = \sigma z^{m+n+1-\delta-2\nu+\Delta} bg/\overline{bg} \qquad (\Delta \geq 0)$$

for some outer function $g \in H^2$. Noting that $z^{\Delta} = [(1+z)/(1+\bar{z})]^{\Delta}$ on S, we replace g by $(1+z)^{\Delta} g$ to get

$$\bar{z}^d f - z^{(n-\nu)-(m-\mu)} \overline{bzh} = \sigma z^{(n-\nu)+(n-\nu)-\delta} bg/\overline{bg}.$$

Letting $u_1 = z^{n-\nu-\delta} bg$, $u_2 = z^{\delta} u_1$, and $u = u_1 + u_2$, it is easily seen that $u \in H^2$, that $f - z^{\mu-\nu} b\bar{h} = \sigma z^d u/\bar{u}$, and that u corresponds to a singular vector U of H with singular value σ. Note that with the preceding arguments, we also get singular values of H by taking best approximations in $\tilde{R}_{m-n+k,k}$, $k \geq 0$.

Conversely, let σ be a singular value for H and U a singular vector. Let u be the corresponding function. Then $\bar{z}^d f\bar{u} - \overline{zh} = \sigma u$ for some $h \in H^2$. Let $u = Bg$, where B is inner and g is outer. Then $f - z^{m-n} B\bar{h}/\bar{g} = \sigma z^d Bg/\overline{Bg}$, so $\bar{h}/\bar{g} \in L^{\infty}$. Since H is compact, B must be a finite Blaschke product, for if it is not, let $B = B_1 B_2$ with B_j inner, $j = 1, 2$. Letting $v = B_1(1+B_2)^2 g$, we have $v/\bar{v} = u/\bar{u} = B^2 g/\bar{g}$. Thus, the vector V corresponding to v is an eigenvector of $H\bar{H}$ with eigenvalue σ^2. Therefore the eigenspace corresponding to σ^2 is infinite dimensional, which is impossible by the compactness of H.

Now let N be the order of B. We have $g/\bar{g} \in \bar{H}^{\infty} + C$ with nonnegative index, by the remark following Lemma 3. We then have $z^{m-n} B\bar{h}/\bar{g} \in \tilde{R}_{m-n+N,N}$, with the index of $z^d B^2 g/\bar{g}$ greater than or equal to $m-n+1+2N = (m-n+N)+N+1$. By Theorems 7 and 8, we have just constructed the best approximation to f from $\tilde{R}_{m-n+N,N}$. Therefore each singular value σ of H is a distance from f to some space $\tilde{R}_{m-n+N,N}$, and conversely. Let $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_n \geq \cdots$ be a listing of the singular values. Because the distance from f to $R_{m-n+k,k}$ is a nonincreasing function of k, we must have that $k = n$ corresponds with σ_n, i.e., $\sigma_n = \inf\|f - \tilde{r}^*\|$ and $\tilde{r}^*_{mn} = f - \sigma_n z^d u/\bar{u}$, where u corresponds to *any* singular vector of H for the value σ_n.

All together, we have proved:

Theorem 9 (CF Table for Continuous f). *Any $f \in C$ has a unique best approximant $\tilde{r}^*$ in $\tilde{R}_{mn}$, which is characterized by*

(1) $|f - \tilde{r}^*| =$ *constant a.e. on* S

and

(2) $\operatorname{Ind}(f - \tilde{r}^*) \geq m+n+1-\delta$.

Moreover, $\tilde{r}^*$ is given by

$$(f - \tilde{r}^*)(z) = \sigma_n z^{m-n+1} \frac{u(z)}{\bar{u}(z)} \tag{6.2}$$

and satisfies

$$\|f - \tilde{r}^*\| = \sigma_n, \tag{6.3}$$

where σ_n, u, etc., are defined above. The CF table for f breaks into square blocks of identical entries, as described in Theorem 3.

The CF table for $f \in C$ looks the same as for $f \in \mathscr{W}$, already illustrated in Fig. 3.

7. Weighted CF Approximation on a Jordan Region

It is well known that the minimal-norm interpolation theorem of Carathéodory and Fejér for the unit disk can be generalized to Jordan domains by a conformal transplantation. This technique is also applicable in a more general setting and is the basis of the Faber-CF method [8], [9], [16]. Another straightforward but useful generalization of the CF method consists in introducing a weight function [16]-[18]. In this section these two ideas are combined in a weighted Faber-CF approximation. We do not strive for maximum generality.

Let Σ be a Jordan domain with rectifiable boundary Γ, as shown in Fig. 4. Let Ω denote the exterior of Γ (including ∞), and let E denote the exterior of the unit circle S. Let ψ be the conformal map of E onto Ω normalized by $\psi(\infty)=\infty$ and $\psi'(\infty)>0$, and set $\phi=\psi^{-1}$. Since Σ is a Jordan domain, ψ and ϕ can be extended continuously to the boundaries S and Γ, respectively. Let $C(\Gamma)$ denote the space of continuous functions on Γ, and let $A(\bar{\Omega})$ denote the space of functions continuous on $\bar{\Omega}$ and analytic in Ω (including at ∞). Both spaces are Banach spaces under the supremum norm.

With a function $F \in C(\Gamma)$ we can now associate the transplanted function $f := F \circ \psi \in C$, and vice versa. Likewise, with a function G defined on $\bar{\Omega}$ we can associate the function $g := G \circ \psi$ defined on $\bar{E}$. In particular, if $G \in A(\bar{\Omega})$, then $g \in A(\bar{E})$, and vice versa. In analogy to $\tilde{R}_{mn}$, we introduce the following space of functions defined on Ω:

$$\tilde{R}_{mn}(\Omega) := \left\{ \frac{\tilde{P}}{Q} : \tilde{P} \in z^m H^\infty(\Omega),\ Q \in P_n,\ Q(\zeta) \neq 0 \text{ on } \bar{\Sigma} \right\}.$$

($H^\infty(\Omega)$ is the space of bounded analytic functions on Ω.) Every $\tilde{R} \in \tilde{R}_{mn}(\Omega)$ has a nontangential limit almost everywhere on Γ. Therefore $\|\tilde{R}\|_\Gamma := \sup\{|\tilde{R}(\zeta)| : \zeta \in \Gamma\}$ is well defined, and clearly $\|\tilde{R}\|_\Gamma = \|\tilde{R} \circ \psi\|$, the latter norm being the sup-norm on S. But in fact much more is true [16, Lemma 7.1]:

Lemma 4. *The map ψ induces an isometric isomorphism between $C(\Gamma)$ and C and*

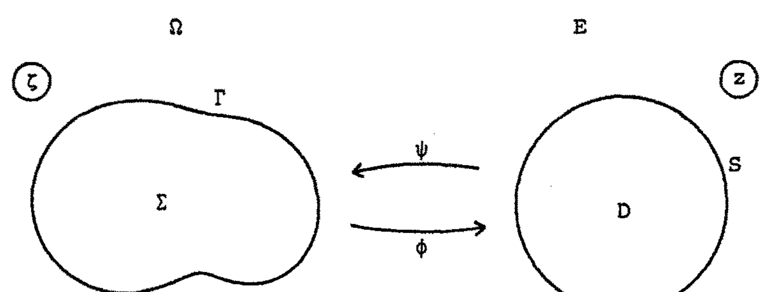

Fig. 4. Exterior conformal maps between a Jordan region and the unit disk.

between $\tilde{R}_{mn}(\Omega)$ and $\tilde{R}_{mn}$. In particular:

$$\tilde{R} \in \tilde{R}_{mn}(\Omega) \quad \Rightarrow \quad \tilde{r} := \tilde{R} \circ \psi \in \tilde{R}_{mn},$$
$$\tilde{r} \in \tilde{R}_{mn} \quad \Rightarrow \quad \tilde{R} := \tilde{r} \circ \phi \in \tilde{R}_{mn}(\Omega).$$

Proof. Assume $\tilde{R} \in \tilde{R}_{mn}(\Omega)$. Since $\tilde{R}$ is meromorphic in Ω, $\tilde{r} := \tilde{R} \circ \psi$ is meromorphic in E. The poles of $\tilde{R}$ in Ω are transplanted by ψ to the poles of $\tilde{r}$ in E, the multiplicity being preserved since $\psi'(z) \neq 0$ in E. In particular, the order at ∞ is also preserved. Therefore $\tilde{r} \in \tilde{R}_{mn}$. The proof for the reverse direction is the same. For the spaces $C(\Gamma)$ and C the result is a consequence of the continuity of ϕ and ψ on the boundary. ∎

As a consequence of Lemma 4 we get immediately [16, Corollary 7.2]:

Lemma 5. *$\tilde{R}^*$ is a best approximation to $F \in C(\Gamma)$ in $\tilde{R}_{mn}(\Omega)$ if and only if $\tilde{r}^* = \tilde{R}^* \circ \psi$ is a best approximation to $f = F \circ \psi \in C$ in $\tilde{R}_{mn}$. The same holds with $C(\Gamma)$ and C replaced by $L^\infty(\Gamma)$ and L^∞.*

From Theorem 9 and Lemma 5 we conclude:

Theorem 10 (Faber-CF Table for Continuous F). *Any $F \in C(\Gamma)$ has a unique best approximant $\tilde{R}^*$ in $\tilde{R}_{mn}(\Omega)$, which is characterized by*

(1) $|F - \tilde{R}^*| = $ *constant a.e. on* Γ

and

(2) $\operatorname{Ind}(F \circ \psi - \tilde{R}^* \circ \psi) \geq m + n + 1 - \delta$.

Moreover, $\tilde{R}^* = \tilde{r}^* \circ \phi$, where $\tilde{r}^*$ is given by (6.2), and $\|F - \tilde{R}^*\|_\Gamma = \sigma_n$. The Faber-CF table for F breaks into square blocks of identical entries, as described in Theorem 3.

There is a further relation between F and f if F belongs to $A(\tilde{\Sigma})$. The Fourier coefficients c_k with index $k \geq 0$ of $f = F \circ \psi$ are then called the *Faber coefficients* of F, and the formal series

$$\sum_{k=0}^{\infty} c_k \phi_k(\zeta)$$

of Faber polynomials ϕ_k is the *Faber series* of F [10], [25]. The *Faber polynomials* ϕ_k for Σ are defined as the analytic parts of the Laurent series of ϕ^k at ∞:

$$\phi_k(\zeta) = [\phi(\zeta)]^k + O(\zeta^{-1}) \qquad \text{as} \quad \zeta \to \infty.$$

The Faber coefficients c_k of F are of course also the Taylor coefficients of the analytic part Pf of f. The mapping

$$\mathcal{T}: Pf = \sum_{k=0}^{\infty} c_k z^k \quad \mapsto \quad F \sim \sum_{k=0}^{\infty} c_k \phi_k(\zeta)$$

is called the *Faber transform*. If $m \geq n-1$, the coefficients c_k with $k \geq m-n+1$ that appear in the Hankel matrix of $f = F \circ \psi$ are just the Faber coefficients of F.

Assume now in addition that Γ has bounded rotation ρ [10], [25]. Kövari and Pommerenke have shown that the Faber transform $\mathcal{T}$ is thn a bounded linear map of $A(\bar{D})$ into $A(\bar{\Sigma})$:

$$\|\mathcal{T}\| \leq 1 + \frac{\rho}{\pi}. \tag{7.1}$$

(The bound $1+2\rho/\pi$ in [10] can be improved; see [9].) For F to be in $\mathcal{T}(A(\bar{D}))$ it is then sufficient that the conjugate function of $t \mapsto F(\psi(e^{it}))$ also be continuous. Moreover, if $F \in \mathcal{T}(A(\bar{D}))$, the Faber series is known to converge uniformly on $\bar{\Sigma}$ to F.

Let us assume that $\sum_{k=1}^{\infty} |c_k| < \infty$, so that, as a consequence of (7.1), the Faber series of F converges absolutely on $\bar{\Sigma}$ (the limit being F by the above more general result of Kövari and Pommerenke). Then $F \in C(\Gamma)$, $f = F \circ \psi \in C$, and $f = f_- + f_+$ with

$$f_-(z) = \cdots + c_{m-n-1} z^{m-n-1} + c_{m-n} z^{m-n} \in C,$$

$$f_+(z) = c_{m-n+1} z^{m-n+1} + c_{m-n+2} z^{m-n+2} + \cdots \in \mathcal{W}.$$

The function $\tilde{r}^*$ depends in a trivial way on f_-, namely

$$\tilde{r}^*[f] = \tilde{r}^*[f_+] + f_-.$$

By Theorem 5, $\tilde{r}^*[f] \in \mathcal{W} \subseteq C$, and consequently $\tilde{R}^* \in C(\Gamma)$. Summarizing, we get:

Theorem 11 (Faber-CF Table for F in a Wiener Algebra, Γ of Bounded Rotation). *Assume that Γ is of bounded rotation. Then for any $F \in A(\bar{\Sigma})$ whose Faber series converges absolutely in $\bar{\Sigma}$, the Faber series of $\tilde{R}^*$ also converges absolutely in $\bar{\Sigma}$, and $\tilde{R}^*$ is characterized as the unique continuous function in $\tilde{R}_{mn}(\Omega)$ whose error curve is a circle about the origin of winding number $\omega \geq m+n+1-\delta$.*

Now let $\gamma \in C(\Gamma)$, $\gamma > 0$, be a prescribed *weight function*, and let us modify our approximation problem: we now aim to find $\tilde{R}^* \in \tilde{R}_{mn}(\Omega)$ such that

$$\|\gamma(F - \tilde{R}^*)\|_\Gamma = \inf_{\tilde{R} \in \tilde{R}_{mn}(\Omega)} \|\gamma(F - \tilde{R})\|_\Gamma.$$

For the next result it suffices for Γ to be any rectifiable Jordan curve, but we require $\log \gamma$ to be the real part of a function $\Lambda \in A(\bar{\Omega})$, resstricted to Γ. Equivalently, $\log \gamma \circ \psi$ is the real part of a function $\lambda \in A(\bar{E})$, which means that λ is the Poisson integral (with respect to E) of $\log \gamma \circ \psi$. (For this it is necessary and sufficient that the conjugate function of $\log \gamma(\psi(e^{it}))$ be continuous.) λ is equal to the coanalytic part plus the constant term in the Fourier series of

$\log \gamma(\psi(e^{it}))$, and thus it can be constructed easily. Λ and λ are again related by transplantation: $\Lambda = \lambda \circ \phi$. We further define:

$$g := e^{\lambda} \in A(\bar{E}), \qquad G := e^{\Lambda} \in A(\bar{\Omega}),$$

$$\tilde{f} := gf \in C, \qquad \tilde{G} := GF \in C(\Gamma),$$

$\tilde{r} :=$ best approximation to f in $\tilde{R}_{mn}$ with weight function $\gamma \circ \psi$,

$\tilde{R}^* :=$ best approximation to F in $\tilde{R}_{mn}(\Omega)$ with weight function γ,

$\tilde{\tilde{r}}^* :=$ best approximation to $\tilde{f}$ in $\tilde{R}_{mn}$ with weight function 1,

$\tilde{\tilde{R}}^* :=$ best approximation to $\tilde{F}$ in $\tilde{R}_{mn}(\Omega)$ with weight function 1.

The function g is known as the *Herglotz transform* of λ (with respect to E). We now have:

Theorem 12 (Weighted Faber–CF Approximation). *The functions* $\tilde{r}^*$, $\tilde{\tilde{r}}^*$, $\tilde{R}^*$, *and* $\tilde{\tilde{R}}^*$ *are related by*

$$\tilde{r}^*(z) = \frac{\tilde{\tilde{r}}^*(z)}{g(z)}, \qquad \tilde{R}^*(\zeta) = \frac{\tilde{\tilde{R}}^*(\zeta)}{G(\zeta)},$$

$$\tilde{r}^* = \tilde{R}^* \circ \psi, \qquad \tilde{R}^* = \tilde{r}^* \circ \phi,$$

$$\tilde{\tilde{r}}^* = \tilde{\tilde{R}}^* \circ \psi, \qquad \tilde{\tilde{R}}^* = \tilde{\tilde{r}}^* \circ \phi.$$

Proof. The connection between $\tilde{\tilde{r}}^*$ and $\tilde{\tilde{R}}^*$ has already been established in Theorem 10. Concerning $\tilde{R}^*$ and $\tilde{\tilde{R}}^*$, the essential point is that $G \in A(\tilde{\Omega})$ is bounded away from 0, so that $1/G \in A(\bar{\Omega})$ also, and therefore any $\tilde{R}$ and $\tilde{\tilde{R}}$ related by $\tilde{R} \times \tilde{\tilde{R}}/G$ satisfy

$$\tilde{\tilde{R}} \in \tilde{R}_{mn}(\Omega) \quad \Leftrightarrow \quad \tilde{R} \in \tilde{R}_{mn}(\Omega)$$

and

$$\|\tilde{F} - \tilde{\tilde{R}}\| = \|G(F - \tilde{R})\| = \||G|(F - \tilde{R})\| = \|\gamma(F - \tilde{R})\|$$

(since $|G(\zeta)| = e^{\mathrm{Re}\,\Lambda(\zeta)} = e^{\log \gamma} = \gamma$). The corresponding result for $\tilde{r}^*$ and $\tilde{\tilde{r}}^*$ follows on replacing Ω by E. ■

Thus, the weighted Faber–CF approximant $\tilde{R}^*$ to F in $\tilde{R}_{mn}(\Omega)$ is obtained by transplanting the unweighted CF approximant $\tilde{\tilde{r}}^*$ of f in $\tilde{R}_{mn}$, divided by g. We leave it to the reader to restate the conclusions of Theorems 10 and 11 concerning characterization and block structure.

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