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Computations of Eigenvalue Avoidance in Planar Domains

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The phenomenon of eigenvalue avoidance is of growing interest in applications ranging from quantum mechanics to the theory of the Riemann zeta function. Until now the computation of eigenvalues of the Laplace operator in planar domains has been a difficult problem, making it hard to compute eigenvalue avoidance. Based on a new method this paper presents the computation of eigenvalue avoidance for such problems to almost machine precision.

1 The Eigenvalue Avoidance Phenomenon

The phenomenon of eigenvalue avoidance is linked to the question of how likely it is that a given operator has multiple eigenvalues. In 1929 von Neumann and Wigner [6] showed that the set of real symmetric $N \times N$ matrices with multiple eigenvalues has codimension 2, which means that this set has at least two degrees of freedom less than the set of all symmetric matrices and is therefore unlikely to be encountered by chance. Let us look at the family of matrices $F(t) := A + tB$, where A and B are real symmetric $N \times N$ matrices and t is a real parameter. If A and B are randomly chosen the eigenvalues $\lambda_k(t), k = 1, \ldots, N$ of $F(t)$ might come very close to each other. But they will probably not intersect since we only have one degree of freedom t but two conditions for a multiple eigenvalue. This eigenvalue avoidance phenomenon is beautifully explained by Peter Lax in his textbook *Linear Algebra* [3], and illustrated by a picture on the cover. Eigenvalue avoidance can not only be observed for finite dimensional operators. Uhlenbeck in 1976 [5] and Teytel in 1999 [4] showed that these results can be generalized to certain classes of self-adjoint operators acting on Hilbert spaces. In particular, Teytel showed by introducing a definition of codimension for inifinite-dimensional spaces that the same argument of codimension 2 from finite-dimensional spaces is valid under certain conditions in the infinite-dimensional case.

The avoidance of eigenvalue crossings plays an important part in several areas of mathematics and physics. A survey of applications in quantum physics was given in 1984 by Berry and Wilkinson [1] who computed the first eigenvalues of the Laplace operator with zero Dirichlet boundary conditions over the whole space of triangles. They were able to find several triangles which have degeneracies in their spectrum (they called them "diabolical points"). In another area of mathematics eigenvalue avoidance is also of growing interest. Deift, Sarnak, Odlyzko and others have shown that there appear to be deep connections between eigenvalue avoidance for random matrices and zero avoidance on the critical line for the Riemann zeta function.

Even though the idea of eigenvalue avoidance is easy to understand, the computations of such effects may be a hard problem for infinite-dimensional operators. In [2] we introduced a new method to compute the eigenvalues and eigenfunctions of the Laplace operator with zero Dirichlet boundary conditions on planar domains to close to machine precision. It is based on minimizing the angle $\sigma(\lambda)$ between the space of particular solutions that satisfy the eigenvalue equation $-\Delta u = \lambda u$ and the space of functions that are zero on the boundary of the domain. With this method we are able to show the eigenvalue avoidance phenomenon in our examples to more than 10 digits of accuracy.

2 Numerical Results

Our example domain is a perturbed rectangle with coordinates in the complex plane $0, L, (L - p) + \frac{i}{L}, \frac{i}{L}$, where $p > 0$ is a small parameter and L ranges from $1/2$ to 2. If $p = 0$ and $L = 1$, the domain is a square and the second eigenvalue is multiple. For $p > 0$ we can expect to observe eigenvalue avoidance as L is varied.

The left plot of Figure 1 shows the curves of the lower eigenvalues for the unperturbed rectangle ($p = 0$). The eigenvalue curves indicate an eigenvalue crossing at $L = 1$ between the second and third eigenvalue. Other eigenvalue crossings for higher eigenvalues also appear in the plot. If a small perturbation of $p = 0.2$ is introduced, the right plot is obtained. The eigenvalues now avoid each other, though still coming very close.

It is interesting to see what happens with very small perturbations. The two plots in Figure 2 show the subspace angles $\sigma(\lambda)$ for various values of λ close to the second eigenvalue in the critical case $L = 1$. In [2] we proved that the true eigenvalues are close to the minima of the subspace angle curve. The left plot now shows the unperturbed case and the right plot shows the perturbed case with $p = 10^{-13}$. In the right plot we have two well separated minima, pointing to two different eigenvalues,

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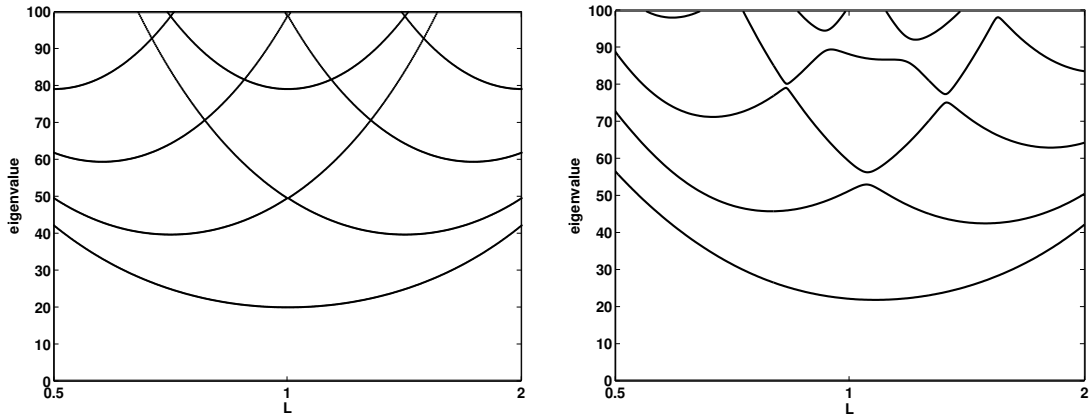

Fig. 1 The left plot shows the lower eigenvalues for rectangles of aspect ratio L, and the right shows the same curves for perturbed rectangles with $p = 0.2$.

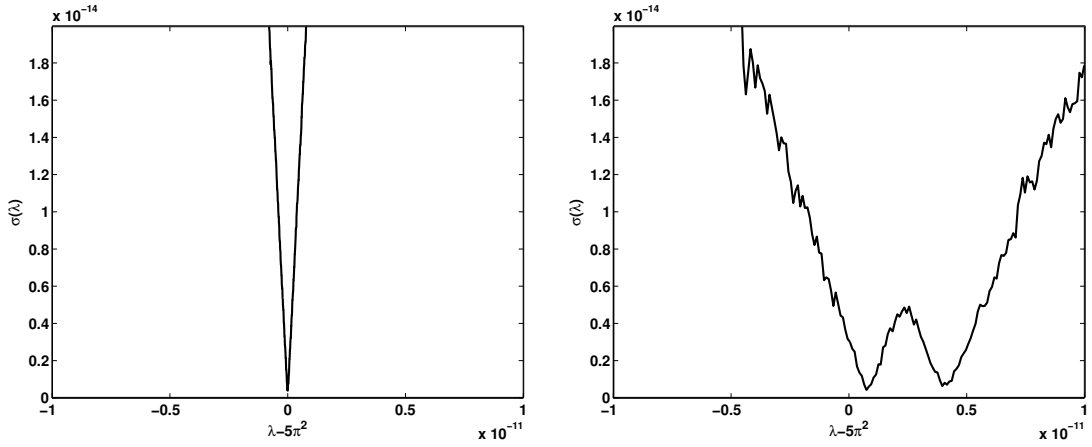

Fig. 2 In the left plot the subspace angle curve close to the second eigenvalue is plotted for the square $(L = 1, p = 0)$. The curve goes sharply into the multiple eigenvalue. In the right plot the same curve is plotted for a small perturbation of $p = 10^{-13}$. Two different minima are now recognizable.

compared to only one minimum in the left plot. The small oscillations in the curve in the right plot are due to small errors in the basis caused by roundoff. But they do not influence the general behavior of the curve. It is interesting to observe that in the symmetric case with $p = 0$ these oscillations are much weaker than in the perturbed case with $p = 10^{-13}$.

The figures in this paper show that our new method is capable of computing eigenvalue avoidance even if the perturbation and therefore the gap between the eigenvalues is almost on the order of machine precision. The same method also works for more complicated polygons and other regions with piecewise smooth boundaries.

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