Notes of a Numerical Analyst

What's the degree of x^n?

NICK TREFETHEN FRS

The degree of x^n, of course, is n. But computational mathematicians keep running up against the fact that its effective degree on a real interval, as defined by approximations, is only $O(\sqrt{n})$ as $n \to \infty$. This effect was made precise in a 1976 paper by Newman and Rivlin [3], and another treatment has been sent to me by Nicholas Marshall and Vladimir Rokhlin (unpublished).

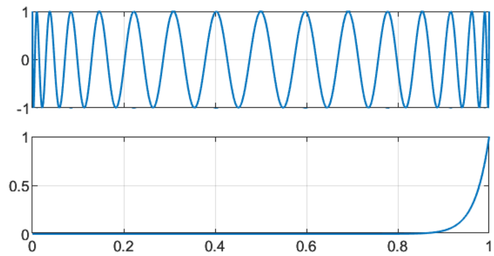

Figure 1. The Chebyshev polynomial $T_n(2x-1)$ and the monomial x^n for $n = 32$, both considered on $[0,1]$. The first has degree n by any measure, but x^n has effective degree only $O(\sqrt{n})$.

For example, in the Chebfun system for numerical computing with functions, a function f is approximated to about 15-digit accuracy by polynomials expressed as Chebyshev series (after transplantation of the interval of interest to $[-1,1]$). The function e^x on $[0,1]$ becomes a polynomial of degree 12. For x^n, the Chebfun degree is equal to n up to $n = 26$, but after this it is smaller, approximately $5\sqrt{n}$. For $n = 64$, 256 and 1024, the degrees are 43, 90 and 177.

Intuitively, what's going on is that on $[0,1]$, all high powers of x look the same, so that in the set $\{1, x, \ldots, x^n\}$, the higher powers can be well approximated by lower ones. The flip side of this observation is the phenomenon that to expand a more general degree n polynomial in this basis, you may need huge coefficients, potentially of size $O(C^n)$ with C as large as $3 + 2\sqrt{2} \approx 5.8$. In particular this is true for the transplanted Chebyshev polynomial $T_n(2x-1)$ shown in Figure 1 for $n = 32$. It is of degree n by any measure; it cannot be approximated by lower degree polynomials. However, its leading coefficient when expanded in the basis $\{1, x, \ldots, x^n\}$ is $\frac{1}{2}4^n$, and its largest coefficient in this expansion is even bigger.

To cook up even worse bases, we need look no further than the *Müntz approximation theorem* [1]. This theorem asserts that a necessary and sufficient condition for an infinite set of monomials x^{α_k} with unbounded exponents $0 = a_1 < a_2 < a_3 < \cdots$ to be dense in $C([0,1])$ is

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k} = \infty.$$

For example, the set $\{1, x^2, x^4, \ldots\}$ is dense in $C([0,1])$. Now, suppose you want to approximate the function $f(x) = x$ on $[0,1]$ in this basis to 6-digit accuracy. This is equivalent to the classic problem of polynomial approximation of $|x|$ on $[-1,1]$. It turns out you'll need 140,000 terms in the series, with coefficients as large as $10^{100{,}000}$.

The effective *rational* as opposed to polynomial degree of x^n is much less than $O(\sqrt{n})$: just $O(1)$, for the best rational approximants converge exponentially. But that is another story [2].

FURTHER READING

[1] J. M. Almira, Müntz type theorems. I, *Surv. Approx. Theory* 3 (2007), 152–194.
[2] Y. Nakatsukasa and L. N. Trefethen, Rational approximation of x^n, *Proc. AMS*, 146 (2018), 5219–5224.
[3] D. J. Newman and T. J. Rivlin, Approximation of monomials by lower degree polynomials, *Aequationes Math.* 14 (1976), 451–455.

Nick Trefethen
Trefethen is Professor of Numerical Analysis and head of the Numerical Analysis Group at the University of Oxford.