Notes of a Numerical Analyst

PDEs and integrals

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Last spring I taught a course on PDEs. A theme struck me as I collected my thoughts towards the end of the course. PDEs are defined by differentiation, but how often in analysing them we make use of integrals!

An example is the theory of *pseudo-differential operators* (and its cousin microlocal analysis). In one dimension for simplicity, we can write a function as an integral of its Fourier transform:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)}u(y)\,dy\,dk. \quad (1)$$

Since the derivative of e^{ikx} is $(ik)e^{ikx}$, it follows that u' can be written

$$u'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ik\,e^{ik(x-y)}u(y)\,dy\,dk. \quad (2)$$

Similarly, u'' corresponds to multiplication by $(ik)^2$, and so on. In other words, applying a constant-coefficient linear differential operator to a function u corresponds to multiplying its Fourier transform $\hat{u}(k)$ by a polynomial $p(k)$, which is called the *symbol*. But now the magic comes from allowing p to be more general than a polynomial. If $p(k) = (ik)^{1/2}$, we have taken 'half a derivative' (omitting technical details). And we can let the symbol depend on x too, so a pseudo-differential operator is defined by

$$Lu(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,k)\,e^{ik(x-y)}u(y)\,dy\,dk. \quad (3)$$

Remarkably, differentiation has become a special case of integration. Note that pseudo-differential operators are usually non-local.

Water waves — say, ripples on a pond — illustrate these ideas. Approximately speaking, they are governed by an equation $\partial u/\partial t = Lu$ where L is not a differential but a pseudo-differential operator with symbol $p(k) = i|k|^{1/2}$. We can see the physics of the non-locality by noting that a stationary flat patch of surface may accelerate upward because it is pushed from below by the pressure due to a higher surface elsewhere.

Figure 1. Water waves are governed by an integral rather than differential operator, related to the notion of "half a derivative." (Photo from iStock.)

Another example of integrals at the heart of PDE theory is the *theory of distributions*. If a function is smooth, we can define it pointwise, but we lose smoothness as we take derivatives. How can we rigorously define the Dirac delta function, for example, which should in some sense be the derivative of a step function? The answers come from integrals. A distribution u is defined as a linear functional acting on C^∞ test functions ϕ with compact support: $u : \phi \mapsto (u,\phi)$. When u is an ordinary function, the functional is just $(u,\phi) = \int_{-\infty}^{\infty} u(x)\phi(x)\,dx$. Integration by parts gives $u' : \phi \mapsto -(u,\phi')$, $u'' : \phi \mapsto (u,\phi'')$ and so on. We, thus, find that, viewed as a distribution, every function is infinitely differentiable.

From distributions, it is a small step to *weak solutions* of PDEs, a standard tool of PDE theory and practice. In the end, the very definition of PDE problems thereby comes down to integrals. Perhaps this brings the science full circle, since PDEs are so often derived by taking limits $\Delta x \to 0$ of conservation and balance laws expressed in integral form.

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