Nick Trefethen, 7 November 1997

TRANSITION TO TURBULENCE IN CHANNELS AND PIPES

Phenomenon to be explained. Why do laminar flows invariably become turbulent at high Reynolds numbers R, even though the flows are stable in the sense that all eigenmodes decay?

Our explanation. In the state space of velocity fields $u = u(x,y,z)$, as implied by those decaying eigenmodes, the laminar flow is a stable fixed point. However, the basin of attraction of this point is extraordinarily narrow, having width $O(R^\alpha)$ for some $\alpha < -1$. Thus extraordinarily small finite velocity perturbations may move the flow into the basin of attraction of a turbulent state (or quasi-state).

Non-normality + nonlinearity = narrow basin of attraction. The narrow basin of attraction comes about as follows. In state space, if we take u to be the deviation from the laminar velocity field, the Navier–Stokes equations reduce to

$$\frac{du}{dt} = Lu + N(u), \tag{1}$$

where L is linear but far from normal and N is nonlinear (quadratic) but energy-conserving (Reynolds–Orr equation). The term Lu produces transient amplification of amplitudes by a factor $O(R)$. This amplification is non-modal, meaning that inputs to the amplifier (right singular vectors, streamwise vortices...) are distinct from outputs (left singular vectors, streamwise streaks...). The nonlinear interactions, however, recycle some energy from outputs back to inputs, completing a nonlinear amplification loop.

History. The ingredients of this explanation date in a general way to Kelvin (1887) and Orr (1907), but these men did not know that the linear non-modal amplification was as large as $O(R)$, and for most of the 20th century, the emphasis in the fluid mechanics community has been on eigenmodes. The first paper to propose this "linear amplification plus nonlinear mixing" explanation of transition was by Boberg and Brosa, *Zeitschrift für Naturforschung,* 1988. Related important papers were by Butler and Farrell, *Physics of Fluids,* 1992 and Reddy and Henningson, *Journal of Fluid Mechanics,* 1993. The $\alpha < -1$ prediction first appeared in Trefethen, Trefethen, Reddy, and Driscoll, *Science,* 1993. It was verified by Navier–Stokes numerical simulations by Lundbladh, Reddy, and Henningson in 1994 and 1995, who obtained estimated upper bounds on the exponents of about $\alpha = -7/4$ for plane Poiseuille flow and $\alpha = -5/4$ for plane Couette flow, and by laboratory experiments, possibly (this is still under discussion), by Darbyshire and Mullin, *Journal of Fluid Mechanics,* 1995.

Illustration by a system of 2 ODEs. The mechanism of non-normal amplification + nonlinear mixing = narrow basin of attraction can be illustrated by a system of 2 ODEs,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -R^{-1} & 1 \\ 0 & -2R^{-1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \sqrt{u^2+v^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

(Trefethen et al., 1993, p. 582). The basin of attraction of $(0,0)^T$ for $R = 4$, of width about 0.11, is the narrow spiral shown on the right. As R increases for this system, the width decreases as $O(R^{-3})$; thus for $R = 40$ the width would be about 0.0001. Of course, such a simple caricature does not contain any of the complexity of the Navier–Stokes equations, and certainly not the physics of the turbulent state. A 3D model is enough for a chaotic attractor (Gebhardt and Grossmann, *Physical Review E,* 1994; Baggett, Driscoll and Trefethen, *Physics of Fluids,* 1995). Further and increasingly more physical models have been constructed by various authors of dimensions 3, 4, 6, 8, 12, 20, 30, and more (Baggett and Trefethen, *Physics of Fluids, 1997*), but when it comes to explaining how non-normality can combine with energy-conserving nonlinearity to generate extraordinary sensitivity to perturbations, we think (2) captures the essence of the matter.