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Book Reviews

Edited by Robert E. O'Malley, Jr.

Featured Review: Visual Complex Functions: An Introduction with Phase Portraits. *By Elias Wegert.* Birkhäuser/Springer Basel AG, Basel, 2012. \$59.95. xiv+360 pp., softcover. ISBN 978-3-0348-0179-9.

I don't read many mathematics books cover to cover, but I just finished Elias Wegert's *Visual Complex Functions*. Four young colleagues and I studied it section by section over a period of a few months. What an experience! Anyone who works with complex variables should read this book.

Wegert's theme is that to understand functions in the complex plane, we should *look at them*. Now it is not news that complex analysis is a geometric subject, and in fact, Riemann's investigations in this area played a role in the creation of the field of topology. But when it comes to actually plotting a complex function $f(z)$, the tradition is sparse. Part of the problem is that since the domain and range spaces are each two-dimensional, the graph of $f(z)$ is four-dimensional. So what exactly can we plot?

There are some classic answers, like the beautiful figures in *Tables of Functions with Formulae and Curves* by Jahnke and Emde in 1909, which became the very first book republished by Dover Publications after they set up operations in 1941. Figure 1 shows Jahnke and Emde's iconic image of the absolute value of the gamma function in the complex plane, produced in an era of hand calculations and expert technical artists.

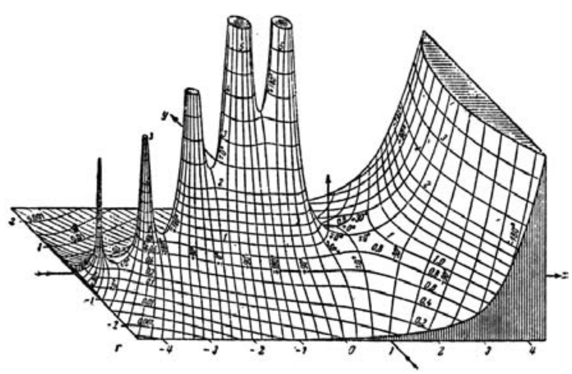

Fig. 1 *Gamma function from Jahnke and Emde.*

Times have changed, and computers with omnipotent color graphics are everywhere. Wegert's book is built on the idea of what he calls a *phase portrait* of a function. (He is not the first to work with such plots, but has carried the idea the

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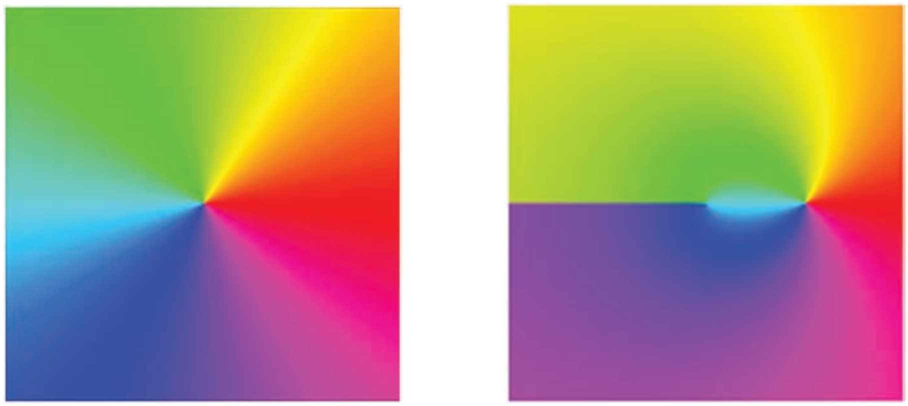

Fig. 2 *Phase portraits of z and $\log(z)$.*

furthest.) This is a 2D plot in which the color at each point z shows the phase of $f(z)$, that is, the number θ for which $f(z) = r\exp(i\theta)$ with $r > 0$. To explain the code, Figure 2(left) depicts a phase portrait of $f(z) = z$. Red corresponds to values of $f(z)$ that are positive real, light green to positive imaginary, cyan to negative real, and violet to negative imaginary.

In Figure 2(right), we see the function $f(z) = \log(z)$. Note the zero at $z = 1$, with the same sequence of colors around it as for $f(z) = z$ at $z = 0$. To the right the color is red, since $\log(z)$ is positive real for $z > 1$, and to the left it is cyan, since $\log(z)$ is negative real for $0 < z < 1$. Further left we see the branch cut along $(-\infty, 0]$. Above the branch cut, the predominant color is green, indicating a phase in the upper half-plane, and below it is violet, showing a phase in the lower half-plane.

After this warmup we can begin to make sense of more complicated functions, and my colleagues and I soon grew happily skilled at this. Consider, for example, the function f plotted in Figure 3(left). In the upper middle of the frame, f has a triple zero, that is, a zero of multiplicity 3. One sees this because the colors appear three times in sequence as one traces around the center point, with yellow to the left of red. In the right part of the frame it has a double pole, a pole of multiplicity 2. This is apparent because the colors are traced out twice in the reverse order, with yellow to the right of red. Near the bottom, it has an essential singularity, an isolated point singularity that is not a pole. The function shown in Figure 3(right) is $f(z) = \exp(1/z)$, the simplest example of a function with an essential singularity. I remember as a graduate student sensing a deep mystery in Picard's theorem, which asserts that all complex values except possibly one are attained in every neighborhood of an essential singularity. When you study figures like these, Picard's theorem begins to seem less mysterious.

Phase portraits don't show magnitudes directly, but they show them indirectly. We know that if yellow is to the left of red, this signifies a zero, implying that $|f|$ grows as one moves away from the zero. This happens on the left of the origin in the plot of $\exp(1/z)$, indicating the growth of this function there from values near zero as z moves leftward. To the right of the origin, yellow is to the right of red, indicating the decay of the function from values near infinity. Sometimes one might prefer to see the modulus directly, but Wegert makes a compelling case that there is usually more precise information in a phase portrait.

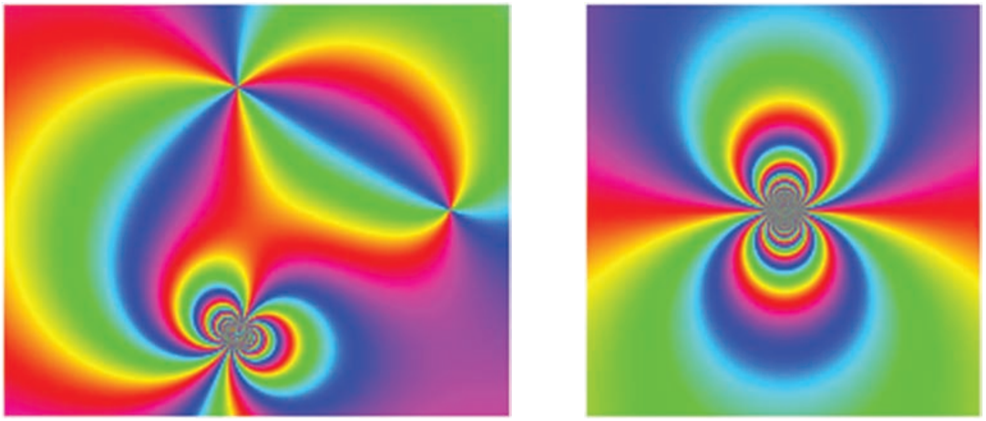

Fig. 3 *Poles, zeros, and essential singularities.*

In Figure 4(left), we see $f(z) = \sin(z)$. (This image is what Wegert calls an "enhanced phase portrait," with additional shading to further mark phase and amplitude.) The 2π-periodicity in the complex plane is evident, and one is reminded that $\sin(z)$ maps half-strips to half-planes. The point $z = 0$ at the center of the plot, for example, lies at the bottom of a half-strip that maps to the upper half-plane, whose predominant colors are orange, yellow, and green. The experienced eye can also infer from the colors the exponential increase of $|\sin(z)|$ as one moves away from the real axis, since yellow lies to the left of red in each parallel channel.

The right image in Figure 4 shows the Riemann zeta function $\zeta(z)$ in the region $-40 < \mathrm{Re}z < 10$, $-2 < \mathrm{Im}z < 48$. Notice that yellow is to the left of red at every rainbow point except one: these are the trivial zeros on the negative real axis and the nontrivial ones with $\mathrm{Re}z = \frac{1}{2}$. The exceptional point has yellow to the right of red, and this is the simple pole of $\zeta(z)$ at $z = 1$. In the left half-plane we see exponential growth in modulus as one moves leftward, since yellow again lies to the left of red in each channel.

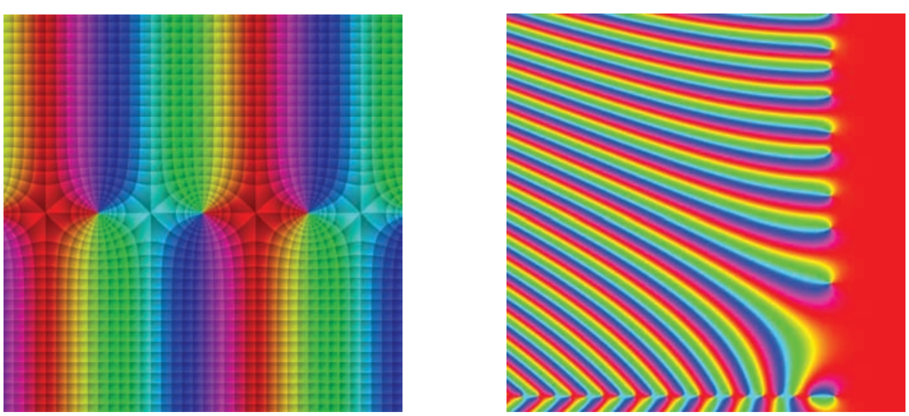

Fig. 4 $\sin(z)$ *and* $\zeta(z)$.

Portraits like these can give added depth to our understanding of many of the functions that have been studied by mathematicians in the past 200 years. Some of the functions explored in Wegert's book, at his Phase Plot Gallery at www.visual.wegert.com, and in his annual calendars of "Complex Beauties" coauthored with Gunter Semmler, include Bessel, Neumann, and Airy functions; theta, Bernoulli, and Painlevé functions; Fresnel integrals, elliptic functions, and continued fractions.

But Wegert's purpose is more than just the exploration of particular functions, for phase portraits can teach us fundamental principles of complex analysis. The two images shown in Figure 5 explore the idea of multivaluedness, on the left by following a chain of function elements twice around a square root singularity, and on the right with a 3D plot of the Riemann surface for the cube root. If you circle around twice on the left, or three times on the right, you return to your starting color.

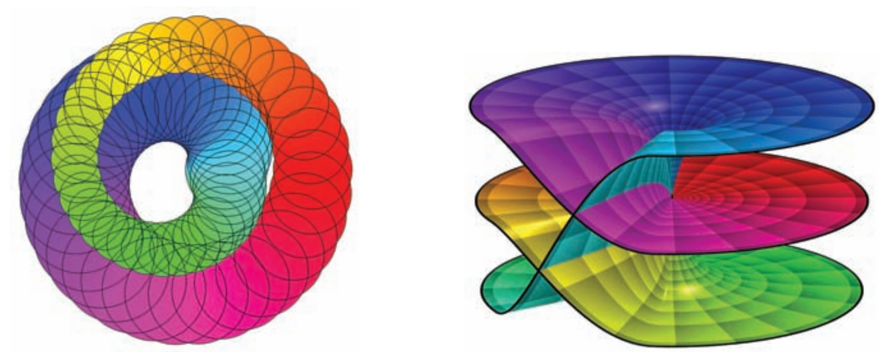

Fig. 5 *Multivalued functions* $z^{1/2}$ *and* $z^{1/3}$.

The plot in Figure 6(left) shows an analytic function f in the unit disk with a countably infinite set of zeros lying along a curve that spirals out to the unit circle. This picture makes it easy to understand that the circle is a natural boundary for this function, meaning that f cannot be analytically continued anywhere outside. This function happens to be an infinite Blaschke product, with constant modulus

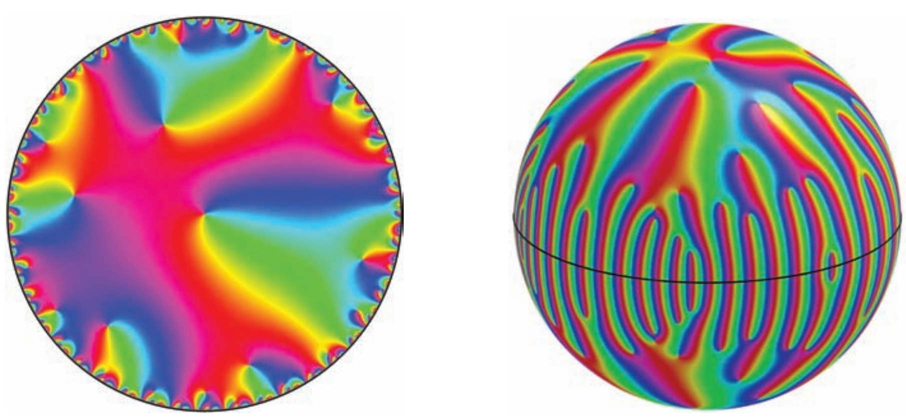

Fig. 6 *Blaschke products in the disk and on the Riemann sphere.*

in the limit as $|z| \to 1$, as is reflected in the color boundaries meeting the circle at right angles. In Figure 6(right), a simpler Blaschke product is shown, a finite one, analytically continuable to a meromorphic function on the whole Riemann sphere, i.e., a rational function. One sees an elegant symmetry of zeros in one hemisphere, poles symmetrically located in the other.

Phase portraits reveal fascinating properties of Taylor series as they converge to analytic functions. In Figure 7(left), $f(z)$ is the degree 60 Taylor polynomial for $1/(1-z)$. Inside the unit circle, the series closely represents the limit function, but outside it becomes a circus tent growing at the rate $|z|^{60}$ (yellow to the left of red in each channel). The transition from one behavior to the other is marked by a line of zeros that converges to the circle of convergence; this is *Jentzsch's theorem*. In Figure 7(right), the middle of the image looks like $\exp(z)$: the stripes are $2\pi i$-periodic and the growth is exactly exponential as one moves to the right, since the stripes are parallel with yellow to the left of red. Outside a certain curve of zeros, however, the behavior changes completely, and in fact, what is plotted here is not $\exp(z)$ but its Taylor polynomial of degree 60. Since $\exp(z)$ is entire, Jentzsch's theorem is inapplicable, and the curve in question moves out to ∞ as the degree increases. This curve was analyzed by Szegő and later generalized for Padé rational approximations of $\exp(z)$ by Saff and Varga.

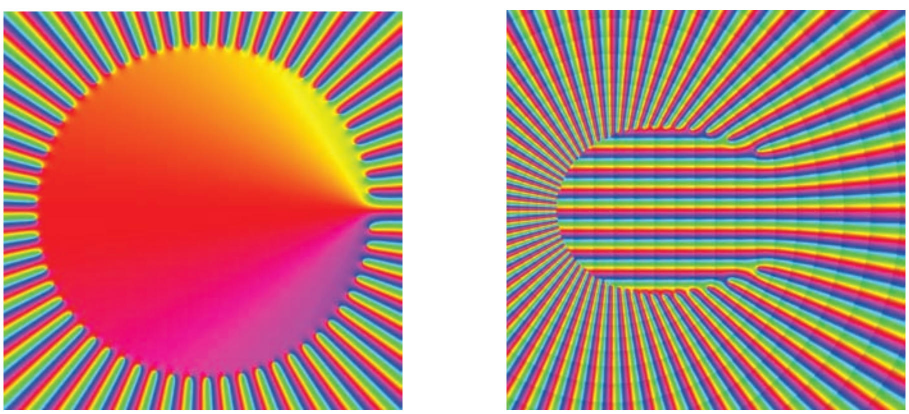

Fig. 7 *Taylor polynomials of* $1/(1-z)$ *and* $\exp(z)$.

One of the central techniques of complex variables is the construction of functions via Cauchy integrals. For example, suppose we consider a Cauchy integral defined by integrating the constant function 1 over the unit interval $[-1, 1]$. The resulting function $f(z)$, plotted in Figure 8(left), is a branch of $\log((z-1)/(z+1))/2\pi i$ with a branch cut along $[-1, 1]$; the Plemelj–Sokhotsky jump relation tells us that f jumps by exactly 1 across the interval. What if the same function 1 is used along a different contour? We get a less standard branch of $\log((z-1)/(z+1))/2\pi i$, as shown in Figure 8(right). Until I saw such pictures, Cauchy integrals never seemed so simple.

For a numerical analyst like me, these ideas become especially intriguing when Cauchy integrals are combined with discretization. The plot shown in Figure 9(left) shows a certain Cauchy integral approximated by the 200-point trapezoid rule over a contour that winds around twice. The resulting function $f(z)$ is a rational approximation to the function defined by the exact Cauchy integral, with 200 poles along the contour and corresponding zeros lining up nearby outside. In Figure 9(right),

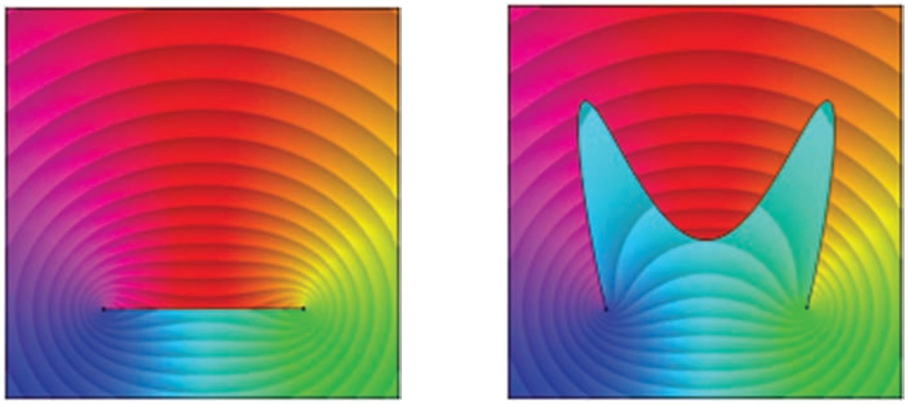

Fig. 8 *Two branches of* $\log((z-1)/(z+1))/2\pi i$.

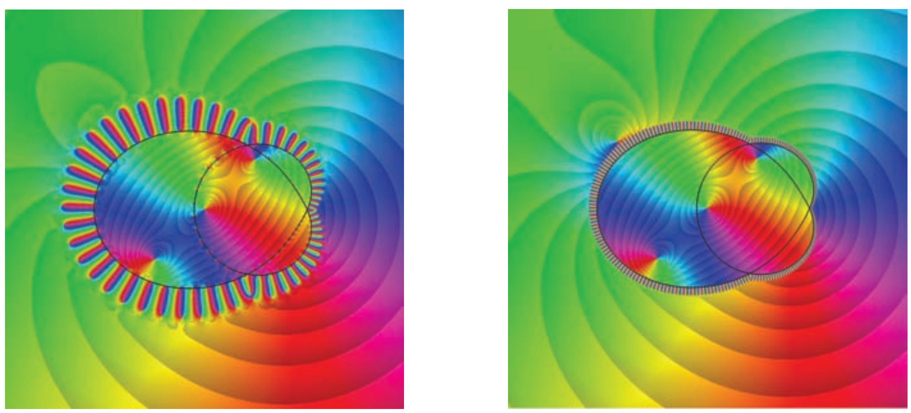

Fig. 9 *Discretized Cauchy integrals.*

the number of quadrature points has increased to 800, and the pole-zero pairs have come closer together. In the limit they define a layer potential of the kind familiar in the field of integral equations, and pictures like these reveal connections between integral equations, complex variables, quadrature theory, rational functions, potential theory, digital filtering, the theory of hyperfunctions, ... it is hard to know where to terminate the list!

Visual Complex Functions is a beautiful and careful presentation of an entire advanced introduction to complex analysis based on phase portraits and, where appropriate, other kinds of computer-generated pictures. The book is mathematically complete, with theorems and proofs as well as pictures. It treats series, products, singularities, analytic continuation, conformal mapping, Riemann surfaces, and many other topics always from a fresh and compelling point of view. My understanding of many ideas and phenomena deepened through reading this book.

Although Wegert's aim was to produce a volume that could be used as a main or companion text for an introductory course, my colleagues and I felt that this aim is not realized. It is hard to imagine learning complex variables for the first time with this book, for the treatment—however enlightening and exciting—is just too idiosyncratic.

A few of the definitions (for example, of paths, curves, arcs, and traces), while carefully thought through, seem not quite standard and potentially confusing for a beginner; occasionally one finds a term imperfectly translated from German, like *ring* instead of *annulus* or *neutral element* instead of *identity*. But for anyone who has already had a first exposure to complex variables, this book offers a thrilling second journey, or a third. Remarkably, despite 360 pages of beautiful glossy color, it sells for less than $50. An epilogue shows how to draw your own phase portraits, and MATLAB-based software is available at www.visual.wegert.com.

I would like to thank my young colleagues for their contributions to this review: Anthony Austin, Mohsin Javed, Georges Klein, and Alex Townsend.

When I went off to graduate school in 1977, a good fraction of the way back to the era of Jahnke and Emde, I loved complex variables and wanted to work with them on the computer. Sometime during my first semester a friend from undergraduate days wrote and asked, "Are you skiing on the complex functions yet?" With Wegert's extraordinary book, at last I am skiing on the complex functions.

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Combinatorics of Permutations. Second Edition. *By Miklós Bóna.* CRC Press, Boca Raton, FL, 2012. $89.95. xiv+458 pp., hardcover. ISBN 978-1-4398-5051-0.

In a first combinatorics or discrete mathematics course a student will learn the formula $n!$ for the number of permutations of n objects. Aside from using this as an annoying correction factor in counting unordered arrangements of various types, the next and last exposure to the combinatorics of permutations will often be to the formula for the number of derangements presented as an illustration of the use of the principle of inclusion-exclusion. This is a shame because the combinatorial theory of permutations contains a wealth of interesting and accessible results using methods from all the drawers of the combinatorial tool chest, and often suggesting surprising connections and relationships between apparently unrelated structures.

This book attempts to fill in some of those gaps while acknowledging that the area as a whole is still so broad that complete coverage is impossible. Of course this means that the selection of topics is somewhat according to the author's tastes, but certainly Bóna has done an excellent job of choosing a variety of areas to discuss. The introduction suggests that the material was developed for a graduate class in combinatorics, but it could just as easily be adapted to an advanced undergraduate course, or for independent study by a motivated student.

Chapters 1 and 2 deal with permutations thought of as sequences from the ordered set $1 < 2 < \cdots < n$. First, descents, i.e., adjacent inversions are discussed, and then inversions in general are considered. The results discussed are largely enumerative, but the mix of techniques (surjections and bijections, generating functions, recurrences) already illustrates one of the main themes of the book. Chapter 3 turns to the familiar decomposition of permutations as products of cycles and investigates various questions one might ask about properties of such decompositions, or the permutations whose representations are of a particular type. In particular we return to the derangements, but in a much more general setting which permits one to see striking results like Example 3.58, that the exponential generating function for permutations whose cycle decomposition consists of cycles of odd length only is $\sqrt{(1+x)/(1-x)}$.

Chapters 4 and 5 (and the reviewer must declare an interest) turn to the less familiar area of pattern avoidance. This again deals with permutations thought of as sequences