# A Generalized Miraculous Cancellation Formula

MARKUS UPMEIER

## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Notation</td>
<td>6</td>
</tr>
<tr>
<td>1.</td>
<td>Bundles And Characteristic Classes</td>
<td>7</td>
</tr>
<tr>
<td>1.1.</td>
<td>Symmetric Polynomials</td>
<td>7</td>
</tr>
<tr>
<td>1.2.</td>
<td>Fiber Bundles</td>
<td>8</td>
</tr>
<tr>
<td>1.3.</td>
<td>Principal Bundles</td>
<td>11</td>
</tr>
<tr>
<td>1.4.</td>
<td>Associated Bundles</td>
<td>11</td>
</tr>
<tr>
<td>1.5.</td>
<td>Reduction of Structure Group</td>
<td>13</td>
</tr>
<tr>
<td>1.6.</td>
<td>Classification of Vector Bundles. The Splitting Principle</td>
<td>13</td>
</tr>
<tr>
<td>1.7.</td>
<td>Stiefel-Whitney Classes</td>
<td>16</td>
</tr>
<tr>
<td>1.8.</td>
<td>Chern Classes</td>
<td>17</td>
</tr>
<tr>
<td>1.9.</td>
<td>Pontrjagin Classes</td>
<td>18</td>
</tr>
<tr>
<td>2.</td>
<td>Connection And Curvature</td>
<td>19</td>
</tr>
<tr>
<td>2.1.</td>
<td>Connections on Principal Bundles</td>
<td>19</td>
</tr>
<tr>
<td>2.2.</td>
<td>Connections on Vector Bundles</td>
<td>21</td>
</tr>
<tr>
<td>2.3.</td>
<td>Covariant Differentiation</td>
<td>22</td>
</tr>
<tr>
<td>2.4.</td>
<td>Invariant Polynomials</td>
<td>24</td>
</tr>
<tr>
<td>2.5.</td>
<td>Characteristic forms</td>
<td>27</td>
</tr>
<tr>
<td>2.6.</td>
<td>Induced Connections</td>
<td>29</td>
</tr>
<tr>
<td>3.</td>
<td>A Refined Chern Root Formalism</td>
<td>33</td>
</tr>
<tr>
<td>3.1.</td>
<td>The Splitting Ring</td>
<td>34</td>
</tr>
<tr>
<td>3.2.</td>
<td>Application to Chern Forms</td>
<td>35</td>
</tr>
<tr>
<td>3.3.</td>
<td>Generalization of a Theorem of Borel and Hirzebruch</td>
<td>36</td>
</tr>
<tr>
<td>4.</td>
<td>Spin Structures on Vector Bundles</td>
<td>39</td>
</tr>
<tr>
<td>4.1.</td>
<td>Clifford Algebras</td>
<td>39</td>
</tr>
<tr>
<td>4.2.</td>
<td>Pin and Spin as Two-Sheeted Coverings</td>
<td>41</td>
</tr>
<tr>
<td>4.3.</td>
<td>Čech cohomology</td>
<td>42</td>
</tr>
<tr>
<td>4.4.</td>
<td>Spin Structures on Vector Bundles</td>
<td>43</td>
</tr>
<tr>
<td>4.5.</td>
<td>Spin* Structures</td>
<td>46</td>
</tr>
<tr>
<td>5.</td>
<td>The Thom Class</td>
<td>48</td>
</tr>
<tr>
<td>5.1.</td>
<td>The Thom Isomorphism Theorem</td>
<td>48</td>
</tr>
<tr>
<td>5.2.</td>
<td>The Thom Isomorphism and Poincaré Duality</td>
<td>50</td>
</tr>
<tr>
<td>5.3.</td>
<td>The Gysin Sequence</td>
<td>52</td>
</tr>
<tr>
<td>5.4.</td>
<td>The Thom Space</td>
<td>52</td>
</tr>
<tr>
<td>5.5.</td>
<td>The Fundamental Cohomology Class of a Submanifold</td>
<td>53</td>
</tr>
<tr>
<td>5.6.</td>
<td>The Chern class of a Codimension 2 Submanifold</td>
<td>54</td>
</tr>
<tr>
<td>6.</td>
<td>Multiplicative Sequences and the Virtual Index</td>
<td>56</td>
</tr>
<tr>
<td>6.1.</td>
<td>Multiplicative Sequences</td>
<td>56</td>
</tr>
</tbody>
</table>
Introduction

This thesis deals with characteristic classes and integrality theorems for manifolds of dimension $8k + 4$.

For a vector bundle $E$ over a smooth closed spin manifold $M$ of dimension $8k + 4$, a well-known theorem of Atiyah and Hirzebruch [AH59] asserts that

$$\langle \hat{A}(TM) \, \text{ch}(E \otimes \mathbb{C}), [M] \rangle \in 2\mathbb{Z} \quad (1)$$

One of the fundamental theorems in the theory of manifolds is due to Rokhlin and states that the signature of a smooth closed spin 4-manifold is divisible by 16, a result which was generalized by Ochanine [Och81] to dimensions $8k + 4$. Later, Landweber [Lan86] used elliptic genera and modular forms to show that it is also possible to derive Ochanine's result directly from (1).

For a 4-dimensional manifold $M$ it is easy to show that

$$\hat{L}(TM) = \frac{1}{3}p_1(TM) = -8 \left( -\frac{1}{24}p_1(TM) \right) = -8\hat{A}(TM) \quad (2)$$

Of course (1) and (2) imply Rokhlin's original divisibility result, i.e. the case $k = 0$. The “miraculous cancellation” formula of Alvarez-Gaumé and Witten [AGW84] may be viewed as an extension of (2) to 12-dimensional manifolds, i.e. for $k = 1$. This formula states that in degree 12 we have

$$\{ \hat{L}(TM) \}^{(12)} = 8 \{ \hat{A}(TM) (\text{ch}(TM) - 4) \}^{(12)}$$

Again, the divisibility of the signature in dimension 12 follows now from (1). By refining Landweber's modularity arguments to differential forms, Liu [Liu95] was able to generalize the miraculous cancellation formula to higher dimensions, thus deducing Ochanine's result entirely from (1).

In the case of a smooth closed oriented 4-manifold $M$ not necessarily spin, Rokhlin generalized his divisibility result to a congruence formula

$$\frac{\text{sgn}(M) - \text{sgn}(B \cdot B)}{8} \equiv \phi(B) \mod 2\mathbb{Z} \quad (3)$$

Here, $B$ is a codimension 2 submanifold of $M$ realizing the homology class Poincaré-dual to the second Stiefel-Whitney class of $M$, $B \cdot B$ denotes the self-intersection of $B$ in $M$, and $\phi$ is a spin cobordism invariant, called Ochanine invariant (or Arf invariant). This congruence formula was generalized again by Ochanine [Och81] to $8k + 4$-dimensional manifolds $M$ that are Spin$^c$. Note here that the congruence formula generalizes the original divisibility results.

Han and Zhang showed in [HZ04] that it is possible to further generalize Liu's miraculous cancellation formula and to allow an additional twist from a coefficient line bundle. This enables them to prove also the congruence formula (3) directly from (1). In their cancellation formula they identify a certain characteristic form of interest as a linear combination with some integer coefficients of forms, to which (1) applies.
In this diploma thesis I will present the derivation of this twisted generalized miraculous cancellation formula and its application to the congruence formula (3). In addition, as the main result of the thesis, I prove a cancellation formula where all the integer coefficients occurring in the linear combination are explicitly determined and expressed in terms of familiar number theoretic functions. As a consequence, these (rather complicated) coefficients may be calculated using a computer. In this sense, the "miraculous cancellation" formula, for all dimensions $8k+4$ and including line bundle twisting, is now available in a completely explicit form, because of (i) the use of differential forms instead of cohomology classes, and (ii), as a result of this thesis, the explicit arithmetic nature of the integer coefficients occurring in the characteristic class.

As an application I prove an explicit miraculous cancellation formula in dimension $20$ (i.e. for $k = 2$)

$$\left\{ \hat{L}(TM) \right\}_{(20)} = 8 \left\{ \hat{A}(TM) \left( 3 \text{ch}(T_{\mathbb{C}} M) - 6 - \text{ch}(\Lambda^2 T_{\mathbb{C}} M) \right) \right\}_{(20)}$$

This formula seems to be new, and its proof requires a bit of knowledge on theta functions and modular forms. In fact, the miraculous cancellation formula may be viewed as a characteristic class version of a "hidden" theta identity. Is is tempting to speculate that multi-variable theta functions could be used to prove such identities in the more difficult case where the coefficients lie in a vector bundle instead of a line bundle.

This thesis also formalizes a method sometimes called "new Chern root formalism", which may be regarded as a kind of splitting principle for characteristic forms. Roughly, the refined Chern root formalism states that in order to establish general relations among characteristic forms of vector bundles with connections it suffices to carry out the calculations merely in the case of a direct sum of line bundles with "diagonal" connection. The discussion culminates in a generalization of a theorem of Borel and Hirzebruch regarding Chern roots of associated bundles.

The thesis is organized as follows. In section 1, a quick review of the prerequisites such as characteristic classes is given. Axioms uniquely determining the Chern and Stiefel-Whitney classes are stated, whose existence is proven in later sections.

In the next section we deal with connections on principal bundles, invariant polynomials, and the Chern-Weil theory of characteristic forms.

Section 3 refines the Chern root formalism to the level of differential forms, and "new Chern roots" of associated bundles needed later in the development are computed.

Spin and Spin$^c$ structures on vector bundles are introduced in Section 4, and criteria for their existence are proven. A very brief definition of Čech cohomology can also be found here.

In section 5 the Thom class of a vector bundle is studied, mainly in order to be able to prove a "formula for the virtual index" (cf. [Hir66]) in section 6. This
formula will be used in the application of the cancellation formula to Ochanine's congruence theorem in section 9.

The number theoretic results presented in section 7 constitute an essential ingredient in the proof of the generalized miraculous cancellation formula in section 8. Here one also finds the explicit computation of the coefficients, their relationship to arithmetic functions and the specialization to dimension 20 stated above.

Finally, I would like to express my gratitude to Professor Schick for his valuable advice and numerous suggestions.
### Notation

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_k )</td>
<td>( k )-th elementary symmetric polynomial</td>
</tr>
<tr>
<td>( A^{sym}[X_1, \ldots, X_n] )</td>
<td>subring of symmetric polynomials</td>
</tr>
<tr>
<td>( \text{Homeo}(F) )</td>
<td>group of homeomorphisms of a topological space ( F )</td>
</tr>
<tr>
<td>( \text{Diff}(F) )</td>
<td>group of diffeomorphisms of a manifold ( F )</td>
</tr>
<tr>
<td>( \mathcal{F} \text{Bun} )</td>
<td>category of fiber bundles</td>
</tr>
<tr>
<td>( \text{VectBun} )</td>
<td>category of vector bundles (either real or complex, adapted to the case under consideration)</td>
</tr>
<tr>
<td>( \text{Vect} )</td>
<td>category of finite-dimensional (either real or complex) vector spaces</td>
</tr>
<tr>
<td>( \text{Prin}(G) )</td>
<td>category of principal ( G )-bundles</td>
</tr>
<tr>
<td>( P \times_p )</td>
<td>associated bundle</td>
</tr>
<tr>
<td>( P[\lambda] )</td>
<td>associated ( H )-bundle of a ( G )-bundle ( P ) under a homomorphism ( \lambda : G \to H )</td>
</tr>
<tr>
<td>( E_1 \oplus E_2 )</td>
<td>Whitney sum of two vector bundles</td>
</tr>
<tr>
<td>( E_C = E \oplus \mathbb{C} )</td>
<td>complexification of a real vector bundle ( E )</td>
</tr>
<tr>
<td>( \mathbb{R}P^n, \mathbb{C}P^n )</td>
<td>real and complex projective spaces</td>
</tr>
<tr>
<td>( G_k(\mathbb{R}^n), G_k(\mathbb{C}^n) )</td>
<td>real (resp. complex) Grassmannian manifold of ( k )-planes</td>
</tr>
<tr>
<td>( \langle \cdot \rangle )</td>
<td>Kronecker product ( H^<em>(X,A) \otimes H_</em>(X,A) \to \mathbb{R} )</td>
</tr>
<tr>
<td>( \lambda_p )</td>
<td>( \lambda_p : G \to P, g \mapsto pg ) for a principal ( G )-bundle ( P )</td>
</tr>
<tr>
<td>( \rho_g )</td>
<td>( \rho_g : P \to P, p \mapsto pg ) for a principal ( G )-bundle ( P )</td>
</tr>
<tr>
<td>( l_g, r_g )</td>
<td>left-translation (resp. right-translation) by ( g \in G ) in a Lie group ( G )</td>
</tr>
<tr>
<td>( \mathfrak{g} )</td>
<td>Lie algebra of ( G )</td>
</tr>
<tr>
<td>( \text{Hom}(V_1, \ldots, V_n; W) )</td>
<td>abelian group of multilinear maps for modules ( V_i, W )</td>
</tr>
<tr>
<td>( \mathcal{O}(V)^G )</td>
<td>ring of ( G )-invariant polynomials on ( V )</td>
</tr>
<tr>
<td>( S(V) )</td>
<td>symmetric algebra</td>
</tr>
<tr>
<td>( T(-) )</td>
<td>tangent bundle functor</td>
</tr>
<tr>
<td>( f_{*,x} : T_xX \to T_{f(x)}Y )</td>
<td>push-forward or derivative of a map ( f : X \to Y )</td>
</tr>
<tr>
<td>( \Omega^*(M) )</td>
<td>algebra of differential forms on ( M )</td>
</tr>
<tr>
<td>( C^\infty(X) )</td>
<td>algebra of smooth real-valued (resp. complex-valued) functions on a manifold ( X )</td>
</tr>
<tr>
<td>( \Gamma(E) )</td>
<td>( C^\infty(X) )-module of sections of a vector bundle ( E \to X )</td>
</tr>
<tr>
<td>( w_k, p_k, c_k )</td>
<td>( k )-th Stiefel-Whitney, Pontrjagin, and Chern class, respectively</td>
</tr>
<tr>
<td>( U_{i_1 \cdots i_k} )</td>
<td>( U_{i_1 \cap \ldots \cap U_{i_k}} ) for an open cover ( {U_i} )</td>
</tr>
</tbody>
</table>
1. Bundles And Characteristic Classes

The first half of this section deals with the theory of fiber bundles, in particular associated bundles and the reduction of structure groups. These concepts will prove useful when describing how continuous functors on vector spaces yield corresponding functors for bundles. We will revisit this point later when considering vector bundles equipped with connections.

Next, we will provide an axiomatic description for the Stiefel-Whitney, Chern, and Pontrjagin classes. On the one hand, only the existence of the first two Stiefel-Whitney classes $w_1, w_2$ will be proven (which will actually be the only ones we will make use of), and on the other, we shall give several constructions (in terms of curvature, using the Thom/Euler class, Čech cohomology) for the Chern classes. It is then convenient to have axioms showing that all of these definitions must coincide. A construction of the Stiefel-Whitney classes using Steenrod squares may be found in [MS74], which is also a good general reference for all of the material presented in the second half of this section. [Ste51] may be consulted for a more detailed treatment of fiber bundles.

1.1. Symmetric Polynomials. We will need symmetric polynomials throughout the text. Let us therefore recall their definition.

**Definition 1.1.** Let $A$ be a commutative ring with unit. The elementary symmetric polynomials $\sigma_k$ in $n$ variables $X_1, \ldots, X_n$ are defined by the relation

$$\prod_{i=1}^{n}(t - X_i) = t^n - \sigma_1(X_1, \ldots, X_n)t^{n-1} + \cdots + (-1)^n\sigma_n(X_1, \ldots, X_n) \quad (1.1)$$

$\sigma_k$ is thus a homogeneous polynomial of degree $k$.

**Remark 1.2.** Let $\sigma_k^{(n)}$ and $\sigma_k^{(m)}$ be the elementary symmetric polynomials in variables $m > n$. Then from (1.1)

$$\sigma_k^{(m)}(X_1, \ldots, X_n, 0, \ldots, 0) = \sigma_k^{(n)}(X_1, \ldots, X_n) \quad (k \leq n) \quad (1.2)$$

For the convenience of the reader we will state here the most important properties of the elementary symmetric polynomials. For proofs we refer to [Lan02].

**Theorem 1.3.** Let $f(X_1, \ldots, X_n)$ be a symmetric polynomial, i.e. $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = f(X_1, \ldots, X_n)$ for any permutation $\sigma$. Then $f$ may be written as a polynomial in the elementary symmetric polynomials: $f = p(\sigma_1, \ldots, \sigma_n)$ for $p$ a polynomial over $A$. Moreover, the $\sigma_k$ are algebraically independent over $A$.

**Remark 1.4.** If we write $A^{sym}[X_1, \ldots, X_n]$ for the sub ring of symmetric polynomials over $A$, the theorem states that

$$A^{sym}[X_1, \ldots, X_n] = A[\sigma_1, \ldots, \sigma_n]$$
1.2. Fiber Bundles. Everything in this subsection is valid also in the smooth setting, replacing the topological groups by Lie groups, homeomorphisms by diffeomorphisms, and topological spaces by smooth manifolds.

Definition 1.5. A fiber bundle $F \hookrightarrow E \xrightarrow{\pi} X$ with typical fiber $F$ and structure group $G$ consists of a (continuous) map $\pi : E \to X$, a topological group $G$ acting (continuously from the left) on a space $F$, and an equivalence class of $G$-trivializations.

Here, by a $G$-trivialization $\{h_i\}_{i \in I}$ we mean a family of local trivializations $E|_{U_i} := \pi^{-1}(U_i) \xrightarrow{h_i} U_i \times F$ (i.e. homeomorphisms $h_i$ with $\text{pr}_1 \circ h_i = \pi$) for an open cover $U_i \subseteq X$ such that for overlapping $U_{ij} := U_i \cap U_j \neq \emptyset$ the change of trivialization $h_i h_j^{-1} : U_{ij} \times F \to U_{ij} \times F$ is given by continuous transition functions $g_{ij} : U_{ij} \to G$ and the $G$-operation on $F$

$$h_i h_j^{-1}(x, f) = (x, g_{ij}(x)f) \quad (1.3)$$

Two such $G$-trivializations $\{h_i\}, \{h'_i\}$ are considered to be equivalent if their union is again a $G$-trivialization.

Remark 1.6. (different definition of fiber bundles) There is a technical point here. An operation $\rho : G \times F \to F$ may be viewed as a group homomorphism $Ad(\rho) : G \to \text{Homeo}(F)$. Equip $\text{Homeo}(F)$ with the CO-topology. Then the continuity of $\rho$ implies that of $Ad(\rho)$ (the converse of this statement however is false in general).

A fiber bundle in the sense of Ehresmann-Feldbau (cf. [Ste51]) with typical fiber $F$ and structure group $G \subseteq \text{Homeo}(F)$ (without topology) is a continuous map $\pi : E \to X$ that may be trivialized $h_i : \pi^{-1}(U_i) \to U_i \times F$ over an open cover $U_i \subseteq X$ such that the change of trivialization may be written as $h_i h_j^{-1}(x, f) = (x, g_{ij}(x)f)$ for continuous “$E$-$F$ transition functions” $g_{ij} : U_{ij} \to \text{Homeo}(F)$ which are to take values only in $G$. The main difference here is that we have dropped the topology of $G$ (and always use subspace topology of $\text{Homeo}(F)$ instead).

Our definition of a fiber bundle is strictly broader than the Ehresmann-Feldbau definition (for transition functions $g_{ij}$ the maps $Ad(\rho) \circ g_{ij}$ may be chosen as $E$-$F$ transition functions). Conversely, if $G \subseteq \text{Homeo}(F)$ is given the subspace topology of CO-topology, the continuity of the transition functions $g_{ij} : U_{ij} \to G$ in (1.3) is automatic from the continuity of $h_i h_j^{-1}$. For subgroups $G \subseteq \text{Homeo}(F)$ that have subspace topology the above two notions thus coincide.

Definition 1.7. Let $E_1 \xrightarrow{\pi_1} X_1$, $E_2 \xrightarrow{\pi_2} X_2$ be fiber bundles having typical fiber $F$ and structure group $G$. Chose $G$-trivializations $\{h_{1i}\}, \{h_{2j}\}$. A bundle morphism $(\phi, \varphi)$ consists of continuous maps $\phi, \varphi$ such that

$$\begin{array}{ccc}
E_1 & \xrightarrow{\phi} & E_2 \\
\pi_1 & \downarrow & \pi_2 \\
X_1 & \xrightarrow{\varphi} & X_2
\end{array}$$
commutes. We require moreover that $h_j^{(2)} \circ \phi \circ (h_i^{(1)})^{-1} : (U_i \cap \varphi^{-1}(U_j)) \times F \to U_j \times F$ (whenever defined) is given in terms of a continuous map $g_{ij}^{\phi} : U_i \cap \varphi^{-1}(U_j) \to G$ and the $G$-operation

$$h_j^{(2)} \circ \phi \circ (h_i^{(1)})^{-1}(x, f) = \left(\varphi(x), g_{ij}^{\phi}(x)f\right)$$

We shall call a continuous pair $(\phi, \varphi)$ that only satisfies $\varphi_1 = \pi_2 \phi$ (without any referral to the structure group) a bundle morphism in the weak sense.

Remark 1.8. In the above situation it is commonly said that $\phi$ is a bundle morphism covering $\varphi$. By (1.3) the definition is independent of the choice of representative $G$-trivializations. We have described above the category of fiber bundles $\mathcal{F}Bun$ along with its subcategory $\mathcal{F}Bun_X$ of fiber bundles over $X$ (both groupoids). Also we have the subcategory $\mathcal{F}Bun(G \times F \xrightarrow{\rho} F)$ of fiber bundles with fixed typical fiber, structure group and operation $\rho$, and then again of course $\mathcal{F}Bun_X(G \times F \xrightarrow{\rho} F)$ for a fixed base $X$.

Example 1.9. A vector bundle is a fiber bundle with typical fiber a finite-dimensional real or complex vector space $V$ and structure group $GL(V)$ (and the obvious operation). We may then define on every fiber $E_x := \pi^{-1}(x)$ a vector space structure by requiring every local trivialization $h_i$ to map each fiber isomorphically. The category of vector bundles will be denoted by $\text{VectBun} = \mathcal{F}Bun(GL(V) \times V \to V)$. The subcategory of vector bundles over a fixed base $X$ will be written $\text{ VectBun}_X$.

We will usually work with a slightly broader definition of vector bundle morphisms, namely that of a fiberwise linear map $f : E_1 \to E_2$ covering some $\varphi : X_1 \to X_2$ which maps each fiber isomorphically (we do not require the typical fibers to be equal).

Proposition 1.10. Let $\{U_i\}_{i \in I}$ be an open cover of $X$, and $G$ a topological group acting on a space $F$. Then, given a set of transition functions $g_{ij} : U_{ij} \to G$ satisfying the “Čech cocycle relation”

$$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x) \quad (\forall x \in U_{ijk} := U_i \cap U_j \cap U_k)$$

there exists a fiber bundle $E \to X$ with structure group $G$, having precisely the $g_{ij}$ as transition functions.

Proof. This is done by gluing local products $U_i \times F$ using the equivalence relation generated by

$$(x, f) \sim (x, g_{ij}(x)f)$$

See [Ste51] for further details, and also figure 1.1.

Proposition 1.11. Let $E_1 \xrightarrow{\varphi_1} X, E_2 \xrightarrow{\varphi_2} X$ be fiber bundles over the same base $X$ and having the same typical fiber $F$ and structure group $G$. Chose $G$-trivializations $\left\{h_i^{(1)}\right\}, \left\{h_i^{(2)}\right\}$ over a common open cover. A family $\lambda_i : U_i \to G$ of continuous maps satisfying

$$g_{ij}^{(2)}(x) = \lambda_i(x) \cdot g_{ij}^{(1)}(x) \cdot \lambda_j(x)^{-1} \quad (\forall x \in U_{ij})$$

(1.4)

defines an isomorphism $E_1 \to E_2$ over $\text{id}_X$. 
Proof. Condition (1.4) ensures that the locally defined $\phi_i(x,f) = (x, \lambda_i(x)f)$

\[ \begin{array}{c}
U_i \times F \xrightarrow{h_i^{(1)}} E_1|U_i \\
\phi_i \downarrow \\
U_i \times F \xrightarrow{h_i^{(2)}} E_2|U_i 
\end{array} \]

piece together the required map $E_1 \to E_2$. \qed

**Corollary 1.12.** The fiber bundle $E \to X$ constructed in proposition 1.10 having
the $g_{ij}$ as transition functions is unique up to isomorphism. If we restrict the $g_{ij}$ to
a refinement $V_r \subset U_i(r)$, $r \in R$ of $U_i$ (so $\tilde{g}_{rs} := g_{i(r)j(s)}|_{V_r \cap V_s}$) to construct $E \to X$
by proposition 1.10, then $E$ and $\tilde{E}$ are isomorphic.

Proof. The first assertion is clear by taking $\lambda_i(x) \equiv 1$ in (1.4). Next, restricting the
 corresponding local trivializations of $E$ to the refinement $V_r$ by $\tilde{h}_r := h_{i(r)}|_{V_r}$ gives
the transition functions $\tilde{g}_{rs}$ which completes the proof (by the first assertion).

\qed

**Remark 1.13.** The above considerations can be stated in terms of Čech cohomology
(cf. subsection 4.3, and also for [Hir66], theorem 3.2.1). Write $H^1(X; G)$ for the
first Čech cohomology of a space $X$ with coefficients in a group $G$. There is an
isomorphism from $H^1(X; G)$ to the isomorphism classes of fiber bundles over $X$
with structure group $G$ given by the construction of proposition 1.10. In the last
corollary we have checked that this map is compatible with taking refinements, and
in (1.4) we have seen that two Čech 1-cocycles differing only by a coboundary yield
isomorphic bundles.
1.3. Principal Bundles.

Definition 1.14. Let $G$ be a topological group. A principal $G$-bundle consists of a (continuous) map $\pi: P \to X$ and a right operation $\rho: P \times G \to P$ such that $G$-equivariant local trivializations (i.e. homeomorphisms $h_i$ with $\text{pr}_1 \circ h_i = \pi$ and $h_i(\rho(p, g)) = h_i(p)g$ $(\forall p \in P,\ g \in G)$)

$$h_i : \pi^{-1}(U_i) \to U_i \times G$$

for an open cover $U_i \subset X$ may be chosen. Here $U_i \times G$ is equipped with the trivial right operation. In particular, $\rho$ preserves fibers and acts freely and transitively on them.

Remark 1.15. (alternative definition of principal bundles) Let $\mu : G \times G \to G$ denote group multiplication. The $G$-equivariance of the local trivializations implies

$$h_i h_j^{-1}(x, g) = (\text{id} \times \mu)(h_i h_j^{-1}(x, 1), g) = (x, \mu(g_i(x), g)) \quad (1.5)$$

A principal $G$-bundle is thus a fiber bundle with typical fiber $G$ and structure group $G$ acting by the group multiplication $\mu$ (commonly also referred to as “acting by left-translations”). Conversely, and such fiber bundle is a principal $G$-bundle. The right operation may simply be defined in local trivializations \{h_i\} by

$$\rho(h_i^{-1}(x, h), g) := h_i^{-1}(x, hg)$$

The category of principal $G$-bundles $\text{Prin}(G)$ may thus be described as

$$\text{Prin}(G) = \mathcal{F}
\begin{array}{c}
\text{Bun}(G \times G \overset{\mu}{\to} G)
\end{array}$$

and a morphism of principal $G$-bundles is just a $G$-equivariant bundle map $P_1 \to P_2$.

1.4. Associated Bundles.

Definition 1.16. Given a principal $G$-bundle $P \to X$ and a $G$-operation $\rho : G \times F \to F$ on a space $F$, the associated fiber bundle $P \times \rho F = E$ is defined as the quotient of $P \times F$ under the equivalence relation generated by

$$(pg, f) \sim (p, \rho(g)f), \ g \in G \quad (1.6)$$

The bundle projection is induced by $\text{pr}_1$. The equivalence class of $(p, f)$ is usually denoted $[p, f]$. In case of the standard operation $\rho : \text{GL}(V) \times V \to V$, $(A, v) \mapsto Av$ for a vector space $V$, we will simply write $P \times_{\text{GL}} V$.

Lemma 1.17. Let $P$ be a principal $G$-bundle and $F$ a $G$-space as above. Assume $P$ has local trivializations $h_i : P|_{U_i} \to U_i \times G$ and transition functions $g_{ij} : U_{ij} \to G$. The associated fiber bundle $P \times_{\rho} F$ is then a fiber bundle with typical fiber $F$ and structure group $G$. Moreover, we may chose the $g_{ij}$ as transition functions.

Proof. The map

$$G_i : U_i \times F \to P \times_{\rho} F, \ (x, f) \mapsto [h_i^{-1}(x, 1), f] \quad (1.7)$$

is fiber-preserving and continuous. Another continuous map $P|_{U_i} \times F \to U_i \times F$ may be defined by the requirement $(h_i^{-1}(x, g), f) \mapsto (x, gf)$. Since the restriction of the canonical map $P|_{U_i} \times F \to (P \times_{\rho} F)|_{U_i}$ is an open map, this shows the
continuity of the inverse \( G_i^{-1} = H_i : (P \times \rho F) | U_i \to U_i \times F \). The assertion for the transition functions follows now from the calculation

\[
H_i H_j^{-1} (x, f) = H_i \left[ h_j^{-1} (x, 1), f \right] (1,5) = H_i \left[ h_i^{-1} (x, g_{ij} (x)), f \right] = (x, g_{ij} (x) f)
\]

\[ \square \]

**Definition 1.18.** Let \( \rho : G \times F \to F \) be the operation of a \( G \)-space \( F \). For a morphism \( f : P_1 \to P_2 \) of principal \( G \)-bundles, let \( f \times \rho F : P_1 \times \rho F \to P_2 \times \rho F \), \( [p_1, f] \mapsto [f(p_1), f] \) be the continuous map induced by \( f \times \text{id} : (p_1, x) \mapsto (f(p_1), x) \). We obtain a covariant functor

\[- \times \rho F : \text{Prin}(G) \to \text{F Bun}(G \times \rho F)\]

**Definition 1.19.** Given a fiber bundle \( F \hookrightarrow E \xrightarrow{\pi} X \) with \( G \)-trivialization \( \{ h_i \}_{i \in I} \) and transition functions \( g_{ij} : U_{ij} \to G \) (and \( G \) operating on \( F \) given by \( \rho : G \times F \to F \), let \( P \) be the fiber bundle constructed via proposition 1.10 using the same transition functions, but with \( G \) viewed as acting on the typical fiber \( G \) by the group multiplication. By remark 1.15 this defines a principal \( G \)-bundle which is said to be associated to the fiber bundle \( F \hookrightarrow E \xrightarrow{\pi} X \).

**Remark 1.20.** A different choice of open cover and transition functions in the definition 1.19 of the associated principal bundle will yield bundles that are isomorphic (cf. corollary 1.12). The associated principal bundle is therefore defined only up to isomorphism. The original fiber bundle may be recaptured as an associated fiber bundle

\[ E \cong P \times \rho F \]

This is immediate from proposition 1.11.

**Example 1.21.** The associated principal \( GL \)-bundle of a vector bundle \( E \to X \) of rank \( n \) may be taken as the bundle of frames

\[ P_{GL}(E) = \{ (e_1, \ldots, e_n) \in E^n \mid e_1, \ldots, e_n \text{ forms a basis of } E_x, x \in X \} \]

topologized as a subbundle of the product bundle \( E^n \). For vector bundles we will usually pick this representative in the isomorphism class of associated principal bundles. A Riemannian vector bundle may be considered as a vector bundle with structure group \( O(n) \), an oriented vector bundle as having structure group \( GL_n^+ \), and of course an oriented Riemannian bundle as having \( SO(n) \) as structure group. The “reduced structure group” guarantees here that we may move from locally defined structure on \( U_i \times V \) to the global case. A Hermitian vector bundle may be viewed as a complex vector bundle with structure group \( U(n) \). The corresponding associated principal bundles of (oriented) orthonormal frames will be denoted by

\[ P_O(E), P_{GL^+}(E), P_{SO}(E), P_U(E) \]
1.5. Reduction of Structure Group.

**Definition 1.22.** Let $\lambda : G \rightarrow H$ be a homomorphism of topological groups. Then we may assign to each principal $G$-bundle $P$ the principal $H$-bundle $P[G \xrightarrow{\lambda} H] = P[\lambda]$ by defining $P[\lambda]$ as the quotient of $P \times H$ under the equivalence relation

$$(pg, h) \sim (p, \lambda(g) \cdot h), \ g \in G$$

Given local trivializations $h_i$ of $P$ with corresponding transition functions $g_{ij}$, the bundle $P[\lambda]$ is trivialized by $H_i^{-1} : U_i \times H \rightarrow P[\lambda] |_{U_i}, \ (x, h) \mapsto \left[ h_i^{-1}(x, 1), h \right]$. It follows that we may chose $\lambda \circ g_{ij}(x)$ as transition functions for $H$. Our construction is actually just the associated bundle construction from above, $P \times_\lambda H$ for $H$ acting on itself by left-translations, but viewing $P \times_\lambda H$ as a principal $H$-bundle from the right (so the structure group is $H$, not $G$). For greater clarity, we will write $P[G \xrightarrow{\lambda} H]$ instead of $P \times_\lambda H$. Of course, we have the same functoriality as in definition 1.18: A morphism $f : P_1 \rightarrow P_2$ of principal $G$-bundles induces as morphism

$$f[\lambda] = f \times_\lambda H : P_1[\lambda] \rightarrow P_2[\lambda], \ [p_1, h] \mapsto [f(p_1), h]$$

of principal $H$-bundles. This yields again a covariant functor

$$-[\lambda] : \text{Prin}(G) \rightarrow \text{Prin}(H)$$

**Definition 1.23.** Let $P$ be a principal $H$-bundle, and $\lambda : G \rightarrow H$ a homomorphism of topological groups. A reduction of structure group from $H$ to $G$ consists of a principal $G$-bundle $Q$ together with an isomorphism $\phi : Q[G \xrightarrow{\lambda} H] \cong P$ of principal $H$-bundles. Two such reductions $(Q_1, \phi_1), (Q_2, \phi_2)$ are called equivalent if there is an isomorphism $f : Q_1 \rightarrow Q_2$ of principal $G$-bundles such that $\phi_1 = \phi_2 \circ f[\lambda]$. It is important to incorporate the isomorphism $f$ into the definition (cf. Milnor’s warning in figure 4.1).

A fiber bundle with structure group $H$ is said to admit a reduction of structure group to $G$ in case its associated principal $H$-bundle does.

**Example 1.24.** Assume $\rho$ is a representation $G \rightarrow GL(V)$ on a vector space $V$. For any principal $G$-bundle $P$, the associated bundle $P \times_\rho V$ may be viewed as a vector bundle (which could be emphasized by writing $P[G \xrightarrow{\rho} GL(V)] \times_{GL} V$ instead).


**Proposition 1.25.** Let $\mathcal{F} : Vect \times \cdots \times Vect \rightarrow Vect$ be a continuous functor\(^2\) on the category of finite-dimensional vector spaces (which needs only to be defined for isomorphisms). Then $\mathcal{F}$ induces a corresponding functor on $\text{VectBun}(X)$.

---

\(^1\)The associated principal bundle is defined only up to isomorphism. If however one of them admits a reduction of structure group, any isomorphic one does as well.

\(^2\)This means that the function $\text{Iso}(V_1 \times \cdots \times V_k, W_1 \times \cdots \times W_k) \rightarrow \text{Iso}(\mathcal{F}(V_1, \ldots, V_k), \mathcal{F}(W_1, \ldots, W_k))$ is continuous for any choice of vector spaces $V_i, W_i$. Note here that the (finite-dimensional) vector space of homomorphisms has a natural (norm) topology.
Proof. Fix “once and for all” isomorphisms $\mathcal{F} (\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \cong \mathbb{R}^{d(n_1, \ldots, n_k)}$. Then the functor $\mathcal{F}$ yields continuous representations $\rho : GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_k}(\mathbb{R}) \to GL(\mathbb{R}^{d(n_1, \ldots, n_k)})$. It may be extended to vector bundles as follows. Let $E_1, \ldots, E_k$ be vector bundles over $X$ with associated principal $GL$-bundles $P_1, \ldots, P_k$ of frames (as in example 1.21). Then $P_1 \times \cdots \times P_k$ is a principal $GL_{n_1} \times \cdots \times GL_{n_k}$-bundle. We may now set

$$\mathcal{F}(E_1, \ldots, E_k) := (P_1 \times \cdots \times P_k) \times_\rho \mathbb{R}^{d(n_1, \ldots, n_k)}$$

The definition on isomorphisms is given similarly by

$$\mathcal{F}(f_1, \ldots, f_k) = (f_1 \times \cdots \times f_k) \times_\rho \mathbb{R}^{d(n_1, \ldots, n_k)}, \quad [(p_1, \ldots, p_k), \vec{v}] \mapsto [(f_1(p_1), \ldots, f_k(p_k)), \vec{v}]$$

where the map induced on the associated principal bundles was again denoted by $f_i$.

Remark 1.26. Let $\mathcal{F}$ and $\mathcal{G}$ be naturally isomorphic functors on the category of finite-dimensional vector spaces. This means in particular that the corresponding representations $\rho^F, \rho^G$ are equivalent, i.e. $\rho^F = \phi^{-1} \circ \rho^G \circ \phi$ for a family of isomorphisms $\phi : \mathbb{R}^{d(n_1, \ldots, n_k)} \to \mathbb{R}^{d(n_1, \ldots, n_k)}$. The induced functors on $\text{Vect Bun}_X$ are then too naturally isomorphic. For any set of vector bundles $E_1, \ldots, E_k$ (with associated frames $P_1, \ldots, P_k$) the natural isomorphism is defined by

$$(P_1 \times \cdots \times P_k) \times_\rho \mathbb{R}^{d(n_1, \ldots, n_k)} \to (P_1 \times \cdots \times P_k) \times_\eta \mathbb{R}^{d(n_1, \ldots, n_k)}$$

where

$$[(p_1, \ldots, p_k), \vec{v}] \mapsto [(p_1, \ldots, p_k), \phi(\vec{v})]$$

(1.8)

We will follow this approach in proposition 2.31 and remark 2.33 to generalize the above statements to the case in which all vector bundles have fixed connections.

Example 1.27. For (real or complex) vector bundles $E, F \to X$ we may form the tensor product $E \otimes F$, exterior powers $\Lambda^p E$, symmetric powers $S^p E$, and the Whitney sum $E \oplus F$. See also example 2.34.

For a complex vector bundle $E$ we may form the conjugate bundle $\bar{E}$ which is defined as having the same underlying additive structure, but scalar multiplication given by $\mathbb{C} \times E \to E, \ (\lambda, v) \mapsto \lambda v$. If we have a Hermitian form on $E$ there is a complex linear isomorphism $E \cong \text{Hom}(E, \mathbb{C})$.

Definition 1.28. Denote by $G_n(\mathbb{R}^k)$ (resp. $G_n(\mathbb{C}^k)$) in the complex case) the Grassmannian manifolds of $n$-dimensional subspaces of $\mathbb{R}^k$ (resp. $\mathbb{C}^k$). They are defined also for $k = \infty$ by $\lim_k G_n(\mathbb{R}^k)$.

By the tautological real $n$-plane bundle over $G_n(\mathbb{R}^k)$ for $k \in \mathbb{N} \cup \infty$ we shall mean the following subbundle of the product bundle

$$(\gamma_{\mathbb{R}})^n_k := \{(v, X) \in \mathbb{R}^k \times G_n(\mathbb{R}^k) \mid v \in X\} \subset \mathbb{R}^k \times G_n(\mathbb{R}^k)$$

Similarly, let $(\gamma_{\mathbb{C}})^n_k$ denote the tautological complex $n$-plane bundle over $G_n(\mathbb{C}^k)$. For $k = \infty$ we will simply write $\gamma_{\mathbb{R}}^n := (\gamma_{\mathbb{R}})^n_\infty, \gamma_{\mathbb{C}}^n := (\gamma_{\mathbb{C}})^n_\infty$.

The bundles are $\gamma_{\mathbb{R}}^n, \gamma_{\mathbb{C}}^n$ often called universal $n$-plane bundles. This comes from the following theorem whose proof can be found in [MS74]. See also figure 1.2.
A GENERALIZED MIRACULOUS CANCELLATION FORMULA

The tautological 1-plane bundle contains any kind of twist an arbitrary rank 1-bundle can have.

Theorem 1.29. (Classification of Vector Bundles) Let $X$ be a paracompact space. “Pullback of the universal $n$-plane bundle” defines a bijection between the homotopy classes of maps $[X, G_n(\mathbb{R}^\infty)]$ and the isomorphism classes of rank $n$ real vector bundles over $X$.

$$[X, G_n(\mathbb{R}^\infty)] \xhookrightarrow{=} \text{Iso}_n(X), \ [f] \mapsto f^*\gamma^n$$

There is a similar result in the complex case and the infinite complex Grassmannian $G_n(\mathbb{C}^\infty)$.

Theorem 1.30. (Splitting Principle) Let $E \to X$ be a rank $n$ vector bundle over a paracompact base $X$. There exists a space $Y$ and a map $q : Y \to X$ such that $q^*E$ is isomorphic to a direct sum of line bundles. Moreover, the map $q^*$ induced in cohomology is injective.

Proof. This is proven by inductively splitting off line bundles; taking $Y$ as the projectivization of $E$, the pullback $q^*E$ has a canonical line subbundle. See [BT82] for further details.

Definition 1.31. Let $R$ be commutative ring with unit. By a characteristic class for real (resp. complex) vector bundles we shall mean the assignment $c(E) \in H^*(X; R)$ of a cohomology class to every isomorphism class of real (resp. complex) vector bundles $E \to X$ in a natural way: $f^*c(E) = c(f^*E)$ for any $f : Y \to X$.

Remark 1.32. Let $\pi : E \to X$, $\pi' : F \to Y$ be vector bundles. By definition, a bundle morphism $\phi : E \to F$ covering a continuous map $\varphi : X \to Y$ satisfies $\pi'\phi = \varphi\pi$ and maps each fiber isomorphically. Two vector bundles are said to be isomorphic over $X$ if there exists a bundle morphism covering $\text{id}_X$. Both concepts are related by the pullback construction: If $(\phi, \varphi)$ is a bundle morphism, the bundles $E \cong \varphi^*F$ are isomorphic over $X$. It follows that if $\varphi : X \to Y$ is covered by a bundle morphism $E \to F$, then $\varphi^*c(F) = c(\varphi^*F) = c(E)$. This is an alternative form of the naturality requirement.
1.7. **Stiefel-Whitney Classes.** From theorem 1.29 it is clear that the Stiefel-Whitney classes are characterized by the following axioms

**SW1:** For every real vector bundle \( E \to X \) we are given cohomology classes 
\[ w(E) = w_0(E) + w_1(E) + \cdots \in H^*(X; \mathbb{Z}_2) \]
such that \( w_k(E) \in H^k(X; \mathbb{Z}_2) \) for any \( k \geq 0 \).

**SW2:** (Naturality) \( w(f^*E) = f^*w(E) \) for \( f : Y \to X \), while isomorphic bundles are to have the same Stiefel-Whitney class.

**SW3:** Let \( \gamma^k_R \) be the universal \( n \)-plane bundle over \( G_n(\mathbb{R}^\infty) \). We have \( w(\gamma^k_R) = 1 + w_1(\gamma^k_R) + \cdots + w_n(\gamma^k_R) \) for the (unique) generators \( w_k(\gamma^k_R) \) of \( H^k(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2 \).

The element \( w_k(E) \) is called the \( k \)-th Stiefel-Whitney class of the bundle, while \( w(E) \) is called the total Stiefel-Whitney class. Instead of axiom (SW3) we may also require the following two axioms

**SW3*: We have \( w(E \oplus F) = w(E) \cup w(F) \) (for the Whitney sum \( \oplus \) of vector bundles and the cohomological cup product \( \cup \) which we will often suppress from our notation).

**SW3**: For the tautological line bundle \( \gamma^1_R \) over \( \mathbb{R}P^\infty \) the Stiefel-Whitney class equals the non-zero element \( w(\gamma^1_R) = 1 + g \) for the generator \( g \) of \( H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}_2[g] \).

**Proof.** We will show that (SW3*) and (SW3**) imply \( H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n] \) for the Stiefel-Whitney classes \( w_k \) of the universal \( n \)-plane bundle \( \gamma^k_R \). This will verify (SW3). Let us first show that there are no polynomial relations between the \( w_k(\gamma^k_R) \), \( k = 1, \ldots, n \). Clearly, by (SW2) any such relation would imply a corresponding relation for an arbitrary vector bundle. It therefore suffices to exhibit an example of a bundle for which no polynomial relations exist. Consider the \( n \)-fold product of the tautological line bundle \( E = \gamma^1_R \times \cdots \times \gamma^1_R \cong \text{pr}_1^* \gamma^1_R \oplus \cdots \oplus \text{pr}_n^* \gamma^1_R \) over \( \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \). Then by (SW3*) and (SW3**) 
\[ w(E) = \text{pr}_1^*(1 + g) \cup \cdots \cup \text{pr}_n^*(1 + g) = (1 + g_1) \cup \cdots \cup (1 + g_n) \]
where we have set \( g_k = \text{pr}_k^*(g) \). It follows that 
\[ w_k(E) = \sigma_k(g_1, \ldots, g_n) \]
for the \( k \)-th elementary symmetric polynomial \( \sigma_k \). Clearly \( H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[g_1, \ldots, g_n] \) so we may view the \( g_k \) as indeterminants over \( \mathbb{Z}_2 \). Our assertion now follows from the fact that the elementary symmetric polynomials over \( \mathbb{Z}_2 \) are algebraically independent (theorem 1.3).

We conclude that \( H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) \) contains the free polynomial algebra \( \mathbb{Z}_2[w_1, \ldots, w_n] \).

Using the cell-structure for the Grassmannians (cf. [MS74]) to compare the dimensions of the corresponding vector spaces in each degree, it follows that both algebras are in fact equal.

\[ \square \]

**Remark 1.33.** A characteristic class \( c \) (with values in \( H^k(-; \mathbb{Z}_2) \)) for real vector bundles which has the property that \( 0 \neq c(\gamma^0_R) \in H^k(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) \) for \( n \leq k \) and \( 0 = c(\gamma^0_R) \) for \( n > k \) is equal to the \( k \)-th Stiefel-Whitney class \( w_k \). This follows from naturality together with theorem 1.29, since by assumption we have \( c(\gamma^0_R) = w_k(\gamma^0_R) \).
for every $n$. We could have also used this description as a set of axioms for solely the $k$-th Stiefel-Whitney class $w_k$.

Assuming the existence of the Stiefel-Whitney classes, we have shown $H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n]$. Conversely, this result could also be used to define the Stiefel-Whitney classes satisfying the axioms (SW1)-(SW3) by

$$w_k(E) := f^*(w_k)$$

where $f : X \to G_n(\mathbb{R}^\infty)$ is the classifying map of the bundle $E \to X$.

1.8. Chern Classes. Everything in this subsection may be carried out also for real/rational coefficients. Since the integer Chern class viewed as real/rational cohomology class still satisfies all of the corresponding axioms it must equal the real/rational Chern class. A set of axioms for the Chern classes is given by

**C1:** For every complex vector bundle $E \to X$ we are given cohomology classes $c(E) = c_0(E) + c_1(E) + \cdots \in H^*(X; \mathbb{Z})$ such that $c_k(E) \in H^{2k}(X; \mathbb{Z})$ for any $k \geq 0$.

**C2:** The Chern class behaves naturally $c(f^*E) = f^*(c(E))$, and isomorphic bundles have the same Chern class.

**C3:** (Whitney product theorem) $c(E \oplus F) = c(E) \cup c(F)$

**C4:** (Normalization) For the tautological complex line bundle $(\gamma_1)_{\mathbb{C}P^1}$ over $\mathbb{C}P^1$ we have $c((\gamma_1)_{\mathbb{C}P^1}) = 1 - g$. Here, $g$ denotes the cohomology class characterized by the relation

$$\langle \gamma_1, [\mathbb{C}P^1] \rangle = 1$$

for the fundamental homology class of $\mathbb{C}P^1$ (canonically oriented by the complex structure).

We have the classifying map $f : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ for $(\gamma_1)_{\mathbb{C}P^1}$. Also $H^2(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z} \cong H^2(\mathbb{C}P^n; \mathbb{Z}) \forall n \in \mathbb{N} \cup \{\infty\}$ and the map $H^2(f)$ is an isomorphism. In particular, (C4) determines

$$c_1((\gamma_1)_{\mathbb{C}P^1}) = H^2(f)^{-1}(-g) \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$$

(1.10) uniquely. Write $h_\infty = H^2(f)^{-1}(g)$ and $h_n = h_\infty|_{\mathbb{C}P^n} \in H^2(\mathbb{C}P^n; \mathbb{Z})$ so that

$$c_1((\gamma_1)_{\mathbb{C}P^n}) = -h_n$$

Note that (C4) is slightly more complicated than (SW3) since a canonical generator of $H^2(\mathbb{C}P^1) \cong \mathbb{Z}$ has to be chosen.

**Proof.** We will show that (C1)-(C4) characterize the Chern class uniquely. By the splitting principle it is possible for a complex vector bundle $E \to X$ to choose a map $f : Y \to X$ such that $f^*E \cong L_1 \oplus \cdots \oplus L_n$ is the direct sum of line bundles. Moreover, $f^*$ induces an injective map in cohomology. Each line bundle $L_k$ may be classified by a map $g_k$ which means $g_k^*\gamma_1^C \cong L_k$. Thus

$$f^*(c(E)) \cong c(L_1 \oplus \cdots \oplus L_n) \cong c(L_1) \cdots c(L_n) \cong \prod_{k=1}^n g_k^*(1 - c_1(\gamma_1^C))$$

from which our assertion follows. 

\[\square\]
Remark 1.34. Let $c$ be a characteristic class for complex vector bundles (with values in $H^2(\mathbb{CP}^1;\mathbb{Z})$) with the property that $c((\gamma_{\mathbb{C}})^1) = -g \in H^2(\mathbb{CP}^1;\mathbb{Z})$. Then by theorem 1.29 we have $c(E) = c_1(E)$ for complex line bundles $E$. From the splitting principle it follows that $c = c_1$.

1.9. Pontrjagin Classes. Using the Chern classes we may define the Pontrjagin classes of a real vector bundle by

$$p_k(E) := (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(X;\mathbb{Z})$$

for the complexification $E \otimes \mathbb{C}$ of $E$. Since for a real vector bundle $E$ we have $E \otimes \mathbb{C} \cong E \otimes \overline{\mathbb{C}}$, $e \otimes z \mapsto e \otimes \overline{z}$ the odd Chern classes are 2-torsion (we will see in (2.27) after defining the Chern classes that $c_k(\overline{E}) = (-1)^k c_k(E)$). The Pontrjagin classes are clearly natural. The Whitney product theorem now takes the form

$$p(E \oplus F) = p(E) \cup p(F) \text{ modulo 2 torsion}$$

Here we have written $p(E) = 1 + p_1(E) + \cdots$ for the total Pontrjagin class. The Pontrjagin classes are sometimes viewed as rational cohomology classes. Then of course the Whitney product theorem holds.

Let $V$ be a complex vector space. There is a $\mathbb{C}$-linear isomorphism

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}, \quad v \otimes (x + iy) \mapsto \left( \frac{x + iy}{2} v, \frac{x - iy}{2} v \right)$$

(for the $\mathbb{C}$-linearity it is important to use the conjugate vector space $\overline{V}$ in the second summand)

Assume $E \to X$ is a complex vector space. The Pontrjagin class of $E$ is defined as the Pontrjagin class of the underlying real vector bundle. It follows that

$$p_k(E) = (-1)^k c_{2k}(E \oplus \overline{E}) = (-1)^k \sum_{l=0}^{2k} c_l(E) \cup c_{2k-l}(\overline{E})$$

$$= \sum_{l=0}^{2k} (-1)^{k+l} c_l(E) c_{2k-l}(E)$$
2. Connection and Curvature

2.1. Connections on Principal Bundles. Let us first fix some notation. Assume $P \to X$ is a smooth principal $G$-bundle. The orbits $pG$ through an element $p \in P$ may be identified with $G$ by

$$\lambda_p : G \to pG, \ g \mapsto pg$$

Differentiating, we obtain an isomorphism

$$\left(\lambda_p\right)_* : \mathfrak{g} \to T_p(pG) =: \mathcal{V}_p, \ V \mapsto \tilde{V}_p$$

onto the “vertical subspace” $\mathcal{V}_p \subset T_p(P)$. Here, $\mathfrak{g}$ denotes the Lie algebra of $G$. Let $\mu : P \times G \to P$ denote the right principal $G$-operation of $P$. Also, set for any $g \in G$

$$\rho_g : P \to P, \ p \mapsto pg$$

In a Lie group $G$ we will write $l_g$ for left translation and $r_g$ for right translation. From $(\lambda_p)_*(Y_g) = \mu_*(p,g)(0_p,Y_g)$ and $(\rho_g)_*(X_p) = \mu_*(p,g)(X_p,0_g)$ we obtain

$$\mu_*(p,g)(X_p,Y_g) = (\rho_g)_*(X_p) + (\lambda_p)_*(Y_g) \quad (2.2)$$

**Definition 2.1.** A connection on $P$ is a smooth $\mathfrak{g}$-valued 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ such that

$$\omega_p(\tilde{V}_p) = V, \quad (\forall V \in \mathfrak{g}) \quad (2.3)$$

$$g^*\omega = Ad_{g^{-1}} \circ \omega, \quad (\forall g \in G, p \in P) \quad (2.4)$$

Here, $g^*$ denotes the pullback of the right-translation $\rho_g$.

A $\mathfrak{g}$-valued $p$-form is by definition an element of $\omega \in \Gamma(\Lambda^p(TP) \otimes \mathfrak{g})$. We will view the evaluation $\omega_p$ at a point $p \in X$ as a multilinear alternating map

$$TP \times \cdots \times TP \to \mathfrak{g}$$

from $p$ copies of $TP$ to the Lie algebra $\mathfrak{g}$.

**Remark 2.2.** From (2.3) $\omega_p$ restricts to an isomorphism $\mathcal{V}_p \cong \mathfrak{g}$ inverse to (2.1). We thus obtain a splitting

$$T_pP = \ker(\omega_p) \oplus \mathcal{V}_p$$

$$\tau_p := \ker(\omega_p)$$

may then be viewed as complementary “horizontal subspaces” to the vertical subspace $\mathcal{V}_p$. 
Definition 2.3. Let $\omega$ be a connection on $P \to X$. Following Kobayashi/Nomizu define the curvature of the connection $\omega$ as the $g$-valued 2-form
\[
\Omega := d\omega + [\omega, \omega] \quad (2.5)
\]
where $[\omega, \omega]_p(v_p, w_p) := [\omega_p(v_p), \omega_p(w_p)]$.

Definition 2.4. (Pullback connection) Let $P \to X$, $P' \to X'$ be principal $G$-bundles and assume $\phi : P' \to P$ is a $G$-equivariant bundle morphism covering $\varphi : X' \to X$. Given a connection $\omega$ on $P$, the pullback $\phi^* \omega := \omega \circ \phi^*$ defines a connection on $P'$. This follows since by $G$-equivariance the $g^*$ of (2.4) commute with $\phi^*$, and since by definition (2.1) the tangent vector $\tilde{V}_p$ in $P$ of (2.3) is related with $\tilde{V}_{\phi(p)} = \phi^* \tilde{V}_p$, $\phi(p) = p'$ (this comes from differentiating $\lambda_{\varphi(p)} = \phi \circ \lambda_p$).

Lemma 2.5. Let $s : X \to P$ be a global section of a principal $G$-bundle $P \to X$. Clearly, $s$ determines a $G$-equivariant trivialization $\varphi : X \times G \to P$, $(x, g) \mapsto s(x)g$. For a connection $\omega$ on $P$ set $\tilde{\omega} = \omega s^*$. We then have for the pullback on the trivial bundle
\[
\varphi^* \omega(v_x \oplus w_y) = Ad_{g^{-1}} \circ \tilde{\omega}(v_x) + (\lambda_{g^{-1}})_* (w_y) \quad (\forall v_x \in T_x X, \ w_y \in T_y G) \quad (2.6)
\]
Conversely, any $g$-valued differential 1-form $\tilde{\omega} \in \Gamma(T^* X \otimes g)$ determines a connection on $P$ by the above equation.

Proof. From the equation $\varphi(x, y) = \mu(s(x), g)$ and (2.2) we have
\[
\omega \varphi^*(v_x \oplus w_y) = \omega ((\rho_g)_* s^*(v_x) + (\lambda_{s(x)})_* w_y) \quad (2.4)
\]
\[
= Ad_{g^{-1}} \omega s^*(v_x) + \omega ((\lambda_{s(x)}g)_* (l_{g^{-1}})_* w_y) \quad (2.3)
\]
For converse, note that $\varphi_*$ is an isomorphism, so we may define $\omega$ by (2.6). The requirements (2.3), (2.4) now follow directly from the definition of $\omega$. \hfill \Box
2.2. Connections on Vector Bundles. We will now restrict our attention to the case of vector bundles. In particular, all the cases of example 1.21 will be considered. That is, let \( G = GL_n(\mathbb{R}), GL_n(\mathbb{C}), O_n, SO_n, U_n \) and consider fiber bundles with structure group \( G \). The corresponding Lie algebras \( \mathfrak{g} \) are

\[
\begin{align*}
\mathfrak{gl}_n(\mathbb{R}) &= \mathbb{R}^{n \times n} \\
\mathfrak{gl}_n(\mathbb{C}) &= \mathbb{C}^{n \times n} \\
\mathfrak{so}_n &= \{ \text{skew-symmetric } n \times n \text{ matrices} \} \\
\mathfrak{u}_n &= \{ \text{skew-hermitian } n \times n \text{ matrices} \}
\end{align*}
\]

The respective associated bundle of frames will be denoted by \( P \).

**Definition 2.6.** A connection on a vector bundle \( E \to X \) is defined as a connection on the associated principal bundle \( P \) of frames.

**Remark 2.7.** Using the canonical “inclusion” \( \mathfrak{u}_n \hookrightarrow \mathfrak{so}_{2n} \) any connection on a Hermitian bundle induces a connection on the underlying real vector bundle (which is of course canonically oriented and Riemannian). In example 2.30 we will see that the complexification of a real vector bundle also inherits a connection.

A connection on a vector bundle may be considered as a matrix of differential 1-forms on the associated principle bundle

\( \omega_{ij} \in \Omega^1(P) \)

Similarly, the curvature may be viewed as a matrix of 2-forms. Of course, the Lie bracket on all our above groups is given by \( [A, B] = AB - BA \). It follows that

\[
[\omega, \omega](v, w) = \omega(v)\omega(w) - \omega(w)\omega(v) = \sum_{k=1}^{n} \omega_{ik}(v)\omega_{kj}(w) - \sum_{k=1}^{n} \omega_{ik}(w)\omega_{kj}(v)
\]

\[
= \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}(v, w)
\]

Therefore the curvature has components \( \Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} \) or in matrix notation

\( \Omega = d\omega + \omega \wedge \omega \)

Differentiating, we obtain the so called **Bianchi identity**

\[
d\Omega = \Omega \wedge \omega - \omega \wedge \Omega
\]
2.3. Covariant Differentiation. A connection on a vector bundle is essentially the same thing as a covariant derivative.

**Definition 2.8.** A covariant derivative on a real (resp. complex) vector bundle $E \to X$ is a $\mathbb{R}$-linear map (resp. $\mathbb{C}$-linear)

$$\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$$

which satisfies the Leibniz rule for any $f \in C^\infty(X), s \in \Gamma(E)$:

$$\nabla(fs) = df \cdot s + f \nabla(s)$$  \hspace{1cm} (2.9)

The Leibniz rule implies that a covariant derivative is a local operator: If a section $s$ vanishes on an open subset $U \subset X$, then so does $\nabla(s)$. For the proof let $p \in U$ and choose $\chi$ equals to 1 in a neighborhood of $p$ and with $\chi \big|_{X \setminus U} = 0$. Then

$$0 = \nabla(0)(p) = \nabla(\chi s)(p) = d\chi(p)s(p) + \chi(p)\nabla(s)(p) = \nabla(s)(p)$$

In particular, the restriction of a covariant derivative to an open subbundle is again equipped with a (well-defined) covariant derivative $\nabla|_U : (s)(p) := \nabla(\chi s)(p)$ for $\chi$ as above. It follows also that a covariant derivative $\nabla$ is already determined by its restrictions to an open cover of $X$. Conversely, a covariant derivative may be specified by defining it on an open cover in a compatible manner.

**Proposition 2.9.** Let $E \to X$ be a real or complex vector bundle. Connection 1-forms $\omega$ on the associated bundle of frames $P_{\text{GL}}(E)$ correspond bijectively to covariant derivatives $\nabla$ on $E$.

**Proof.** Assume we are given a connection $\omega$. A local frame $s_1, \ldots, s_n : U \to E$ may be viewed as a section of the associated principal bundle $P_{\text{GL}}(E)$. $\tilde{\omega} = s^*\omega$ is then a matrix of 1-forms on $U$. We contend that it is possible to define a covariant derivative by requiring

$$\nabla s_i := \sum_{j=1}^n \tilde{\omega}_{ji} \otimes s_j$$  \hspace{1cm} (2.10)

for any such frame. This definition may be uniquely extended\(^3\) to a covariant derivative on $U$, so by the remarks preceding the proposition it remains to check that the definition is independent of $s$ on overlaps. Let $r = (r_1, \ldots, r_n) : V \to P_{\text{GL}}(E)$ be another local frame with $U \cap V \neq \emptyset$. Then on $V$ we have for $\tilde{\omega} = r^*\omega$

$$\nabla r_i = \sum_{j=1}^n \tilde{\omega}_{ji} \otimes r_j$$  \hspace{1cm} (2.11)

We must show that both definitions of $\nabla$ coincide on $U \cap V$. We may write $s = r \cdot H$ for $H : U \cap V \to GL_n$, i.e. $s_i(x) = \sum_{j=1}^n r_j(x)h_{ji}(x)$ for any $x \in U \cap V$. While (2.10) gives

$$\nabla(s_i) = \sum_j \tilde{\omega}_{ji} s_j = \sum_j \tilde{\omega}_{ji} \sum_k r_k h_{kj} = \sum_{k,j} h_{jk} \tilde{\omega}_{ki} r_j$$

\(^3\)Any section may uniquely be decomposed as $s = \sum f_i s_i$. By (2.9) we then must have $\nabla s = \sum df_i s_i + f_i \nabla(s_i) = \sum_i \left(df_i + \sum_j \tilde{\omega}_{ij} f_j \right) s_i$. Conversely, this expression may be used as a definition for $\nabla$. 

---

**A GENERALIZED MIRACULOUS CANCELLATION FORMULA**

22
from (2.11) we have
\[
\nabla(s_i) = \sum_j (dh_{ji} \cdot r_j + h_{ji} \nabla(r_j)) = \sum_j \left( dh_{ji} \cdot r_j + h_{ji} \sum_k \tilde{\omega}_{kj} r_k \right) = \sum_j \left( dh_{ji} + \sum_k h_{ki} \tilde{\omega}_{jk} \right) r_j
\]

Using matrix notation, what we need to prove therefore is that
\[
H \tilde{\omega} = dH + \tilde{\omega} H
\]  
(2.12)

Let \( \tilde{\varphi} : U \times GL_n \to PGL(E), (x, g) \mapsto s(x)g \) and \( \tilde{\varphi} : V \times GL_n \to PGL(E), (x, g) \mapsto r(x)g \) be the local trivializations of \( PGL(E) \) determined by the sections \( s \) and \( r \). For the change of trivialization \( \psi(x, g) := \tilde{\varphi}^{-1} \tilde{\varphi}(x, g) = \tilde{\varphi}^{-1}(s(x)g) = \tilde{\varphi}^{-1}(r(x)H(x)g) = (x, H(x)g) \) we compute
\[
\psi^*(x \oplus w) = (\tilde{\varphi}^{-1})^* (\tilde{\varphi})^*(v_x \oplus w) = v_x \oplus ((r_g)_* H_v) + (l_{H(x)})_*(w_g))
\]

But then from (2.6), applied to both sides of \( \omega \tilde{\varphi}^*(v_x \oplus w) = \omega \tilde{\varphi}^* (v_x \oplus ((r_g)_* H_v) + (l_{H(x)})_*(w_g)) \), it follows that
\[
Ad_{g^{-1}} \circ \tilde{\omega}(v_x) + (l_{g^{-1}})(w_g)
\]  
(2.13)

\[
= Ad_{(H(x)g)^{-1}} \circ \tilde{\omega}(v_x) + (l_{(H(x)g)^{-1}})_*(r_g)_* H_v + (l_{H(x)})_*(w_g))
\]

From which we obtain (2.12).

If conversely we are given a covariant derivative on \( E \) define \( \tilde{\omega} \) on \( U \) by (2.10) for every trivializing neighborhood \( U \) and section \( s : U \to PGL(E) \). Invoking lemma 2.5 we obtain local connection 1-forms that fit together precisely when (2.13) holds. But (2.13) follows immediately from (2.12)

\[\square\]

As it is important in its own right, let’s record (2.12) in a lemma

**Lemma 2.10.** Let \( \omega \) be a connection on a vector bundle. Denote by \( \tilde{\omega} = s^* \omega, \tilde{\omega} = r^* \omega \) the associated matrix of 1-forms for local frames \( r, s \) defined on \( U, V \). Assume \( s = r \cdot H \) on \( U \cap V \neq \emptyset \). We then have the transformation formula

\[
\tilde{\omega} = H^{-1} dH + H^{-1} \tilde{\omega} H
\]  
(2.12)

**Corollary 2.11.** Given a connection \( \omega \) on a vector bundle, let \( \Omega \) denote the corresponding curvature. Let \( r, s \) be local frames as in the preceding lemma. Both \( \tilde{\Omega} = s^* \Omega \) and \( \tilde{\Omega} = r^* \Omega \) may be viewed as matrices of 2-forms on \( U \) and \( V \) respectively. We have the transformation formula

\[
\tilde{\Omega} = H^{-1} \tilde{\Omega} H
\]  
(2.14)

**Proof.** Recall that the curvature for vector bundles can be written in matrix notation as \( \Omega = d\omega + \omega \wedge \omega \). It follows that \( \tilde{\Omega} = s^*(d\omega + \omega \wedge \omega) = d(s^*\omega) + \tilde{\omega} \wedge \tilde{\omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} 

and similarly  \( \tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} \). Note that  \( d(H^{-1}) = -H^{-1}dH^{-1} \). Viewing all products as  \( \wedge \) we compute from (2.12)

\[
\begin{align*}
  d\tilde{\omega} &= (-H^{-1}dHH^{-1}) dH + (-H^{-1}dHH^{-1}) \tilde{\omega}H + H^{-1}d\tilde{\omega}H - H^{-1}\tilde{\omega}dH \\
  \tilde{\omega} \wedge \tilde{\omega} &= H^{-1}dHH^{-1}dH + H^{-1}\tilde{\omega}dH + H^{-1}dHH^{-1}\tilde{\omega}H + H^{-1}\tilde{\omega}dH
\end{align*}
\]

which implies  \( \tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \tilde{\omega}H = H^{-1}\tilde{\omega}H \).

\[ \square \]

2.4. Invariant Polynomials.

**Definition 2.12.** Let  \( V \) be vector space over some field  \( K \). The symmetric algebra  \( S(V) \) is the free commutative algebra containing both  \( V \) and  \( K \). More explicitly,  \( S(V) \) may be defined as a quotient of the tensor algebra:

\[
S(V) = TV/J
\]

for the ideal  \( J \) generated by elements of the form  \( v \otimes w - w \otimes v, v, w \in V \).

The grading of the tensor algebra  \( TV \) decomposes  \( S(V) \) into the so-called symmetric powers  \( S^k(V) \). From the universal property of the tensor algebra we immediately conclude

**Proposition 2.13.** (Universal Property of the Symmetric Algebra) Any linear map  \( f : V \to A \) into an algebra  \( A \) satisfying  \( f(v)f(w) = f(w)f(v) \)  \( (\forall v, w \in V) \) has a unique extension  \( f : S(V) \to A \) to an algebra homomorphism.

**Definition 2.14.** A polynomial on  \( V \) is an element of the symmetric algebra  \( S(V^\ast) \).

There is an evaluation map

\[
S(V^\ast) \times V \to K, \ (f, v) \mapsto f(v)
\]

(2.15)

**Remark 2.15.** Assume a basis of  \( V \) is given by  \( v_1, \ldots, v_n \). It can be shown (cf. [Lan02]) that  \( v_1, \ldots, v_n \) viewed as elements of  \( S(V) \) are algebraically independent. As they certainly generate  \( S(V) \), elements of  \( S(V) \) may thus be viewed as polynomials in the indeterminants  \( v_1, \ldots, v_n \).

**Definition 2.16.** Let  \( G \) be a group acting linearly on  \( V \). A polynomial  \( p \in S(V^\ast) \) is said to be  \( G \)-invariant if

\[
p(gv) = p(v)
\]

for all  \( g \in G, \ v \in V \). The set of all such polynomials forms a subalgebra  \( \mathcal{P}(V)^G \subset S(V^\ast) \).

\[ ^4\text{Noticing that } \text{Hom}(S(V^\ast), V; K) \cong \text{Hom}(S(V^\ast); \text{Hom}(V; K)) \text{ this map may be constructed by applying the universal property to } V^\ast \subset K^V \]
Let $G$ be a Lie group with associated Lie algebra $\mathfrak{g}$. We will be concerned only with $G$-invariant polynomials $P$ on $\mathfrak{g}$, the operation of $G$ on $\mathfrak{g}$ being the adjoint representation, especially those of (2.7). In every case, the adjoint representation is simply conjugation $Ad_A(X) = AXA^{-1}$ $(A \in G, \ X \in \mathfrak{g})$.

**Example 2.17.** Define $s_k : GL_n(\mathbb{C}) \to \mathbb{C}$ as the $k$-homogeneous part of $A \mapsto \det(1 + A)$ viewed as a polynomial in the matrix entries $a_{ij}$:

$$\det(1 + A) =: 1 + \sum_{k=1}^{n} s_k(A)$$  \hfill (2.16)

Clearly the $s_k$ are $\mathfrak{gl}_n(\mathbb{C})$ invariant. We define

$$c_k(A) := s_k \left( \frac{i}{2\pi} \Lambda \right) \ k\text{-th Chern class}$$  \hfill (2.17)
$$c(A) := \sum_{k=0}^{n} c_k(A) = \det \left( 1 + \frac{i}{2\pi} \Lambda \right) \ \text{total Chern class}$$  \hfill (2.18)

Let $E \to X$ be a complex vector bundle. If $X$ is paracompact a Hermitian structure may always be chosen (cf. [MS74]). As it is the case of most interest to us, we will consider in the following mainly Hermitian vector bundles. As a special case of Chevalley’s Theorem, we prove

**Theorem 2.18.** The unitary group $U(n)$ has maximal torus $T = \mathbb{T}^n$ identified with unitary diagonal matrices. The associated Lie algebra $\mathfrak{t} = (i\mathbb{R})^n = \{(t_1, \ldots, t_n) \in (i\mathbb{R})^n\}$ may then be identified with skew-Hermitian diagonal matrices.

Restricting a polynomial to the diagonal defines an isomorphism to the symmetric polynomials on the diagonal

$$\mathcal{P}(\mathfrak{u}_n)^{U_n} \cong \mathcal{P}(\mathfrak{t})^T \cong \mathbb{C}^{\text{sym}}[t_1, \ldots, t_n]$$

**Proof.** Injectivity is clear since for $P, Q \in \mathcal{P}(\mathfrak{u}_n)^{U_n}$ agreeing on diagonal matrices we have

$$P(A) = P(UAU^{-1}) = Q(UAU^{-1}) = Q(A)$$

were $U$ is chosen so that $UAU^{-1}$ is diagonal (note here that $A \in \mathfrak{u}_n$ is normal $A^*A = AA^*$). The map is surjective since the elementary symmetric polynomials $\sigma$ of (1.1), which generate $\mathbb{C}^{\text{sym}}[t_1, \ldots, t_n]$ by theorem 1.3, may be extended to $\mathfrak{u}_n \to \mathbb{C}$ by the $s_k$ defined in (2.16).

**Remark 2.19.** The ring $\mathbb{C}^{\text{sym}}[t_1, \ldots, t_n]$ of symmetric polynomials over $\mathbb{C}$ is generated by the algebraically independent elements $\sigma_1, \ldots, \sigma_n$ (cf. theorem 1.3). It follows that $\mathcal{P}(\mathfrak{u}_n)^{U_n}$ is generated as a ring by the algebraically independent elements $s_1, \ldots, s_n$. Therefore, any $P \in \mathcal{P}(\mathfrak{u}_n)^{U_n}$ may be written uniquely as

$$P(A) = f(s_1, \ldots, s_n)(A)$$  \hfill (2.19)

for a polynomial $f$ over $\mathbb{C}$ in $n$ variables. Moreover, the theorem asserts that in order to define an invariant polynomial it suffices to define it on skew-Hermitian
diagonal matrices.

**Example 2.20.** (complex case) We may define invariant polynomials on $u_n$ by defining them for diagonal $\Lambda = \begin{pmatrix} t_1 & \cdots & \cdots & t_n \end{pmatrix} \in \mathbb{C} \cong (i\mathbb{R})^n$. Using the notation $x_j := \frac{i}{2\pi} t_j$ let

$$\text{ch}(\Lambda) = \sum_{k=1}^{n} e^{x_k} \quad \text{Chern character} \quad (2.20)$$

The definition of the Chern classes (2.18), (2.17) on diagonal matrices reads

$$c_k(\Lambda) := \sigma_k (x_1, \ldots, x_k) = s_k \left( \frac{i}{2\pi} \Lambda \right) \quad (2.17)$$

$$c(\Lambda) := \sum_{k=0}^{n} c_k(\Lambda) = \prod_{l=1}^{n} (1 + x_l) = \det \left( 1 + \frac{i}{2\pi} \Lambda \right) \quad (2.18)$$

We define also

$$\hat{A}(\Lambda) = \prod_{k=1}^{n} \frac{x_k}{2 \sinh (x_k/2)} \quad \text{A-roof genus} \quad (2.21)$$

$$\hat{L}(\Lambda) = \prod_{k=1}^{n} \frac{x_k}{\tanh (x_k/2)} \quad \text{L-roof genus} \quad (2.22)$$

$$L(\Lambda) = \prod_{k=1}^{n} \frac{x_k}{\tanh (x_k)} \quad \text{L genus} \quad (2.23)$$

**Remark 2.21.** The last two examples are not really invariant polynomials. They are invariant formal power series $P \in \mathcal{P}_\infty (u_n)^{U_n}$. This is a formal power series of the matrix entries $P : u_n \rightarrow \mathbb{C}$ which is invariant under conjugations $P(A) = P(UAU^{-1})$. Alternatively, a $G$-invariant formal power series may be defined as $P = \sum_{i=0}^{\infty} P_i$ for $G$-invariant homogeneous degree $i$ polynomials $P_i$ (for an element of $S(V^*)$ being homogeneous of degree $i$ means being an element of $S^i(V^*)$). Theorem 2.18 still holds: Let $\mathcal{P}_r(u_n)^{U_n}$ denote the set of invariant polynomials which are homogeneous of degree $r$. Then the following commutative diagram shows that $\mathbb{C}^{sym}[t_1, \ldots, t_n] \cong \mathcal{P}_\infty (u_n)^{U_n}$:

$$\begin{array}{ccc}
\mathbb{C}^{sym}[t_1, \ldots, t_n] & \xrightarrow{\pi} & \mathcal{P}_\infty (u_n)^{U_n} \\
\prod_{r=0}^{\infty} \mathbb{C}^{sym}[t_1, \ldots, t_n] \cong & & \prod_{r=0}^{\infty} \mathcal{P}_r(u_n)^{U_n}
\end{array}$$
2.5. Characteristic forms. An invariant polynomial $P \in \mathcal{P}(g)^G$ defines for every vector bundle with connection having the appropriate structure (i.e. fiber bundle with structure group $G$ for the cases (2.7)) a closed differential form $P(\Omega)$ on the base space. They are often called characteristic forms.

By remark 2.15 such an invariant polynomial $P$ may be viewed as a polynomial expression $P(A)$ in the matrix entries $A = [a_{ij}]_{i,j=1}^n$. We will write $\frac{\partial P}{\partial a_{ij}}$ for the formal partial derivative of $P$.

Definition 2.22. Let $P$ be an invariant polynomial and let $E \to X$ be a vector bundle equipped with a connection $\omega$ and corresponding curvature $\Omega$. Set $\Omega = s^*\Omega$ as above for a local frame $s = (s_1, \ldots, s_n)$ over $U$. Since even differential forms commute we may define $P(\tilde{\Omega})$ by formally replacing the indeterminants $a_{ij}$ for the components $\tilde{\Omega}_{ij}$ of the matrix of 2-forms $\tilde{\Omega}$. Covering $X$ by trivializing neighborhoods, it is clear from the invariance of $P$ and (2.14) that the locally defined differential forms $P(\tilde{\Omega})$ agree on overlaps and piece together a well-defined differential form on $X$ which we will denote by $P(\Omega)$.

Remark 2.23. For invariant formal power series $P = \sum_{i=0}^{\infty} P_i$ with $P_i$ homogeneous of degree $i$ set

$$P(\Omega) = \sum_{i=0}^{\infty} P_i(\Omega)$$

This sum is in fact finite.

Theorem 2.24. $P(\Omega)$ defines a closed differential form.

Proof. Clearly the problem is local, so it suffices to consider $P(\tilde{\Omega})$. Moreover, it suffices to prove the assertion for invariant polynomials. Let $P'(\tilde{\Omega})$ denote the matrix which is transpose of the matrix of formal partial derivatives $\frac{\partial P}{\partial a_{ij}}(\Omega)$. Then, by the chain rule

$$dP(\tilde{\Omega}) = \sum_{i,j} \frac{\partial P}{\partial a_{ij}}(\tilde{\Omega}) \wedge d\tilde{\Omega}_{ij} = \text{tr} \left( P'(\tilde{\Omega}) d\tilde{\Omega} \right)$$

By the Bianchi identity (2.8) this is equal to

$$dP(\tilde{\Omega}) = \text{tr} \left( P'(\tilde{\Omega}) \tilde{\Omega} \omega - P'(\tilde{\Omega}) \omega \tilde{\Omega} \right)$$

It is not hard to show (cf. [MS74]) that $P'(\tilde{\Omega})$ and $\tilde{\Omega}$ commute. It follows that

$$dP(\tilde{\Omega}) = \text{tr} \left( \Omega \left( P'(\tilde{\Omega}) \omega \right) - \left( P'(\tilde{\Omega}) \omega \right) \tilde{\Omega} \right)$$

We are working in the commutative ring of even forms, so from the general relation $\text{tr}(AB) = \text{tr}(BA)$ we conclude $dP(\tilde{\Omega}) = 0$. □

Proposition 2.25. Different choices of connections on a vector bundle $E \to X$ yield characteristic forms that are equal in cohomology.
We have to pull back for $g \in \mathfrak{g}$.

Given covariant derivatives $\nabla_0, \nabla_1$ (with respective curvatures $\Omega_0, \Omega_1$), we may form on the pullback of the bundle to $X \times \mathbb{R}$ the covariant derivative (whose curvature will be denoted by $\Omega$)

$$(\nabla s)(x,t) = t\nabla_0 s(x) + (1-t)\nabla_1 s(x)$$

Let $\iota_t(x) = (x,t)$. Then $P(\Omega_0)$ may be identified with $\iota_t^* P(\Omega)$ while $P(\Omega_1)$ may be identified with $\iota_t^* P(\Omega)$. But $\iota_0$ and $\iota_1$ are homotopic. A more elaborate discussion can be found in [MS74].

This last proposition enables us to prove the following.

**Proposition 2.26.** The definitions (2.17), (2.17) of $c_k$ (viewed as elements in cohomology) satisfy all the axioms of the (real) Chern classes.

**Proof.** As our definition (2.18) is clearly natural (using the pullback connection) and also satisfies the Whitney product axiom (use the direct sum of connections induced on $E_1 \oplus E_2$ discussed in the next section), our main task is to show that

$$c_1((\gamma_C)^1) = -g$$

for the tautological complex line bundle $(\gamma_C)^1 \to \mathbb{C}P^1$. Recall that $g$ was defined in (1.9) by the relation $(g, [\mathbb{C}P^1]) = 1$. The associated principal $GL_1(\mathbb{C}) = \mathbb{C}^\times$-bundle of $(\gamma_C)^1$ is the canonical map

$$\mathbb{C}^2 \setminus 0 \to \mathbb{C}P^1$$

A connection on this bundle is given by a $\mathbb{C}$-valued $1$-form $\omega$ on $\mathbb{C}^2 \setminus 0$. Denote by $(z_0, z_1)$ the standard coordinates on $\mathbb{C}^2 \setminus 0$. Clearly

$$\omega = \frac{\bar{z}_0 dz_0 + z_1 dz_1}{|z_0|^{2} + |z_1|^{2}}$$

satisfies the axioms (2.3), (2.4) for a connection on $\mathbb{C}^2 \setminus 0$. Note that $\omega$ is a complex valued differential form on $\mathbb{C}^2 \setminus 0$ viewed as a real manifold. Rewrite $\omega$ in the coordinates $z = \frac{z_0}{z_1}$ and $z_0$ on $\mathbb{C}^x \times \mathbb{C}^x \subset \mathbb{C}^2 \setminus 0$. From $dz_1 = d(z_0) = dz_0 + zdz_0$ we have

$$\omega = \frac{dz_0}{z_0} + \frac{\bar{z}}{1 + |z|^2} dz$$

Since $\mathbb{C}$ is abelian, (2.5) simplifies to $\Omega = d\omega$ for the curvature. We obtain

$$\Omega = d\omega = -\frac{dz \wedge d\bar{z}}{(1 + |z|^2)}$$

We have to pull back $\Omega$ to the base $\mathbb{C}P^1$ and then integrate the result over $\mathbb{C}P^1$.

Since we may neglect a set of measure zero, it suffices to consider $U_0 = \{(z_0 : z_1) \in \mathbb{C}P^1 \mid z_0 \neq 0\} = \mathbb{C}P^1 \setminus \{0 : 1\} \approx \mathbb{C}$, which is parametrized by the coordinate $z = z_1/z_0$. Introducing polar coordinates $re^{i\varphi} = z$ we have $dz = e^{i\varphi} dr + ire^{i\varphi} d\varphi$ and $d\bar{z} = e^{-i\varphi} dr - ire^{-i\varphi} d\varphi$, so that

$$dz \wedge d\bar{z} = -2i \, dr \wedge d\varphi$$

---

$^5$For any $p = (p_0, p_1) \in \mathbb{C}^2 \setminus 0$ the map $(\lambda_p)_\ast$ from (2.1) is simply given by $(\lambda_p)_\ast : \mathbb{C} \to T_p(\mathbb{C}^2 \setminus 0) \cong (\mathbb{C}^2 \setminus 0) \times \mathbb{C}^2, v \to V_p := (p, (v_0, v_1)).$ From this (2.3) follows. On the other hand, for $g \in \mathbb{C}^\times$ the differential of the map $p_{g}(p) = pg$ is just $(\mathbb{C}^2 \setminus 0) \times \mathbb{C}^2 \to (\mathbb{C}^2 \setminus 0) \times \mathbb{C}^2, (p, v) \to (pg, gv).$ Then (2.4) is immediate from the commutativity of $\mathbb{C}^\times$. 

---

A GENERALIZED MIRACULOUS CANCELLATION FORMULA 28
It follows that
\[
\langle \Omega, [\mathbb{C}P^1] \rangle = \int_{U_0} \frac{-dz \wedge d\bar{z}}{(1 + |z|^2)^2} = \int_0^{2\pi} \int_0^\infty \frac{2\pi i}{(1 + r^2)^2} dr d\varphi = 2\pi i \left[ \frac{-1}{1 + r^2} \right]_0^\infty = 2\pi i
\]
which establishes the result:
\[
\langle -c_1((\gamma C)_1), [\mathbb{C}P^1] \rangle = \langle -\frac{i}{2\pi} \Omega, [\mathbb{C}P^1] \rangle = 1
\]
\[\square\]

2.6. Induced Connections. In the light of proposition 1.25 it is natural to ask whether a smooth functor \( F \) and given connections on each of the vector bundles \( E_1, \ldots, E_n \to X \) will yield a connection on \( F(E_1, \ldots, E_n) \). To answer this we have to analyze the situation for associated bundles (cf. the proof of proposition 1.25). We will write \( T \) for the “tangent bundle” functor.

Let \( G \) be Lie group with multiplication \( \mu : G \times G \to G \) and inversion \( i : G \to G \). Clearly, the tangent bundle \( TG \) is again a Lie group with multiplication \( T\mu \) and inversion \( Ti \) called the tangent group. If \( e \in G \) is the neutral element of \( G \), then \( 0_e \) is the neutral element of \( TG \).

**Proposition 2.27.** Let \( p : P \to X \) be a principal \( G \)-bundle. Then \( TP : TP \to TX \) is a principal \( TG \)-bundle.

**Proof.** Let \( h_i : P|_{U_i} \to U_i \times G \) be \( G \)-equivariant local trivializations of \( P \) over an open cover \( U_i \) with corresponding transition functions \( g_{ij} : U_i \cap U_j \to G \), and \( G \) acting by left-translations (so \( h_i h_j^{-1}(x, g) = (x, \mu(g_{ij}(x), g)) \)). Since \( T \) respects products it is clear that the \( Th_i \) constitute a set of local trivializations of the fiber bundle \( TP \). From
\[
Th_i Th_j^{-1}(v_x, w_g) = (v_x, \mu_*( (g_{ij})_*(v_x)), w_g))
\]
and because \( T\mu = \mu_* \) is the group operation of \( TG \), the transition functions of \( TP \) are \( Tg_{ij} \) (viewing \( TG \) as acting by left-translations). It follows by remark 1.15 that \( TP \) is a principal \( TG \)-bundle. \[\square\]

**Proposition 2.28.** Let \( P \) be a principal \( G \)-bundle, and assume \( \lambda : G \to H \) is a Lie group homomorphism. Then
\[
T(P[G \to H]) \cong TP[TG \to TH]
\]

**Proof.** The bundle \( T(P[G \to H]) \) has transition functions
\[
T(\lambda \circ g_{ij}) = T\lambda \circ Tg_{ij}
\]
as does \( TP[TG \to TH] \). The result now follows from proposition 1.11. \[\square\]
Proposition 2.29. Let $P \to X$ be a principal $G$-bundle with connection $\omega$, and let $\lambda : G \to H$ be a Lie group homomorphism. Write $Tq : TP \times TH \to TP[TG \overset{\lambda}{\to} TH]$ for the canonical map. Define
\[
\hat{\omega}(v_p, w_h) := Ad_{h^{-1}} \circ \lambda_* \circ \omega(v_p) + (l_{h^{-1}})_* w_h, \quad v_p \in T_pP, \ w_h \in T_hH
\] (2.24)
Then $\hat{\omega}$ factors through $Tq$ to a connection $\tilde{\omega} : T(P[H \to G]) \to \mathfrak{h}$ on the associated principal $H$-bundle $P[H \overset{\lambda}{\to} G]$.

Proof. We have for any $u_g \in T_gG$ that (see subsection 2.1 for the notation $\lambda_p, \rho_p, \ldots$)
\[
\hat{\omega}(v_p, u_g, w_h) = \hat{\omega}((\lambda_p)_* u_g + (\rho_g)_* v_p, w_h)
\]
\[
\hat{\omega}(v_p, (\lambda_p)_* u_g, w_h) = \hat{\omega}(v_p, (l_{h^{-1}})_* w_h + (r_{h^{-1}})_* \lambda_*(u_g))
\]
coincide. Here we have used the properties (2.3), (2.4) of the connection $\omega$. Therefore $\hat{\omega}$ factors through $Tq$. Axiom (2.4) is trivial for $\hat{\omega}$. As for (2.3), observe that $(\lambda_{[p,h]})(w) = q_u(0_p, (l_h)_*, w) = w_{[p,h]}$ for $w \in \mathfrak{h}$ and $p \in P, h \in H$. Thus $\hat{\omega}(w_{[p,h]}) = \hat{\omega}(0_p, (l_h)_*, w) = w$.

Example 2.30. Let $E \to X$ be a real vector bundle with given connection $\omega$. Then the complexification $E \otimes \mathbb{C} \cong P_{\text{GL}}(E) \times_{\text{GL}} \mathbb{C}^n$ for $\rho : GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ inherits a connection. This corresponds simply to viewing a real matrix as a complex one.

Let $s : U \to P$ be a section of $P$. Then $\hat{s}(x) = [s(x), e]$ is a section of $P[\lambda]$. For a tangent vector $v_x \in T_xX$ we have by (2.24)
\[
\tilde{\omega}(v_x) = \hat{\omega}(s_*(v_x), 0_e) = \rho_* \omega(s_*(v_x))
\] (2.25)

Proposition 2.31. Let $\mathcal{F} : \text{Vect} \times \cdots \times \text{Vect} \to \text{Vect}$ be a smooth functor on the category of finite dimensional (complex) vector spaces. Assume $E_1, \ldots, E_n \to X$ are vector bundles with connections $\omega_1, \ldots, \omega_n$. Then $\mathcal{F}(E_1, \ldots, E_n)$ has a canonical connection $\omega$ determined by the $\omega_k$.

If $E_1, \ldots, E_n$ are trivialized over $U$, the connections $\omega_k$ correspond (by pulling back to the base space using sections $s_k$ of the bundle $P_k$ of frames) to matrices of differential 1-forms $\tilde{\omega}_k$. Then, $\mathcal{F}(E_1, \ldots, E_n)$ may also be trivialized over $U$ and the induced connection $\omega$ corresponds to $\tilde{\omega}$ (using the section $(s_1, \ldots, s_n)$) defined by
\[
\tilde{\omega} = \rho_*(\tilde{\omega}_1, \ldots, \tilde{\omega}_n)
\] (2.26)

Proof. As in proposition 1.25, the functor $\mathcal{F}$ yields smooth representations $\rho : GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_k}(\mathbb{R}) \to GL(\mathbb{R}^{d(n_1, \ldots, n_k)})$. The associated principal bundle of frames of the vector bundle $\mathcal{F}(E_1, \ldots, E_n)$ is canonically isomorphic to $(P_{\text{GL}}E_1 \times \cdots \times P_{\text{GL}}E_n)[\rho]$. We may therefore just a well work with $(P_{\text{GL}}E_1 \times \cdots \times P_{\text{GL}}E_n)[\rho]$.

We have already seen in proposition 2.29 that $(P_{\text{GL}}E_1 \times \cdots \times P_{\text{GL}}E_n)[\rho]$ inherits a

\[\text{The canonical isomorphism } (P_{\text{GL}}E_1 \times \cdots \times P_{\text{GL}}E_n)[\rho] \cong P_{\text{GL}}((P_1 \times \cdots \times P_k) \times_{\rho} \mathbb{R}^d) \text{ is given by mapping } (\vec{p} = (p_1, \ldots, p_n), A) \text{ for } (A_1, \ldots, A_k) = A \in GL(d) \text{ to the frame } (\vec{p}, A_1, \ldots, \vec{p}, A_k).\]
connection. The connection on $P_{GL}(E_1,\ldots,E_n)$ is now defined as the pullback under the canonical isomorphism.

The last assertion (2.26) follows immediately from the calculation (2.25).

\[\square\]

**Remark 2.32.** Proposition 2.31 can also be stated as follows: Let $\text{VectBun}^\nabla(X)$ be the category of complex (resp. real) vector bundles over $X$ with connections. In this category, a morphism $(E,\omega_E) \to (F,\omega_F)$ is by definition a bundle isomorphism $\phi : E \to F$ so that $\omega_F = \phi^*\omega_E$ for the pullback connection ($\phi$ clearly induces an equivariant map of the associated principal bundles). Proposition 2.31 now states that a smooth functor of vector spaces will induce a corresponding functor on $\text{VectBun}^\nabla(X)$ for any $X$. In remark 2.33 we will see that a natural isomorphism of functors on $\text{Vect}$ will give a natural isomorphism of functors on the category $\text{VectBun}^\nabla(X)$ of bundles with connections.

**Remark 2.33.** Let $\mathcal{F},\mathcal{G}$ be naturally isomorphic functors on vector spaces. In remark 1.26 it was shown that the induced functors on $\text{VectBun}$ will then too be naturally isomorphic. There, we considered the representations $\rho^\mathcal{F},\rho^\mathcal{G}$ corresponding to the functors, which are equivalent by assumption, i.e. $\rho^\mathcal{F} = \phi^{-1} \circ \rho^\mathcal{G} \circ \phi$. The natural isomorphism was then given in (1.8) by

$$
(P_1 \times \cdots \times P_k) \times \rho^\mathcal{F} \mathbb{R}^{d(n_1,\ldots,n_k)} \to (P_1 \times \cdots \times P_k) \times \rho^\mathcal{G} \mathbb{R}^{d(n_1,\ldots,n_k)}
$$

$$
[(p_1,\ldots,p_k),\tilde{v}] \mapsto [(p_1,\ldots,p_k),\phi(\tilde{v})]
$$

It induces an isomorphism of the associated principal bundles

$$(P_1 \times \cdots \times P_k) [\rho^\mathcal{F}] \to (P_1 \times \cdots \times P_k) [\rho^\mathcal{G}], \quad ((p_1,\ldots,p_k),A) \mapsto ((p_1,\ldots,p_k),\phi \circ A)$$

We have $\rho_e^\mathcal{F} = \text{Ad}_{\phi^{-1}} \circ \rho_e^\mathcal{G}$. From (2.24) it is now clear that the connections $\omega^\mathcal{F},\omega^\mathcal{G}$ induced by proposition 2.31 on $(P_1 \times \cdots \times P_k) [\rho^\mathcal{F}]$ and $(P_1 \times \cdots \times P_k) [\rho^\mathcal{G}]$ correspond to each other via pullback under this isomorphism:

$$
\omega^\mathcal{F}(v_{p_1} \oplus \cdots \oplus v_{p_k} \oplus w_A) = \text{Ad}_{A^{-1}} \circ \rho_e^\mathcal{F} \omega(v_{p_1} \oplus \cdots \oplus v_{p_k}) + (l_{A^{-1}})_* w_A
$$

$$
= \omega^\mathcal{G}(v_{p_1} \oplus \cdots \oplus v_{p_k} \oplus (\phi)_* w_A) = \text{Ad}_{(\phi A)^{-1}} \circ \rho_e^\mathcal{G} \omega(v_{p_1} \oplus \cdots \oplus v_{p_k}) + (l_{(\phi A)^{-1}})_* (\phi)_* w_A
$$

**Example 2.34.**

1. **Direct sum.** $\rho : GL(n) \times GL(m) \to GL(n+m)$, $(A_1,A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$

and $\rho_{e,(1,1)}(A,B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

2. **Tensor product.** $\rho : \mathbb{C}^{n_1 \times n_1} \times \cdots \times \mathbb{C}^{n_d \times n_d} \to \mathbb{C}^{(n_1 \cdots n_d) \times (n_1 \cdots n_d)}$, $(A_1,A_d) \mapsto A_1 \otimes \cdots \otimes A_d$ (for the tensor product $(a_{ij}) \otimes (b_{ij}) := (a_{ij}B)_{ij}$ of matrices) is multilinear in each of it’s $d$ variables, so $\rho_{e,(1,\ldots,1)}(A_1,\ldots,A_d) = A_1 \otimes 1 \otimes \cdots \otimes 1 + A_2 \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes A_d$. The same therefore holds for the restriction $GL(n_1) \times \cdots \times GL(n_d) \to GL(n_1 \cdots n_d)$. 


(3) **Conjugate / dual.** $\bar{E}$ is obtained from $E$ by using the representation $\rho : U(n) \to U(n), \ A \mapsto \bar{A}$. From (2.16) it follows immediately that

$$1 + \sum s_k(\bar{A}) = \det \left( 1 + \frac{1}{2\pi} \bar{A} \right) = \det \left( 1 - \frac{1}{2\pi} A \right) = 1 + \sum s_k(-A)$$

for $A$ skew-Hermitian, so $s_k(\bar{A}) = (-1)^k s_k(A)$ by homogeneity. Therefore

$$c_k(\bar{E}) = s_k(\bar{\Omega}) = (-1)^k s_k(\Omega) = (-1)^k c_k(E) \quad (2.27)$$

(4) **Exterior Powers.**

(5) **Symmetric Powers.**
3. A Refined Chern Root Formalism

Let us first recall the original Chern root formalism. By the splitting principle 1.30 there exists for any complex vector bundle $E \rightarrow X$ a map $q : Y \rightarrow X$ such that $q^*E \cong L_1 \oplus \cdots \oplus L_n$ is isomorphic to a direct sum of line bundles. Moreover, the induced map $q^*$ in cohomology is injective. From the axioms of the Chern classes we compute

$$q^*c(E) \cong c(q^*E) \cong c(L_1 \oplus \cdots \oplus L_n) \cong \prod_{i=1}^n (1 + c_1(L_i))$$

If we identify $H^*(X)$ as a subring of $H^*(Y)$ via the injective map $q^*$, it follows that

$$c_k(E) = \sigma_k(w_1, \ldots, w_n)$$

for the so called Chern roots $w_i = c_1(L_i) \in H^*(Y)$.

Assume an invariant polynomial $P \in \mathcal{P}(u_n)^{U_n}$ is defined on diagonal matrices by a symmetric polynomial $p(t_1, \ldots, t_n) \in \mathbb{C}^{sym}[t_1, \ldots, t_n]$. In all the examples 2.20 it was customary to write out the polynomial $p(t_1, \ldots, t_n)$ as $q \left( \frac{i}{2\pi}t_1, \ldots, \frac{i}{2\pi}t_n \right)$ for some symmetric polynomial $q$ (there we had set $x_k = \frac{1}{2\pi}t_k$ so that $q(x) = p(t)$). By theorem 1.3 we may write

$$q = f(\sigma_1, \ldots, \sigma_n)$$

As in the proof of theorem 2.18, extend $q$ to arbitrary skew-Hermitian matrices by

$$Q = f(s_1, \ldots, s_n)$$

for $s_k$ defined in (2.16). Thus $Q(\Lambda) = q(\Lambda_{11}, \ldots, \Lambda_{nn})$ for every diagonal matrix $\Lambda$. It follows that $Q \left( \frac{i}{2\pi} \Lambda \right) = q \left( \frac{i}{2\pi} \Lambda_{11}, \ldots, \frac{i}{2\pi} \Lambda_{nn} \right) = p(\Lambda_{11}, \ldots, \Lambda_{nn})$, whence $Q \left( \frac{i}{2\pi} \Lambda \right) =: P(\Lambda)$ is the unique extension of $p$ to arbitrary skew-Hermitian matrices (cf. theorem 2.18).

Let $E \rightarrow X$ be a Hermitian vector bundle with curvature $\Omega$. In the ring extension $H^*(Y) \supset H^*(X)$ we compute

$$P(\Omega) = Q \left( \frac{i}{2\pi} \Omega \right) = f \left( s_1 \left( \frac{i}{2\pi} \Omega \right), \ldots, s_n \left( \frac{i}{2\pi} \Omega \right) \right) \stackrel{(2.17)}{=} f(c_1(E), \ldots, c_n(E)) \quad \text{(3.1)}$$

$$= f(\sigma_1, \ldots, \sigma_n)(w_1, \ldots, w_n)$$

$$= q(w_1, \ldots, w_n)$$

Once the Chern roots have been established, we may thus compute $P(\Omega)$ by using only the definition for diagonal matrices. This may also be stated by saying that the splitting principle may be applied to the computation of characteristic classes by allowing us to view everything as a direct sum of line bundles. The drawback of this approach is that it is only valid in cohomology and not on the level of differential forms.

We will show that using an appropriate ring extension of $\Omega^{2\ast}(X)$ this procedure can be refined to the level of differential forms. More precisely, what we will show is that it is algebraically possible to adjoin “new Chern roots” to the ring $\Omega^{2\ast}(M)$.
of even differential forms so that \((3.1)\) is valid. Clearly, the rest of the argument then goes through as above.

3.1. The Splitting Ring.

**Proposition 3.1.** Let \(A\) be a commutative ring and assume \(f(t) = \sum_{i=0}^{n} a_i t^i\) is a monic non-constant polynomial of degree \(n\) (so \(a_n = 1\)). There exists a ring extension \(\hat{A}\) of \(A\) such that \(f\) has \(n\) roots \(w_1, \ldots, w_n\) in \(\hat{A}\):

\[
f(t) = (t - w_1) \cdots (t - w_n)
\]

**Proof.** Let us proceed by induction on \(n\), the case \(n = 1\) being trivial. Assume \(n \geq 2\) and let

\[
B := A[t]/(f)
\]

where \((f)\) denotes the principal ideal in \(A[t]\) generated by \(f\). Write \(p : A[t] \to B\) for the canonical projection. Since \(f(t)\) is monic, the restriction of \(p\) to \(A\) is injective.\(^8\)

We may therefore view \(B\) as a ring extension of \(A\). Defining \(w_1 := p(t)\) we have in \(B\) that

\[
0 = p(f(t)) = \sum_{i=0}^{n} a_i w_1^i
\]

That is, the polynomial \(f\) has the root \(w_1\) in \(B\). Since the leading coefficient of \(f\) is a unit, by division with remainder we may write

\[
f(t) = (t - w_1)g(t)
\]

for a monic polynomial of degree \(n - 1\) over \(B\). Applying the induction hypothesis to \(g\) and \(B\) we obtain another ring extension \(B \subset C\) such that \(g(t) = \prod_{k=2}^{n} (t - w_k)\). This completes the proof. \(\square\)

A splitting ring may be described by a universal property:

**Definition 3.2.** Let \(A\) be a commutative ring and \(f(t) \in A[t]\) a monic polynomial of degree \(n \geq 1\). A ring extension \(A \subset \hat{A}\) is said to be a splitting ring for \(f\) if the polynomial \(f\) splits in \(\hat{A}\), and \(\hat{A}\) is universal with respect to this property. This means that for any ring homomorphism \(\varphi : A \to R\) such that the reduction of \(f\) mod \(\varphi\) splits in \(R\)

\[
(\varphi f)(t) = \prod_{i=1}^{n} (t - v_i), \quad v_i \in R
\]

(3.3)

there exists an extension \(\phi : \hat{A} \to R\), uniquely determined by the requirement \(\phi(w_i) = v_i\).

It follows that the splitting ring of \(f\) is determined up to unique isomorphisms.

**Lemma.** The construction of proposition 3.1 gives a splitting ring.

\(^8\)Assume \(0 \neq a \in (f)\), say \(a = (\sum_{i=0}^{m} b_i t^i) f(t)\) with \(b_m \neq 0\). Since \(f\) is monic, taking the top degree of the equation yields \(0 = b_m t^{n+m}\), a contradiction.
Proof. It remains to verify that the constructed ring extension is universal. We will do this again by induction on the degree \( n \) of the polynomial \( f \), the case \( n = 1 \) being trivial (take \( \phi := \varphi : \hat{A} = A \rightarrow R \)), so assume \( n \geq 2 \). With the notation of proposition 3.1 let \( \varphi : A \rightarrow R \) be given as in (3.3). \( \varphi \) extends to \( A[t] \xrightarrow{\text{mod } \varphi} R[t] \xrightarrow{\text{eval} \_1} R \) and then to \( \varphi_1 : B = A[t]/(f) \rightarrow R \) since \( v_1 \) is a root of \( \varphi f \).

Note also that \( \varphi_1(w_k) = v_k \), \( k \geq 2 \). Uniqueness is clear since \( C \) is generated by \( w_1, \ldots, w_n \) over \( A \).

\[ \square \]

3.2. Application to Chern Forms.

Definition 3.3. Let \( E \rightarrow M \) be a rank \( n \) Hermitian vector bundle with given curvature \( \Omega \). Elements \( w_1, \ldots, w_n \) (possibly in a ring extension on \( \Omega^{2\ast}(M) \)) satisfying

\[ t^n - c_1(E)t^{n-1} + \cdots + (-1)^n c_n(E) = (t - w_1) \cdots (t - w_n) \quad (3.4) \]

will be called new Chern roots of \( E \). Here, \( c_k(E) \) denotes the \( k \)-th Chern form.

Remark 3.4. The new Chern roots \( w_1, \ldots, w_n \) depend on the choice of connection, but they are not unique anyway. Instead we will fix in applications an arbitrary set of them for the given bundles. By comparing coefficients in (3.4) it follows that

\[ c_k(E) = \sigma_k(w_1, \ldots, w_n) \]  

(3.1)

As already remarked, (3.2) remains valid now also for differential forms.

Example 3.5. (complex case) Let \( E \rightarrow M \) be a Hermitian vector bundle. If we apply proposition 3.1 to the commutative ring \( A = \Omega^{2\ast}(M) = \bigoplus_{i \geq 0} \Omega^{2i}(M) \) and the monic polynomial

\[ f(t) = t^n - c_1(E)t^{n-1} + \cdots + (-1)^n c_n(E) \]

we conclude that there always exists a ring extension \( \hat{\Omega}(M) \) of \( \Omega^{2\ast}(M) \) together with \( n \) new Chern roots \( w_1, \ldots, w_n \) so that we may factor

\[ f(t) = (t - w_1) \cdots (t - w_n) = t^n - (w_1 + \cdots + w_n)t^{n-1} + \cdots + (-1)^n w_1 \cdots w_n \]

The so defined Chern roots are more or less unique, owing to the universal property of splitting rings.

Example 3.6. (real case) Let \( E \) be a Riemannian vector bundle of even rank \( 2n \). It will be more convenient to work with a slightly altered construction of the new Chern roots. From \( E \otimes \mathbb{C} \cong E \otimes \mathbb{C} \) and (2.27) it is clear that the odd Chern forms \( c_k(E) \in \Omega^{2k}(E) \) must vanish. The polynomial in (3.4) is thus an even polynomial. Set \( s = t^2 \), and apply proposition 3.1 to

\[ s^n + c_2(E)s^{n-1} + \cdots + c_{2n}(E) \]
In a ring extension therefore the polynomial splits $\prod_{k=1}^n (s - u_k)$. Applying proposition 3.1 once more to $X^2 - u_k$ we obtain $X^2 - u_k = (X - a_k)(X - b_k)$, i.e. $a_k = -b_k = \sqrt{u_k}$. Thus

$$f(t) = t^{2n} + c_2(E)t^{2n-2} + \cdots + c_{2n}(E) = \prod_{k=1}^n(t^2 - u_k) = \prod_{k=1}^n(t - \sqrt{u_k})(t + \sqrt{u_k})$$

It follows that $c_k(E) = \sigma_k(\pm\sqrt{u_1}, \ldots, \pm\sqrt{u_n})$ for any $k$ (for odd $k$ both sides are zero). We have thus obtained formal Chern roots in $\pm w_k = \pm\sqrt{u_k}$ pairs. Moreover, from the last equation it is clear that

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) = \sigma_k(w_1^2, \ldots, w_n^2) \quad \quad (3.5)$$

**Proposition 3.7.** (Naturality of the new Chern roots) Let $E \to M$ be a Hermitian vector bundle of rank $n$ with given connection, and let $q : N \to M$. Chose Chern roots as in example 3.5. Then $q$ induces a ring-homomorphism

$$q^* : \Omega(M) \to \hat{\Omega}(N),$$

where $w_k$ denote the new Chern roots of $E$ and $w_k^*$ those of the pullback bundle $q^*E$.

**Proof.** This follows from the fact that the Chern form $c(E)$ is natural, $q^*c(E) = c(q^*E)$ if we equip $q^*E$ with the pullback connection. Write $f(t) = \sum_{k=0}^n (-1)^k c_k(E)t^{n-k}$. We then have

$$(q^*f)(t) = \sum_{k=0}^n (-1)^k q^*(c_k(E))t^{n-k} = \sum_{k=0}^n (-1)^k c_k(q^*E)t^{n-k} = \prod_{k=1}^n(t - w_k^*)$$

Therefore, by the universal property, pullback $\Omega^*(M) \to \Omega^*(N)$ can $\hat{\Omega}(N)$ extends to the required map. \qed

### 3.3. Generalization of a Theorem of Borel and Hirzebruch

Let $\rho : U(n) \to U(m)$ be a representation. The image of the maximal torus $S^n \subset U(n)$ is again contained in some maximal torus of $U(m)$. Since all maximal tori are conjugate, by replacing $\rho$ by an equivalent representation if necessary, we may assume

$$\rho(S^n) \subset S^m$$

It follows that we may write

$$\rho \left( e^{2\pi i x_1} \cdots e^{2\pi i x_n} \right) = \left( e^{2\pi i y_1} \cdots e^{2\pi i y_n} \right)$$

where

$$y_k = a_{k1}x_1 + \cdots + a_{kn}x_n$$

for some $a_{kl} \in \mathbb{Z}$ which are called the weights of the representation. For the restriction to $S^n$ it follows that $\rho(t_1, \ldots, t_n) = (t_1^{a_{11}} \cdots t_n^{a_{1n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})$ so the
restriction of the derivative $\rho_*$ to the Lie algebra of skew-Hermitian diagonal matrices $(i\mathbb{R})^n$ equals

\[
\begin{pmatrix}
\Omega_{11} \\
\vdots \\
\Omega_{nn}
\end{pmatrix} \mapsto \begin{pmatrix}
a_{11}\Omega_{11} + \cdots + a_{1n}\Omega_{nn} \\
\vdots \\
a_{m1}\Omega_{11} + \cdots + a_{mn}\Omega_{nn}
\end{pmatrix}
\]

Define

\[P_t(\Omega) = \det \left(t \text{id} - \frac{i}{2\pi} \rho_\ast \Omega\right)\]

Note that by (2.26) $\rho_\ast \Omega$ gives the appropriate connection on the associated bundle as needed in the definition of the new Chern roots in (3.4). The polynomial $P_t$ is $U(n)$-invariant since by differentiating $\rho_\ast (A\Omega A^{-1}) = \rho_\ast A(\Omega) = \rho_\ast A \rho_\ast (\Omega)$ we obtain $\rho_\ast (A\Omega A^{-1}) = \rho_\ast \circ Ad_A(\Omega) = Ad_\rho(A) \rho_\ast (\Omega)$. For diagonal $\Omega = \text{diag}(\Omega_{11}, \ldots, \Omega_{nn})$ we have

\[P_t(\Omega) = \prod_{k=1}^m \left(t - \frac{i}{2\pi} (a_{k1}\Omega_{11} + \cdots + a_{kn}\Omega_{nn})\right) =: p(\Omega_{11}, \ldots, \Omega_{nn})\]

so by (3.2) it follows that

\[P_t(\Omega) = q(w_1, \ldots, w_n) = p \left(2\pi \frac{i}{i} w_1, \ldots, 2\pi \frac{i}{i} w_n\right) = \prod_{k=1}^m \left(t - (a_{k1} w_1 + \cdots + a_{kn} w_n)\right)\]

We thus conclude:

**Theorem 3.8.** (Borel/Hirzebruch) Let $E$ be a Hermitian vector bundle with new Chern roots $w_1, \ldots, w_n$, and assume $\rho : U(n) \to U(m)$ is a representation having weights $a_{kl}$ as above. Then the associated bundle $\rho E := P[\rho] \times_{GL} \mathbb{C}^m$ (for the bundle $P$ of orthogonal frames) has new Chern roots

\[y_k = a_{k1} w_1 + \cdots + a_{kn} w_n, \quad k = 1, \ldots, m\]

**Example 3.9.**

1. **Exterior Powers** $\Lambda^k E$

   \[y_{i_1, \ldots, i_k} = w_{i_1} + \cdots + w_{i_k}, \quad \text{for } 1 \leq i_1 < \cdots < i_k \leq n\]

2. **Tensor Product** $E \otimes E$

   \[y_{ij} = w_i + w_j, \quad \text{for } 1 \leq i, j \leq n\]

3. **Symmetric Powers** $S^k E$

   \[y_{i_1, \ldots, i_k} = w_{i_1} + \cdots + w_{i_k}, \quad \text{for } 1 \leq i_1 \leq \cdots \leq i_k \leq n\]
Proof. As an example, consider the case of exterior powers. We have to calculate the weights of the representation $U(n) \rightarrow U\left(\begin{array}{c} n \\ k \end{array}\right)$. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$. Consider diagonal $f$, i.e. $f(e_i) = e^{2\pi i x_i} e_i$. Then

$$(\Lambda^k f)(e_i \wedge \ldots \wedge e_{ik}) = e^{2\pi i (x_{i_1} + \ldots + x_{i_k})} e_{i_1} \wedge \ldots \wedge e_{i_k},$$

so $y_{i_1}, \ldots, y_{i_k} = x_{i_1} + \ldots + x_{i_k}$.

\[\Box\]

**Corollary 3.10.** Let $\Lambda t(E) = \sum_{p \geq 0} \Lambda^p(E)t^p$. Then if $w_1, \ldots, w_n$ denote the new Chern roots of the rank $n$ vector bundle $E$

$$ch(\Lambda tE) = \prod_{i=1}^n (1 + e^{w_i}t) \tag{3.6}$$

**Proof.** We have

$$ch(\Lambda tE) = \sum_{p=0}^n ch(\Lambda^pE)t^p = \sum_{p=0}^n \sum_{1 \leq i_1 < \ldots < i_p \leq n} e^{w_{i_1} + \ldots + w_{i_p}}t^p$$

$$= \sum_{p=0}^n \sigma_p(e^{w_1}, \ldots, e^{w_n})t^p = \prod_{i=1}^n (1 + e^{w_i}t)$$

\[\Box\]
4. Spin Structures on Vector Bundles

In this section we will introduce the notion of a spin structure. A full account on this subject is given in [LM89] to which the presentation here owes much. A feature of the treatment here is that it makes no use of spectral sequences to prove $w_2(E) = 0$ iff $E$ is spin. The cost is that of replacing ordinary cohomology by Čech cohomology. Algebras are assumed to unital in the section.

4.1. Clifford Algebras. Let $V$ be a finite-dimensional real or complex vector space equipped with a non-degenerate symmetric bilinear form $⟨ , ⟩$ (with corresponding quadratic form $q(v) := ⟨v,v⟩$). The Clifford algebra associated to $(V,q)$ is defined as the associative algebra $\text{Cl}(V,q)$ generated by elements of $V$ subject to the relation

$$v \cdot w + w \cdot v = -2⟨v,w⟩ \quad (4.1)$$

By polarization, $2⟨v,w⟩ = q(v+w) - q(v) - q(w)$, it suffices to require only the relation $v \cdot v = -q(v)$. Explicitly, $\text{Cl}(V,q)$ may be defined as a quotient of the tensor algebra $TV = \bigoplus_{n \geq 0} V^\otimes n$ by the ideal $I$ generated in $TV$ by elements of the form $v \otimes v + q(v)$, $v \in V$. There is a canonical map $ι : V \hookrightarrow TV \twoheadrightarrow \text{Cl}(V,q)$.

**Proposition 4.1.** (Universal Property of Clifford Algebras) Given any map $f : V \rightarrow A$ into an associative algebra $A$ satisfying

$$f(v) \cdot f(v) = -q(v)1 \quad (4.2)$$

there exists a unique extension $\text{Cl}(V,q) \rightarrow A$ to an algebra homomorphism $\hat{f}$ with $\hat{f} \circ ι = f$.

**Proof.** By the universal property of the tensor algebra $f$ extends to an algebra homomorphism on $TV$. Assumption (4.2) shows that $f(v \otimes v + q(v)) = 0$ so $f$ descends to the required map $\text{Cl}(V,q) = TV/I \rightarrow A$. Uniqueness follows since $V$ generates $\text{Cl}(V,q)$.

It is clear that this characterizes the Clifford algebra up to unique isomorphism. Moreover, we obtain a functor $\text{Cl}$ from the category of vector spaces with inner products (and linear maps preserving them) into the category of algebras: $\text{Cl}(V \rightarrow W) := \left( V \xrightarrow{f} W \xrightarrow{ι} \text{Cl}(W) \right)^\sim$ is defined as the map induced by the universal property. Let $α := \text{Cl}(V \xrightarrow{\text{antipodal}} V)$. Then $α^2 = \text{id}$ which gives a $±1$ eigenspace decomposition of the Clifford algebra

$$\text{Cl}(V,q) = C^0(V,q) \oplus C^1(V,q)$$

into the so called even part $C^0(V,q)$ and the odd part $C^1(V,q)$. It is clear that $C^i(V,q) \cdot C^j(V,q) \subset C^{i+j \mod 2}(V,q)$ which turns the Clifford algebra into a $\mathbb{Z}_2$-graded algebra. Note that when $q = 0$ the Clifford algebra is just the exterior algebra $Λ(V)$. It is an elementary fact that if $V$ has basis $e_1, \ldots, e_n$ then $Λ(V)$ has basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ of ascending sequences, cf. [Lan02].
Proposition 4.2. The Clifford algebra of a n-dimensional vector space is a vector space of dimension $2^n$. Given an orthonormal basis $e_1, \ldots, e_n$ of $V$, a basis of $\text{Cl}(V,q)$ consists of all increasing sequences (as always the empty sequence being 1)

$$e_{i_1} \cdots e_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n, \quad k = 0, 1, \ldots n$$ (4.3)

In particular, the vector space $V$ is embedded into $\text{Cl}(V,q)$ by the canonical map $\iota : V \hookrightarrow TV \twoheadrightarrow \text{Cl}(V,q)$.

Proof. By definition of $\text{Cl}(V,q)$ as a quotient of the tensor algebra and by relation (4.1) it is clear that these products generate $\text{Cl}(V,q)$. In particular, the Clifford algebra of a $2n$-dimensional vector space is given by sequences $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for $1 \leq i_1, \ldots, i_k \leq n, \ k \geq 0$. We will define $f : B \to \Lambda(V)$ by induction on $k$. For $k = 1$ set $f(e_i) = e_i$ and of course $f(1) = 1$. For $k > 1$ the image of $e_{i_1} \otimes \cdots \otimes e_{i_k}$ is defined as $e_{i_1} \wedge \cdots \wedge e_{i_k}$ in case the $i_s$ are all distinct. Otherwise, let $i_r = i_s$ denote the first occurrence of the double index $(r < s)$, and map the element $e_{i_1} \otimes \cdots \otimes e_{i_k}$ to $(-1)^{k-r} f(e_{i_1} \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})$. The hat indicates omission.

Claim. In case $i_p \notin i_{p+1}$ we have

$$f(e_{i_1} \cdots e_{i_p} \otimes e_{i_{p+1}} \cdots \otimes e_{i_k}) = -f(e_{i_1} \cdots e_{i_{p+1}} \otimes e_{i_p} \cdots \otimes e_{i_k})$$ (4.4)

Using the shorthand notation $e_I := e_{i_1} \otimes \cdots \otimes e_{i_k}$ for $I = (i_1, \ldots, i_k)$, we have also

$$f(e_I \otimes (e_i \otimes 1) \otimes e_J) = 0$$ (4.5)

Proof of Claim. Let us first prove (4.4) by induction. If $k = 2$, $f(e_{i_1} \otimes e_{i_2}) = e_{i_1} \wedge e_{i_2} = -e_{i_2} \wedge e_{i_1} = -f(e_{i_2} \otimes e_{i_1})$. Let $k \geq 3$ and denote again by $i_r = i_s$ the first occurrence of a double indice. We will omit the tensor product symbol. If $s \neq p$

$$f(e_{i_1} \cdots e_{i_p} \cdots e_{i_r} \cdots e_{i_{p+1}} \cdots e_{i_k}) = (-1)^{k-r} f(e_{i_1} \cdots \hat{e}_{i_r} \cdots e_{i_r} \cdots e_{i_{p+1}} \cdots e_{i_k})$$

and the assertion follows by the induction hypothesis. On the other hand, if $s = p$

$$f(e_{i_1} \cdots e_{i_r} \cdots e_{i_{p+1}} \cdots e_{i_k}) = (-1)^{k-r} f(e_{i_1} \cdots \hat{e}_{i_r} \cdots e_{i_r} \cdots e_{i_{p+1}} \cdots e_{i_k})$$

This completes the proof of (4.4). Next, we will show (4.5) by induction on $|I|$, the case $I = \emptyset$ following immediately from the definition. In fact, even the more general case that $I$ contains no double indices and $i \notin I$ follows directly by definition. If $|I| > 0$ and $I$ contains a double indice, (4.5) follows from

$$f(e_I e_i e_i + 1) e_J) = f(e_{i_1} e_i e_i + 1) e_J)$$

and the induction hypothesis. Finally, in case all elements of $I$ are distinct and $i \in I$, say $i = i_r$, we compute

$$f(e_{i_1} \cdots e_{i_r} \cdots e_{i_k} (e_i e_i + 1) e_J)$$

$$= (-1)^{k+1-r} f(e_{i_1} \cdots \hat{e}_{i_r} \cdots e_{i_r} e_i e_J) + f(e_{i_1} \cdots e_{i_r} \cdots e_i e_J)$$

(4.4)

$$= (-1)^{k+1-r} (-1)^{k-r} f(e_{i_1} \cdots e_{i_r} e_i e_J) + f(e_{i_1} \cdots e_{i_r} \cdots e_i e_J) = 0$$

\[ \square \]

9 Which will be an isomorphism as soon as the proof is done.
Continuation of the Proof of Proposition 4.2. Let \( f : TV \to \Lambda(V) \) be the linear extension given by \( f \) on the basis. Since any sequence \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) of distinct elements is mapped onto \( e_{i_1} \wedge \cdots \wedge e_{i_k} \), \( f \) is surjective. It remains to show that the ideal \( I \) lies in the kernel of \( f \).

Every element of the ideal \( I \) may be written as a sum of elements of the form
\[
\varphi \otimes [v \otimes v + q(v)] \otimes \psi
\]
with \( v \in V \) and \( \varphi, \psi \in TV \) which may be assumed to be of the form \( \varphi = fe_I, \psi = pe_J \) for scalars \( f, p \). Since \( e_i \) is an orthonormal basis
\[
\left( \sum_i a_i e_i \right) \otimes \left( \sum_i a_i e_i \right) + q \left( \sum_i a_i e_i \right) = \sum_{i < j} a_i a_j (e_i \otimes e_j + e_j \otimes e_i) + \sum_i a_i^2 (e_i \otimes e_i + 1)
\]
and it follows that we may represent any element of the ideal \( I \) as a sum of elements of the form
\[
\lambda e_I \otimes (e_i \otimes e_j + e_j \otimes e_i) \otimes e_J, \text{ or } \lambda e_I \otimes (e_i \otimes e_i + 1) \otimes e_J
\]
where \( \lambda \) is a scalar. But by (4.4) and (4.5), each such element is mapped by \( f \) onto zero.

\[\square\]

4.2. **Pin and Spin as Two-Sheeted Coverings.** The group of units \( Cl^\mathbb{R}(V, q) \) of the Clifford algebra has two important subgroups
\[
\begin{align*}
\text{Pin}(V, q) & := \{v_1 \cdots v_n \mid \|v_i\| = 1\} \\
\text{Spin}(V, q) & := \{v_1 \cdots v_{2n} \mid \|v_i\| = 1\} = \text{Pin}(V, q) \cap Cl^0(V, q)
\end{align*}
\]
which consist of products of elements of the unit sphere of \( V \) (in the Spin case with an even number factors). These groups may be understood as follows. Consider the twisted adjoint representation
\[
\tilde{Ad} : Cl^\mathbb{R}(V, q) \to \text{Aut}(Cl(V, q)), \varphi \mapsto \tilde{Ad}_\varphi = x \mapsto \alpha(\varphi) x \varphi^{-1}
\]
(The adjoint representation \( Ad_{\varphi}(x) := \varphi x \varphi^{-1} \) clearly coincides with \( \tilde{Ad} \) on Spin.)

A calculation shows for \( v, w \in V \) that
\[
\tilde{Ad}_\varphi(w) = -vwv^{-1} = vw - \frac{1}{\|v\|^2}v = \frac{1}{\|v\|^2}(-wv-2\langle v, w \rangle)v = w-2\frac{\langle v, w \rangle}{\langle v, v \rangle}v = \text{reflection across } v^\perp
\]
It follows that we may restrict \( \tilde{Ad} : \text{Pin}(V, q) \to O(V, q) \). Now it is a well-known fact (cf. [LM89]) that every orthogonal transformation may be written as a composition of reflections across hyperplanes given by some unit normal \( \pm v_i \), defined up to sign:
\[(\pm v_1) \circ \cdots \circ (\pm v_n)\]

**Theorem 4.3.** In the real case, the (twisted) adjoint representation defines a two-fold covering
\[
\begin{align*}
0 & \to \{\pm 1\} \to \text{Spin}(V, q) \xrightarrow{\tilde{Ad}} SO(V, q) \to 1 \\
0 & \to \{\pm 1\} \to \text{Pin}(V, q) \xrightarrow{\tilde{Ad}} O(V, q) \to 1
\end{align*}
\]
For \( n \geq 2 \) the group \( \text{Spin}_n := \text{Spin}(\mathbb{R}^n) \) is path connected and thus is the universal covering space of \( SO_n \) for \( n \geq 3 \).
Proof. The map $\tilde{\text{Ad}}$ is surjective, since any orthogonal transformation may be written as a product of reflections across hyperplanes $v^\perp$, $\|v\| = 1$. Assume $\varphi \in \text{Pin}(V,q)$ lies in the kernel, so that $\alpha(\varphi)v = v\varphi$ ($\forall v \in V$). Let us first show that $\varphi$ must then be a real number. Decompose $\varphi = \varphi^0 + \varphi^1$ into its even and odd part, so for any $v \in V$

$$v\varphi^0 = \varphi^0 v, \quad -v\varphi^1 = \varphi^1 v \quad (\ast)$$

Let $e_1, \ldots, e_n$ be an orthogonal basis of $V$. For any $i$ the relation (4.1) implies that it is possible to write $\varphi^0 = a_0 + e_1a_1$ where $a_0, a_1$ do not involve $e_i$. We then have that $a_0$ is even, while $a_1$ is odd. $(\ast)$ now implies for $v = e_i$ that

$$e_ia_0 + e_1^2a_1 \equiv a_0e_i + e_ia_1e_i = e_ia_0 - e_1^2a_1$$

so $a_1 = 0$. Whence $\varphi^0$ does not involve $e_i$ for any $i$. Analogous reasoning applies to $\varphi^1$ to show that $\varphi^1 = 0$, so $\varphi$ is a scalar. Since $\varphi$ lies in $\text{Pin}(V,q)$ it must solve $\varphi^2 = \pm 1$ which is only possible for $\varphi = \pm 1$. This completes the proof that $\text{Spin}$ is a double covering.

To show that $\text{Spin}_n$ is path connected for $n \geq 2$ (since $SO_n$ is) it suffices to connect $+1$ and $-1$ by a path. But for $e_1, e_2 \in \mathbb{R}^n$ orthogonal, $\gamma(t) := -\cos(t) + e_1e_2\sin(t)$, $t \in [0,\pi]$ does the job. \hfill $\Box$

4.3. Čech cohomology. Several facts from Čech cohomology theory will be used in the next sections, which will therefore be roughly outlined here. We give [Hir66] as a general reference.

Let $G$ be an abelian group (written multiplicatively). For an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a (say paracompact) topological space $X$, a $q$-cochain is a family of maps

$$g_{i_0\cdots i_q} : U_{i_0\cdots i_q} \rightarrow G, \quad i_0, \ldots, i_q \in I$$

These comprise the $q$-th Čech cochain group $C^q(\mathcal{U};G)$. The differential $C^q(\mathcal{U};G) \xrightarrow{\delta} C^{q+1}(\mathcal{U};G)$ is defined by

$$(\delta g)_{i_0\cdots i_{q+1}} := \prod_{k=0}^{q+1} g_{i_0\cdots i_k\cdots i_{q+1}}^{-1} |_{U_{i_0\cdots i_{q+1}}}$$

the hat (as always) indicating omission. The Čech cohomology $H^q(\mathcal{U};G)$ of the open cover $\mathcal{U}$ is defined as the cohomology of this cochain complex. For a refinement $\mathcal{V} \hookrightarrow \mathcal{U}$, i.e., an open cover $V_j, j \in J$ with $V_j \subset U_{\tau(j)}$ for some map $\tau : J \rightarrow I$, we may define a cochain map $C^q(\mathcal{U};G) \rightarrow C^q(\mathcal{V};G), \quad g \mapsto \left( (j_0, \ldots, j_n) \mapsto g_{\tau(j_0)\cdots\tau(j_n)} |_{V_{j_0\cdots j_n}} \right)$ which thus induces a map $H^q(\mathcal{U};G) \rightarrow H^q(\mathcal{V};G)$. The Čech cohomology of $X$ is then defined as the limit

$$H^q(X;G) = \lim_{\mathcal{U}} H^q(\mathcal{U};G)$$

For a homomorphism $f : G \rightarrow H$ we obtain a cochain map $f_* : C^q(\mathcal{U};G) \rightarrow C^q(\mathcal{U};H)$ by post-composition. All in all, this yields a map

$$H^q(X;f) : H^q(X;G) \rightarrow H^q(X;H)$$

Čech cohomology is a covariant functor in the coefficients. For topological groups, $H^q(X;\_)$ is a homotopy invariant functor. A short exact sequence $0 \rightarrow G_1 \rightarrow
$G_2 \to G_3 \to 0$ of topological groups gives a long exact sequence in cohomology.

**Remark 4.4.** For $G$ non-abelian we may still define $H^0$ and $H^1$ as pointed sets. A short exact sequence still induces a "long" exact sequence, but of course one that breaks off as soon as the corresponding $H^0$ is not defined.

### 4.4. Spin Structures on Vector Bundles.

For an oriented $n$-dimensional Riemannian vector bundle $\pi : E \to B$ write as usual $P_{SO}(E)$ for the associated bundle of positively oriented orthonormal frames. We will use the notation $\xi_0 = Ad$ for the double covering of theorem 4.3.

**Definition 4.5.** A spin structure on an oriented Riemannian vector bundle $\pi : E \to B$ is defined to be a reduction of the structure group of the frame bundle $P_{SO}(E)$ by $\xi_0$.

This means that we may chose a principal Spin-bundle $P_{Spin}(E)$ and an isomorphism $P_{Spin}(E)[Spin_n \xrightarrow{\xi_0} SO(n)] \cong P_{SO}(E)$.

**Remark 4.6.** Write $g_{ij} : U_{ij} \to SO(n)$ for the transition functions of $P_{SO}(E)$. From a set of lifted $\tilde{g}_{ij} : U_{ij} \to Spin_n$ satisfying the cocycle relation we may construct a principal Spin-$n$-bundle $P_{Spin}(E)$ and an isomorphism $P_{SO}(n) \cong P_{Spin}(E)[Spin_n \xrightarrow{\xi_0} SO(n)]$ (by propositions 1.10, 1.11) which thus yields a Spin structure. Note however that we do not consider two Spin structures to be equivalent if the corresponding principal Spin-bundles are isomorphic (i.e. if the cocycles of the liftings in $H^1(X; Spin_n)$ are equal). It can be shown that two liftings $\tilde{g}_{ij}^{(1)} , \tilde{g}_{ij}^{(2)}$ define the same spin structure in case they differ by an isomorphism class of a two-fold cover $c_{ij} : U_{ij} \to \mathbb{Z}_2 = \{\pm 1\}$, meaning $c_{ij} \tilde{g}_{ij}^{(2)} = c_{ij} \tilde{g}_{ij}^{(1)}$. This is just an element of $H^1(X; \mathbb{Z}_2)$, which thus stands in bijection with the equivalence classes of Spin structures on $E$.

The exact sequence $0 \to \mathbb{Z}_2 \to Spin_n \xrightarrow{\xi_0} SO(n) \to 0$ yields a natural exact sequence in Čech cohomology

\[
\cdots \longrightarrow H^1(X; \mathbb{Z}_2) \longrightarrow H^1(X; Spin_n) \xrightarrow{(\xi_0)_*} H^1(X; SO(n)) \xrightarrow{\delta} H^2(X; \mathbb{Z}_2) \longrightarrow \cdots \tag{4.6}
\]

We recall here the definition of the first Čech cohomology of $X$ for non-abelian groups. The cocycles $g_{ij} : U_{ij} \to SO(n)$ that are closed (i.e. satisfy the usual cocycle relation) comprise elements of $H^1(X; SO(n))$ which are considered to be equivalent if they differ by exact elements $h_i : U_i \to SO(n)$, meaning that two cocycles $\{g_{ij}\}, \{g'_{ij}\}$ are equivalent iff $h_i \cdot g_{ij} \cdot h_{ij}^{-1} = g'_{ij}$. Let us describe the coboundary $\delta$ explicitly. Lift the $g_{ij}$ to $\tilde{g}_{ij} : U_{ij} \to Spin_n$. Then $(\delta \tilde{g})_{ijk} := \tilde{g}_{jk} \tilde{g}_{ik}^{-1} \tilde{g}_{ij} : U_{ijk} \to \{\pm 1\} = \mathbb{Z}_2 \subset Spin_n$ takes values in $\mathbb{Z}_2$. 

---

A GENERALIZED MIRACULOUS CANCELLATION FORMULA 43
Consider the trivial circle bundle over $S^1$, trivialized over the two neighborhoods $U_1, U_2$, and having transition function $g_{12}$ (the dashed blue section, which is then glued to the solid blue section of $U_2$ in the indicated fashion, thus forming a torus). Any lift of the transition functions to $(\gamma)^2 : S^1 \cong \text{Spin}_2 \to S^1$ yields principal Spin$_2$-bundles that are abstractly equivalent (any principal bundle over $S^1$ admits a global section). There are however two distinct spin structures on this bundle: on each connected component of $U_{12}$ there are two choices of lifting $g_{12}$, namely the red and blue dashed sections. The Spin structure given by dashed red on both $A$ and $B$ is equivalent to that given by dashed blue on both $A$ and $B$. The isomorphism of Spin$_2$ bundles is defined by simply requiring that the dashed blue (resp. red) section along with its obvious prolongation on $U_1$ is mapped onto the dashed red (resp. blue) section and its natural prolongation, while on $U_2$ we take the identical map. The isomorphism is compatible with the $(\gamma)^2$ reduction, which actually just means that its representing section takes values in $\{\pm 1\}$. It is however inequivalent to the Spin structure given by dashed red on $A$ and dashed blue on $B$.

This comes from the simple observation that any section connecting red and blue cannot take values only in $\{\pm 1\}$.

**Figure 4.1. Milnor’s Warning**

**Proposition 4.7.** $E$ carries a Spin-structure precisely when $\delta(P_{SO}(E)) = 0$.

**Proof.** From exactness it is clear that if $P_{SO}(E)$ comes from a Spin-bundle we have $\delta(P_{SO}(E)) = 0$. Suppose conversely that $\delta g \in H^2(\mathcal{U}; \mathbb{Z}_2)$ is exact, i.e. equals the coboundary of a two-fold cover $h_{ij} : U_{ij} \to \mathbb{Z}_2$

$$\delta h_{ij} = h_{jk}^{-1}h_{ik}$$

But then, the $h_{ij} \cdot \hat{g}_{ij}$ satisfy the cocycle relation and $\xi_0(h_{ij} \cdot \hat{g}_{ij}) = g_{ij}$, so we have found a spin structure.

**Theorem 4.8.** $\delta(P_{SO}(E)) = w_2(E)$ equals the second Stiefel-Whitney class. Therefore an oriented Riemannian vector bundle carries a spin structure precisely when its second Stiefel-Whitney class vanishes.

**Proof.** Clearly, from the naturality of the sequence (4.6), $\delta$ is natural. We will show the Whitney product axiom for $\delta$. Assume $\xi$ is the direct sum of two bundles $\xi = \xi_1 \oplus \xi_2$. Choosing transition functions $g^{(1)}_{ij} : U_{ij} \to SO(n), g^{(2)}_{ij} : U_{ij} \to SO(m)$
for \( \xi_1, \xi_2 \) over a common good cover, the transition functions of \( \xi \) are given by

\[
g_{ij} = \begin{pmatrix} g_{ij}^{(1)} & 0 \\ 0 & g_{ij}^{(2)} \end{pmatrix}
\]

Lifts of \( g_{ij}^{(1)}, g_{ij}^{(2)} \) to \( \text{Spin}_n \) and \( \text{Spin}_m \) respectively yield (considering \( \text{Spin}_n, \text{Spin}_m \) as subgroups of \( \text{Spin}_{n+m} \): coming from \( \mathbb{R}^n \subseteq \mathbb{R}^{n+m}, \mathbb{R}^m \subseteq \mathbb{R}^{n+m} \), the bottom arrow is given by multiplication)

\[
\begin{array}{ccc}
SO(n) \times SO(m) & \overset{\zeta}{\longrightarrow} & SO(n+m) \\
\downarrow{\xi_0} & & \downarrow{\xi_0} \\
\text{Spin}_n \times \text{Spin}_m & \overset{\xi_0}{\longrightarrow} & \text{Spin}_{n+m}
\end{array}
\]

a lift of \( g_{ij} \) to \( \text{Spin}_{n+m} \) by \( \tilde{g}_{ij} = g_{ij}^{(1)} \cdot g_{ij}^{(2)} \). It follows that

\[
\delta(\xi_1 \oplus \xi_2) = \delta(\xi_1) \cdot \delta(\xi_2)
\]

for pointwise multiplication of the 2-cocycles.

Next, let us show that \( \delta \) is non-trivial. Consider the complex line bundle \( \gamma^1 \) defined on \( \mathbb{C}P^1 \) by the requirement that it is trivial over each of \( U_i = \{ [z_0 : z_1] \in \mathbb{C}P^1 \mid z_i \neq 0 \} \) and has transition function

\[
g_{01} : U_0 \cap U_1 \to U(1) = S^1 = SO(2), \quad [z_0 : z_1] \mapsto \frac{z_0}{z_1}, \frac{|z_1|}{|z_0|}
\]

(this bundle equals the tautological complex line bundle). A complex line bundle may be considered as an oriented Riemannian two-plane bundle. We contend that it does not carry a spin structure, so that \( \delta(P_{SO(\gamma^1)}) \neq 0 \) by proposition 4.7. Note that under the homeomorphism

\[
\psi : S^2 \cong \mathbb{C}P^1, \quad (x, y, z) \mapsto \begin{cases} [x + iy : 1 - z] & (z \neq 1) \\ [1 : 0] & (z = 1) \end{cases}
\]

\[
\psi^{-1}[z_0 : z_1] = \left( \frac{2z_0 \bar{z}_1}{|z_0|^2 + |z_1|^2}, \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2} \right)
\]

the set \( U_0 \cap U_1 \) corresponds to \( S^2 \setminus \{(0, 0, \pm 1)\} \). It follows that any lift of \( g_{01} \) via \( \xi_0 = \epsilon \) would yield a lift of \( S^1 \to S^2 \setminus \{(0, 0, \pm 1)\} \) \( \tilde{\xi}_0 \mid_{U_0 \cap U_1} \to S^1 \) which is the identical map \( w \mapsto w \). But this is impossible by elementary algebraic topology (consider for instance the exact homotopy sequence of the covering \( \xi_0 = \epsilon \)).

Adding the trivial \( n \)-plane bundle \( \varepsilon^n = \mathbb{R}^n \rightarrow \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 \) (which clearly has a spin structure) to \( \gamma^1 \) we obtain examples of bundles of arbitrary rank \( \geq 2 \) with \( \delta(\gamma^1 \oplus \varepsilon^n) \neq 0 \) by applying (4.8).

By the classification of vector bundles (Theorem 1.29) and the naturality of \( \delta \) we must have in particular that \( \delta(\gamma^n) \neq 0 \) for the tautological real oriented \( n \)-plane bundle, \( n \geq 2 \). We have therefore checked that \( \delta \) satisfies all the axioms for the second Stiefel-Whitney class \( w_2 \) (see also remark 1.33). \( \square \)
Proposition 4.9. Let \( \xi = \xi_1 \oplus \xi_2 \) be the Whitney sum of two vector bundles. If two of the involved bundles carry a spin structure, then third carries a canonical spin structure, too.

Proof. The third bundle must be orientable, for in the case of vector spaces \( V = V_1 \oplus V_2 \) with chosen orientations on two of \( V, V_1, V_2 \) we have a canonical orientation on the third. Moreover, the third bundle has some spin structure: observe that since \( w_1(\xi_1) = w_1(\xi_2) = 0 \) the Whitney product axiom implies

\[
w_2(\xi) = w_2(\xi_1) + w_2(\xi_2)
\]

The canonical choice of spin structure on the third bundle is the following: let \( g_{ij}^{(1)}, g_{ij}^{(2)} \) be transition functions of the bundles \( \xi_1, \xi_2 \) defined over a common open cover. Transition functions of \( \xi \) are then given by \( g_{ij} : U_i \cap U_j \to SO(n + m), \ x \mapsto \begin{pmatrix} g_{ij}^{(1)} & 0 \\ 0 & g_{ij}^{(2)} \end{pmatrix} \). The canonical choice of spin structure on the third bundle is defined to be the one which satisfies

\[
\hat{g}_{ij}(x) = \hat{g}_{ij}^{(1)}(x) \cdot \hat{g}_{ij}^{(2)}(x) \in \text{Spin}_{n+m}, \ x \in U_i \cap U_j
\]

Uniqueness of choice is clear. But it is also always possible to choose such a spin structure on the third bundle: For instance, suppose \( \xi_1 \) and \( \xi_2 \) are equipped with a fixed spin structure. Then define \( \hat{g}_{ij}^{(1)}(x) := \hat{g}_{ij}(x) \left( \hat{g}_{ij}^{(2)}(x) \right)^{-1} \). Since \( \hat{g}_{ij}^{(2)}, \hat{g}_{ij} \) satisfy the cocycle relation, so does \( \hat{g}_{ij}^{(1)} \). Finally, from (4.7) this is the required lift of \( g_{ij}^{(1)} \)

\[
\xi_0(\hat{g}_{ij}^{(1)}) = \xi_0(\hat{g}_{ij}(x)) \xi_0(\hat{g}_{ij}^{(2)})^{-1} = g_{ij}(x) \cdot g_{ij}^{(2)}(x)^{-1} = g_{ij}^{(1)}(x)
\]

\(\Box\)

4.5. Spin\(^c\) Structures. Define the group \( \text{Spin}^c_n := \text{Spin}_n \times_{\mathbb{Z}_2} U(1) = \frac{\text{Spin}_n \times U(1)}{\{(1,-1),(-1,1)\}} \). As shown in theorem 4.3 we have a two-fold covering \( 0 \to \mathbb{Z}_2 \to \text{Spin}_n \xi \to SO(n) \to 0 \). This yields for \( \xi(\varphi, z) := (\xi_0(\varphi), z^2), \ \varphi \in \text{Spin}_n, z \in U(1) \) an exact sequence

\[
0 \to \mathbb{Z}_2 \to \text{Spin}_n^c \xrightarrow{\xi} SO(n) \times U(1) \to 0
\]

We have a commutative diagram with exact horizontal sequences

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}_2 & \to & \text{Spin}_n & \xrightarrow{\xi_0} & SO(n) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathbb{Z}_2 & \to & \text{Spin}_n^c & \xrightarrow{\xi} & SO(n) \times U(1) & \to & 0 \\
\downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow \\
0 & \to & \mathbb{Z}_2 & \to & U(1) & \xrightarrow{(j)^2} & U(1) & \to & 0 \\
\downarrow \text{can} & & \downarrow \text{can} & & \downarrow \text{can} & & \downarrow \text{can} & & \downarrow \\
0 & \to & \mathbb{Z} & \to & \mathbb{R} & \xrightarrow{e^{\pi i x}} & U(1) & \to & 0 \\
\end{array}
\]
Definition 4.10. Let $E \to X$ be an oriented Riemannian vector bundle of rank $n$. A $\text{Spin}^c$-structure on $E$ consists of a principal $\text{Spin}^c$-bundle $P_{\text{Spin}^c}$ as well as a principal $U(1)$-bundle $P_{U(1)}$ together with a bundle map

$$\xi : P_{\text{Spin}^c} \to P_{SO(n)}(E) \times P_{U(1)}$$

which is $\text{Spin}^c$-equivariant, i.e. $\xi(pg) = \xi(p)\xi(g)$ for $p \in P_{\text{Spin}^c}$, $g \in \text{Spin}^c$.

It is a simple fact from Čech cohomology that for $G_1$ and $G_2$ groups with $G_1$ abelian we have an isomorphism $H^1(X; G_1 \times G_2) \cong H^1(X; G_1) \oplus H^1(X; G_2)$. This may easily be proved by considering the split exact sequence $0 \to G_1 \to G_1 \times G_2 \to G_2 \to 0$.

The large diagram above gives a corresponding diagram with exact sequences in Čech cohomology

$$
\begin{array}{cccccccc}
\cdots & H^1(X; \text{Spin}_n) & \longrightarrow & H^1(X; SO_n) & \longrightarrow & H^2(X; \mathbb{Z}_2) & \\
& \text{incl} & & \text{incl} & \text{incl} & & \\
\cdots & H^1(X; \text{Spin}_n) & \longrightarrow & H^1(X; SO_n) \oplus H^1(X; U(1)) & \longrightarrow & H^2(X; \mathbb{Z}_2) & \\
& \text{incl} & & & \text{incl} & & \\
\cdots & H^1(X; U(1)) & \longrightarrow & H^1(X; U(1)) & \longrightarrow & H^2(X; \mathbb{Z}_2) & \\
& & \text{incl} & & \text{incl} & & \\
\cdots & H^1(X; \mathbb{R}) & \longrightarrow & H^1(X; U(1)) & \longrightarrow & H^2(X; \mathbb{Z}_2) & \\
& & & & \text{incl} & & \\
\end{array}
$$

Since $\mathbb{R}$ is contractible we have $H^1(X; \mathbb{R}) = 0 = H^2(X; \mathbb{R})$, and $c_1$ is an isomorphism. From the diagram it follows that $\delta : H^1(X; SO_n) \oplus H^1(X; U(1)) \to H^2(X; \mathbb{Z}_2)$ equals $w_2 + c_1$ for the mod 2 reduction of the “Chern class” $c_1$.

Remark 4.11. In this context, the Chern class of a (Hermitian) complex line bundle may be defined as $c_1(E) := c_1(P_U(E))$. Clearly, $c_1$ so constructed is natural. To prove that $c_1$ satisfies all the axioms for the first Chern class, it has to be shown that $c_1 \left( (g_2)_1 \right)$ equals $-g \in H^2(CP^1; \mathbb{Z})$. The proof is somewhat in the spirit of theorem 4.8. Details may be found in [Hir66], theorem 4.3.1.

Theorem 4.12. A bundle $E$ as above may be given a $\text{Spin}^c$-structure precisely when $w_2(E)$ is the mod 2 reduction $\bar{a}$ of some integer cohomology class $a \in H^2(X; \mathbb{Z})$.

Proof. Since $c_1 : H^1(X; U(1)) \cong H^2(X; \mathbb{Z})$ any such class defines a principal $U(1)$-bundle $\lambda$ with Chern class $a$ (see also 1.10). Then $\delta(P_{SO}(E) \oplus \lambda) = w_2(E) + \bar{a} = 0$ which by the diagram implies that the transition functions of $P_{SO}(E) \oplus \lambda$ come from transition functions with values in $\text{Spin}^c_n$. Thus the bundle will carry a $\text{Spin}^c$ structure under the stated condition. For the converse, choose the integer cohomology class $a := c_1(P_{U(1)})$. The diagram now implies $0 = w_2(E) + \bar{a}$.  \qed
Example 4.13. The above theorem shows that any bundle with spin structure also carries a Spin$^c$-structure. Since for any complex vector bundle $w_2(E) \equiv c_1(E) \mod 2$ any complex vector bundle is Spin$^c$. We will call a manifold Spin$^c$ in case the tangent bundle has a Spin$^c$-structure. In particular, any almost complex manifold is Spin$^c$.

5. The Thom Class

5.1. The Thom Isomorphism Theorem. In this section, $\mathbb{Z}$-coefficients are to be understood. Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and, more generally, let $E_0 = E - 0_x$ denote the total space of a vector bundle $E \to X$ minus the zero section. Define a canonical generator $e^1$ of $H^1(\mathbb{R}, \mathbb{R}_0)$ corresponding to the natural orientation of $\mathbb{R}$ as image of $1 \in H^0([0, \infty])$ under the excision and coboundary isomorphisms\footnote{By considering the exact sequence of the triple $(\mathbb{R}, \mathbb{R}_0, -\infty, 0)$ we see that $\delta$ is an isomorphism since $\{-1\}$ is a deformation retract of both $\mathbb{R}$ and $-\infty, 0]$. Thus $H^*(\mathbb{R}, -\infty, 0) \cong H^*([\mathbb{R}, -\infty, 0]) = 0.$}

$$1 \in H^0([0, \infty]) \xrightarrow{\sim} H^0(\mathbb{R}_0, -\infty, 0] \xrightarrow{\delta} H^1(\mathbb{R}, \mathbb{R}_0) \ni e^1$$

There is a unique dual element $e_1 \in H_1(\mathbb{R}, \mathbb{R}_0)$ with

$$(e^1, e_1) = 1$$

Write $e^n = e^1 \times \cdots \times e^1 \in H^n(\mathbb{R}, \mathbb{R}_0)$ and similarly $e_n = e_1 \times \cdots \times e_1$.

Lemma 5.1. For any pair $(X, A)$ with $A$ open in $X$ the map

$$- \times e^n : H^*(X, A) \xrightarrow{\cong} H^{*-n}((X, A) \times (\mathbb{R}^n, \mathbb{R}_0^n))$$

is an isomorphism. In particular, $e^n$ is a generator of $H^n(\mathbb{R}^n, \mathbb{R}_0^n)$.

Proof. Clearly it suffices to consider $n = 1$. We will also restrict ourselves to the case when $A = \emptyset$, the general case then following by naturality of the cohomological cross product $\times$ and a five lemma argument. We will rewrite the map $- \times e^1$ in terms of maps we already know are isomorphisms. From the naturality of $\times$ and its compatibility with the coboundary, for any $a \in H^1(X)$ the following diagram commutes up to sign

$$\begin{align*}
1 \in H^0([0, \infty]) & \xrightarrow{\sim} H^0(\mathbb{R}_0, -\infty, 0] \xrightarrow{\delta} H^1(\mathbb{R}, \mathbb{R}_0) \ni e^1 \\
\begin{array}{ccc}
H^m(X \times [0, \infty]) & \cong & H^m(X \times (\mathbb{R}_0, -\infty, 0]) \xrightarrow{\delta} H^{m+1}(X \times (\mathbb{R}, \mathbb{R}_0)) \\
\cong & \quad \cong \\
\cong & \quad \\
\end{array}
\end{align*}$$

In particular, $a \times e^1$ equals $(-1)^{d_1} a \times 1$. Also $\text{pr}^* : H^m(X) \cong H^m(X \times [0, \infty])$ is just the correspondence $a \mapsto a \times 1$. Therefore $a \times e^1 = (-1)^d_1 \text{pr}_{*}^*(a)$, and $- \times e^1$ equals the isomorphism $(-1)^{d_1} \text{pr}_{*}^*$. \qed
For an oriented vector bundle $E \to X$ of fiber dimension $n$, every restriction to fibers $E_x$, $x \in X$ has a given orientation. This may be considered as the choice of a preferred generator of

$$H^n(E_x, (E_x)_0) \cong H^n(\mathbb{R}^n, \mathbb{R}_0^n) \cong \mathbb{Z}$$

**Theorem 5.2.** (Thom Isomorphism Theorem) Let $p : E \to X$ be an oriented vector bundle of rank $n$ over a compact base $X$. There exists a unique cohomology class $u \in H^n(E, E_0)$ that restricts fiberwise to each of the preferred generators of $H^n(E_x, (E_x)_0)$. This class is called the Thom class of the bundle. Moreover, we obtain an isomorphism $\phi$

$$H^*(X) \overset{p^*}{\cong} H^*(E) \overset{-\cup u}{\cong} H^{*+n}(E, E_0)$$

**Proof.** The proof is by induction on the number $k$ of open subsets needed to cover $X$ over which the bundle is trivial (as an oriented bundle). Only the cases $k = 1$ and $k = 2$ need special attention. If $k = 1$ and $E = X \times \mathbb{R}^n$ the Thom class must be $u = 1 \times e^n$, for it is the only class that restricts fiberwise to $1(x) \times e^n$. The composition

$$H^*(X) \overset{p^*}{\to} H^*(X \times \mathbb{R}^n) \overset{-\cup u}{\to} H^{*+n}(X \times (\mathbb{R}^n, \mathbb{R}_0^n))$$

is equal to

$$a \mapsto p^*(a) \cup u = (a \times 1) \cup (1 \times e^n) = a \times e^n$$

which is known to be an isomorphism by lemma 5.1. This completes the proof in case $k = 1$. If $k = 2$, assume $X = X_1 \cup X_2$ for trivializing open subsets $X_1, X_2$. Set $X_{12} := X_1 \cap X_2$ and $E_1 = p^{-1}(X_1)$, $E_2 = p^{-1}(X_2)$, $E_{12} = E_1 \cap E_2 = p^{-1}(B_1 \cap B_2)$. Since $E_1, E_2$ are open the triple $(E, E_1, E_2)$ is excisive, and we have an exact Mayer-Vietoris sequence

$$H^0(E_{12}, (E_{12})_0) \overset{\delta}{\to} H^1(E, E_0) \to H^1(E_1, (E_1)_0) \oplus H^1(E_2, (E_2)_0) \to H^1(E_{12}, (E_{12})_0)$$

By the case $k = 1$, we already have unique elements of $u_1 \in H^1(E_1, (E_1)_0)$ and $u_2 \in H^1(E_2, (E_2)_0)$ that restrict fiberwise to the given generators. The same is true for $H^1(E_{12}, (E_{12})_0)$ which means in particular that the restriction of these two elements must coincide there, and are thus mapped onto zero by the map $\text{incl}_{E_{12} \subset E_1} - \text{incl}_{E_{12} \subset E_2}$ of the Mayer-Vietoris sequence. Also, from the case $k = 1$ we have $H^0(E_{12}, (E_{12})_0) = 0$. Therefore there exists a unique Thom class in $H^1(E, E_0)$. Consider the two Mayer-Vietoris sequences

$$\delta$$

The five lemma together with the $k = 1$ isomorphisms complete the proof in case $k = 2$. For $k > 2$ fix $X_k$ and consider the cover of $X$ given by $X_1 \cup \ldots \cup X_{k-1}$ and $X_k$. The inductive hypothesis may be applied to $X_1 \cup \ldots \cup X_{k-1}$, and the remainder of the proof is entirely analogous to the $k = 2$ case. □
5.2. The Thom Isomorphism and Poincaré Duality. Over compact oriented manifolds $X$ there is another way to describe the Thom isomorphism and the Thom class using Poincaré duality. We shall need this description in what follows. Choose some Riemannian metric on $E$. We may then form the associated disk bundle $D(E)$ (which is of course a manifold with boundary) which inherits an orientation from $E$. The sphere bundle $S(E) = \partial D(E)$ is given the boundary orientation. From the commutative diagram

$$
\begin{array}{ccc}
H^*(D(E), S(E)) & \xrightarrow{-\cap[D(E), \partial D(E)]} & H_{n+\dim(X)-*}(D(E)) \\
\cong & & \cong \\
H^*(E, E_0) & \xrightarrow{-\cap[E, E_0]} & H_{n+\dim(X)-*}(E)
\end{array}
$$

and Poincaré duality for manifolds with boundary we obtain an isomorphism

$$-\cap [E, E_0] : H^*(E, E_0) \cong H_{n+\dim(X)-*}(E) \quad (5.2)
$$

We will also write $D$ for the Poincaré isomorphism.

**Proposition 5.3.** Let $p : E \to X$ be an oriented rank $n$ vector bundle over a compact oriented manifold $X$ of dimension $m$. Denote by $s : X \to E$ the zero section. Then the Thom class $u$ is Poincaré dual to $(-1)^mn_*[X]$ under (5.2). Moreover, for the Thom isomorphism $\phi$ we have

$$\phi(a) = (-1)^mn(D^1_{[E, E_0]}s_*D)[X](a), \quad a \in H^*(X)
$$

**Proof.** Assume $X$ is connected, and let $x \in U$ be an open subset of $X$ over which the bundle is trivial. We may then identify $E|_U = U \times \mathbb{R}^n$. The diagram

$$
\begin{array}{ccc}
H^n(U \times (\mathbb{R}^n, \mathbb{R}^n_0)) \otimes H_{n+m}((U, U-x) \times (\mathbb{R}^n, \mathbb{R}^n_0)) \xrightarrow{(-\times e^n) \otimes \text{id}} H^0(U) \otimes H_{n+m}((U, U-x) \times (\mathbb{R}^n, \mathbb{R}^n_0)) \\
\cong & & \\
H_m((U, U-x) \times \mathbb{R}^n) \xrightarrow{\text{id} \otimes (-\times e^n)} H^0(U) \otimes H_m(U, U-x) \\
\xrightarrow{\text{pr}_*} & & \\
H_m(U, U-x)
\end{array}
$$

commutes up to sign. From the choice of $e^n$ we have $[(U, U-x) \times e^n = [(U, U-x) \times (\mathbb{R}^n, \mathbb{R}^n_0)]$. The Thom class $u|_U$ of $E|_U$ is by definition (5.1) the image of

\[\text{applied to } H_d(U, U-x) \cong \begin{cases} 0 & (s \neq m) \\ \mathbb{Z} & (s = m) \end{cases}\] the Kronecker product $\langle , \rangle$ is non-degenerate. Then, from $\langle a, \text{pr}_* b \times (e^n \cap e_n) \rangle = \langle \text{pr}_* a, b \times (e^n \cap e_n) \rangle = \langle a \times 1, b \times (e^n \cap e_n) \rangle = \langle a, b \rangle \langle e^n, e_n \rangle = \langle a, b \rangle$ we may conclude $b = \text{pr}_* (b \times (e^n \cap e_n))$. 

\[\text{For } u \in H^n(U), \quad v \in H_m(U, U-x) \text{ we have } (u \times e^n) \cap (v \times e_n) = (-1)^{mn}(u \cap v) \times (e^n \cap e_n) \text{ which we contend is equal to } (-1)^{mn} \text{pr}_*^{-1}(u \cap v). \] For the proof observe that by universal coefficients we need only show that the diagram above commutes when $u = 0$. This is clear. The Kronecker product $\langle , \rangle$ is non-degenerate.
$1_U \in H^0(U)$ in $H^n(U \times (\mathbb{R}^n, \mathbb{R}^n_0))$ under the isomorphism $- \times c^n$. Following the diagram around\textsuperscript{12} for $1_U \otimes [U, U - x] \in H^0(U) \otimes H_m(U, U - x)$ we obtain

$(-1)^{mn} \text{pr}_+(u|U \cap [(U, U - x) \times (\mathbb{R}^n, \mathbb{R}^n_0)]) = [U, U - x]$

Write $E - 0_x := E - \{0_x\} \subset E - 0_X = E_0$. From the naturality of the cap product we have a commutative diagram (use the excision isomorphisms to simplify the diagram

$H^n(E, E_0) \otimes H_{n+m}(E, E - 0_x) \to H^n(E|U, (E|U)_0) \otimes H_{n+m}(E|U, E - 0_x) \cong H^n(E|U, (E|U)_0) \otimes H_{n+m}(E|U, E|U - 0_x)$

The right part of this diagram is just the left part of the diagram before. Since the bottom arrows are isomorphisms, $[X]$ is the only preimage of $[U, U - x]$. Also, $u \otimes [E, E_0]$ is mapped onto $u \mid_U \otimes [E|U, E|U - 0_x]$ by the upper arrows, so we must have

$(-1)^{mn} p_+(u \cap [E, E_0]) = [X]$

Since $s_*p_* = \text{id}$ this completes the proof in the connected case. If $X$ is not connected, apply the above to each connected component.

The stated equation for the Thom isomorphism now follows by a direct verification

$\phi(a) = p^*(a) \cup u = (-1)^{mn} p^*(a) \cup D_{[E, E_0]}^{-1} s_*[X]$

$= (-1)^{mn} D_{[E, E_0]}^{-1} (p^*(a) \cap s_*[X]) = (-1)^{mn} D_{[E, E_0]}^{-1} s_* (s^* p^*(a) \cap [X])$

$= (-1)^{mn} D_{[E, E_0]}^{-1} s_* D_{[X]}(a)$

\textsuperscript{12}We have $1_U \cap [U, U - x] = [U, U - x]$ again by the non-degeneracy of $\langle, \rangle$: $\langle a, 1 \cap [U, U - x] \rangle = \langle a \cup 1, [U, U - x] \rangle = \langle a, [U, U - x] \rangle$
5.3. **The Gysin Sequence.** Plugging the Thom isomorphism into the long exact sequence of the pair \((E, E_0)\), we obtain the so called Gysin sequence

\[
\cdots \xrightarrow{\delta} H^*(E, E_0) \xrightarrow{j^*} H^*(E) \xrightarrow{\pi^*} H^*(X) \xrightarrow{-\cup e} H^*(X) \xrightarrow{-\cup u} H^*(X) \xrightarrow{\delta} \cdots
\]

where we have set \(e := (\pi^*)^{-1} j^*(u) = s^* j^*(u) \in H^*(X)\) “Euler class”, \(j : E \subset (E, E_0)\). Thus we have an exact sequence

\[
\cdots \xrightarrow{\delta} H^{*-n}(X) \xrightarrow{-\cup e} H^*(X) \xrightarrow{\pi^*} H^*(X) \xrightarrow{-\cup u} H^*(X) \xrightarrow{\delta} \cdots
\]

This sequence may be used to describe the structure of the cohomology of \(\mathbb{C}P^\infty\) as a ring:

\[
H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c]
\]

for the Euler class \(c \in H^2(\mathbb{C}P^\infty)\) of the tautological complex line bundle. Note here that any complex vector bundle has a canonical orientation. All the above applies also in the case that the vector bundle is not orientable, but then only for \(\mathbb{Z}_2\)-coefficients. In that vein

\[
H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}_2[w]
\]

where now \(w \in H^1(\mathbb{R}P^\infty)\) is the Euler class of the tautological real line bundle.

5.4. **The Thom Space.** Let \(E \to X\) be a Riemannian vector bundle. Then we may form the associated disk and sphere bundle

\[
D(E) = \{ e \in E \mid \|e\| \leq 1 \} \\
S(E) = \{ e \in E \mid \|e\| = 1 \}
\]

The Thom space of \(E\) is defined (up to homotopy type) by \(Th(E) = D(E)/S(E)\) or also by \(E/E_0\). Over a compact base \(Th(E)\) may equivalently be viewed as the one-point compactification of the total space \(E\). Since \(E_0 \subset E\) is a neighborhood deformation retract, the canonical map \(can : (E, E_0) \to (E/E_0, *)\) induces an isomorphism in (co)homology \(H_n(E, E_0) \cong H_n(Th(E), *) \cong H_n(Th(E))\), \(n \geq 1\). For an oriented vector bundle \(E\) the Thom space can be given a natural orientation in the case that the Thom space is a manifold (which usually can’t be expected). The fundamental class \([Th(E)]\) is then defined as \(can_*[E, E_0]\).
5.5. The Fundamental Cohomology Class of a Submanifold.

Definition 5.4. Let \( j : V \subset X \) be a codimension \( k \) closed oriented submanifold of a manifold \( X \) that is also assumed to be compact and oriented. The fundamental cohomology class \( v \in H^k(X; \mathbb{Z}) \) of \( V \) in \( X \) is defined by the requirement

\[
v \cap [X] = j_\ast [V] \quad \text{(5.3)}
\]
i.e. by \( v := D_{\{X\}, j_\ast [V]} \in H^k(X; \mathbb{Z}) \).

As an example, recall that the first Chern class is normalized by \( c_1(\eta_n) = h_n \) for the dual \( \eta_n = \text{Hom}(\gamma_n^1, \mathbb{C}) \) of the tautological complex line bundle \( \gamma_n^1 \) and \( h_n \) the fundamental cohomology class of \( \mathbb{C}P^{n-1} \) in \( \mathbb{C}P^n \).

Lemma 5.5. The cup-cap relation gives the following identity for \( a \in H^\ast(X) \)

\[
\langle (a \cup v), [X] \rangle = \langle a, v \cap [X] \rangle = \langle a, j_\ast [V] \rangle = \langle j^\ast a, [V] \rangle \quad \text{(5.4)}
\]

Proposition 5.6. Let \( j : V \subset X \) be submanifold of codimension \( k \) in the \( n \)-dimensional manifold \( X \) (as above) with fundamental cohomology class \( v \) and normal bundle \( \nu \) in \( X \). Write \( u \) for the Thom class of \( \nu \). Then \( u \) is mapped onto \( (-1)^{kn}v \) by the map

\[
H^k(\nu, \nu_0) \xrightarrow{(r^\ast)^{-1}} H^k(X, X - V) \xrightarrow{k^\ast} H^k(X)
\]

Proof. Note here that \( r^\ast \) is an isomorphism by excision. From the tubular neighborhood theorem we may assume \( V \subset \nu \subset X \), so \( s : V \subset \nu \) is the zero section. By proposition 5.3 the Thom class may be written \( u = (-1)^{kn}D_{\nu, \nu_0}^{-1} \ast [V] \) while the fundamental cohomology class is defined as \( v = D_{\{X\}}^{-1}j_\ast [V] \). The result follows immediately from the following commutative diagram (\( u := \dim V \))

\[
\begin{array}{ccccccccc}
(-1)^{kn}u & \in & H^k(\nu, \nu_0) & \xrightarrow{r^\ast} & H^k(X, X - V) & \xrightarrow{k^\ast} & H^k(X) \\
\downarrow D_{\nu, \nu_0} & & \downarrow D_{(X, X - V)} & & \downarrow D_{\nu_0} \\
H_n(\nu) & \xrightarrow{s^\ast} & H_n(X) & \xrightarrow{j^\ast} & H_n(X) \\
\downarrow & & \downarrow & & \downarrow \\
[V] & \in & H_n(\nu) & & H_n(\nu) & & H_n(\nu)
\end{array}
\]

\[\square\]
5.6. The Chern class of a Codimension 2 Submanifold. There is a homeomorphism

\[ h : \mathbb{C}P^n - \{0 : \ldots : 0 : 1\} \to \eta_{n-1}, \ (z_0 : \ldots : z_n) \mapsto \varepsilon \]

defined by \( \varepsilon(z_0, \ldots, z_{n-1}) := z_n \). The inverse of \( h \) maps \( \varepsilon : (z_0 : \ldots : z_{n-1}) \to \mathbb{C} \) to the element \((z_0 : \ldots : z_{n-1} : \varepsilon(z_0, \ldots, z_{n-1}))\). Since this map is complex analytic it preserves the canonical orientations of the underlying manifolds. As a homeomorphism it is also a proper map which may thus be extended to the one-point compactifications by letting \((0 : \ldots : 0 : 1) \mapsto \infty \)

\[
\begin{array}{c}
\mathbb{C}P^n \\
\mathbb{C}P^n - \{0 : \ldots : 1\} \\
\eta_{n-1} \\
\eta_{n-1} \\
\mathbb{C}P^{n-1}
\end{array}
\]

Here, we have written \( s_0 : \mathbb{C}P^{n-1} \to \eta_{n-1} \) for the zero section.

**Proposition 5.7.** Let \( E \to X \) be a complex line bundle with zero section \( s : X \to (E, E_0) \). Then

\[ c_1(E) = s^* s(1) \]

using the notation \( s_1 = D_{[E, E_0]}^* s, D \) for the map dual to \( s \) in the sense of Poincaré.

**Proof.** Consider first the bundle \( \eta_{n-1} \) with zero section \( s_0 \). From the above remarks and definitions we have the commutative diagram

\[ \begin{array}{c}
H^0(\mathbb{C}P^{n-1})^{(s_0)} \xrightarrow{\cong} D_{[\mathbb{C}P^{n-1}]} H^2(\eta_{n-1}, (\eta_{n-1})_0) \xrightarrow{\text{can}^*} H^2(Th(\eta_{n-1})) \xrightarrow{h^*} H^2(\mathbb{C}P^n) \\
\cong D_{[\eta_{n-1}]} H^2(\eta_{n-1}, (\eta_{n-1})_0) \xrightarrow{\cong} D_{[\mathbb{C}P^{n-1}]} H^2(\eta_{n-1}, (\eta_{n-1})_0) \xrightarrow{\cong} D_{[\eta_{n-1}]} \end{array} \]

In particular, we have

\[ h^* (\text{can}^*)^{-1} (s_0)_!(1) = D_{[\mathbb{C}P^n]} j_* D_{[\mathbb{C}P^{n-1}]}(1) = D_{[\mathbb{C}P^n]} j_* [\mathbb{C}P^{n-1}] = h_n = c_1(\eta_n) \tag{5.5} \]

For an arbitrary line bundle \( E \to X \) choose a classifying map \( f : X \to \mathbb{C}P^n \) together with a bundle isomorphism \( F : E \cong f^* \eta_n \). Using the naturality of the Chern class and the fact that \( can F = h f \) we conclude from (5.5) that

\[ c_1(E) = f^* c_1(\eta_n) = f^* h^* (\text{can}^*)^{-1} (s_0)_!(1) = s^* F^* (s_0)_!(1) \]

for the Thom isomorphism \( s \). From the Thom isomorphism theorem we have that \( s_1 \) is natural, meaning that \( F^* (s_0)_! = s_1 F^* \). But since 1 is natural, this completes the proof. \( \square \)
Remark 5.8. This last proposition gives yet another possible definition for the first Chern class of a complex line bundle. If we had used this definition, the last proposition would have been the verification of the normalization axiom.

**Corollary 5.9.** Let $j : V \hookrightarrow X$ be a codimension two closed oriented submanifold of $X$ which is also assumed to be a compact and oriented manifold $X$. Denote by $v$ the fundamental cohomology class of $V$ in $X$. Then, for the normal bundle $\nu$ of $V$ in $X$,

$$c_1(\nu) = j^*v$$

**Proof.** Considering $\nu$ as subset of $X$ be the tubular neighborhood theorem, the zero section $s : V \subset (\nu, \nu_0)$ is just an inclusion. In the notation of proposition 5.6, namely $r : (\nu, \nu_0) \subset (X, X - V), j : V \subset X, k : X \to (X, X - V)$, we have $kj = rs$. It follows now from proposition 5.3, 5.6 and 5.7 that

$$j^*u = j^*k^*(r^*)^{-1}s(1) = s^*s(1) = c_1(\nu) \quad \square$$
6. Multiplicative Sequences and the Virtual Index

6.1. Multiplicative Sequences. We briefly recall here the notion of a multiplicative sequence. It is due to [Hir66] which may be consulted for further details. For a power series \( f(X) = \sum a_i X^i \), the \( i \)-th degree component will be denoted by

\[
\{f(X)\}^{(i)} := a_i X^i
\]

and similarly for any other graded algebra.

**Definition 6.1.** Let \( Q \in \mathbb{Q}[X] \) be a formal power series with constant term 1. Then

\[
Q(X_1) \cdots Q(X_n)
\]

is a power series which is symmetric in the \( X_i \). For each \( l \leq n \), taking the degree \( l \) component leads to a symmetric polynomial in the \( X_i \), which may then (by theorem 1.3) be expressed in terms of the elementary symmetric polynomials \( \sigma_i \), \( i \leq l \).

\[
\{Q(X_1) \cdots Q(X_n)\}^{(l)} = K_l^{(n)}(\sigma_1, \ldots, \sigma_l) \quad \text{(6.1)}
\]

**Remark 6.2.** The \( \sigma_i^{(n)}(X_1, \ldots, X_n) = \sigma_i(X_1, \ldots, X_n) \) actually depend on the number of variables \( n \), but from (1.2) we have \( \sigma_i^{(n+t)}(X_1, \ldots, X_n, 0, \ldots, 0) = \sigma_i^{(n)}(X_1, \ldots, X_n) \).

It follows then that

\[
K_l^{(n+t)}(\sigma_1^{(n)}, \ldots, \sigma_l^{(n)}) = K_l^{(n+t)}(\sigma_1^{(n+t)}, \ldots, \sigma_l^{(n+t)})(X_1, \ldots, X_n, 0, \ldots, 0)
\]

\[
= \{Q(X_1) \cdots Q(X_n)Q(0) \cdots Q(0)\}^{(l)} = \{Q(X_1) \cdots Q(X_n)\}^{(l)}
\]

and therefore (the \( \sigma_i^{(n)} \) are algebraically independent by theorem 1.3) that \( K_l^{(n+t)} = K_l^{(n)} =: K_l \). Thus, the \( n \)-fold product \( Q(X_1) \cdots Q(X_n) \) will suffice to compute \( K_{1}, \ldots, K_n \).

**Proposition 6.3.** To every power series with constant term 1 there is associated a sequence \( \{K_l\} \) of weighted homogeneous degree \( l \) polynomials called the multiplicative sequence associated to the power series \( Q \).

**Proof.** Note first that for an arbitrary formal power series \( f(X) = \sum a_i X^i \) we have the rule

\[
\{f(tX)\}^{(i)} = a_i t^i X^i = t^i \{f(X)\}^{(i)} \quad \text{(6.2)}
\]

We now compute, using the homogeneity of the elementary symmetric polynomials

\[
K_l(t\sigma_1, \ldots, t^i \sigma_l)(X_1, \ldots, X_n) = K_l(\sigma_1, \ldots, \sigma_l)(tX_1, \ldots, tX_n)
\]

\[
= \{Q(tX_1) \cdots Q(tX_n)\}^{(l)} = t^l \{Q(X_1) \cdots Q(X_n)\}^{(l)}
\]

This completes the proof (the multiplicative property is dealt with in proposition 6.5). \( \square \)
Definition 6.4. Let $B^* = \bigoplus_{i=0}^{\infty} B^i$ be a graded commutative $\mathbb{Q}$-algebra, and denote by $B^\wedge$ the subring of $\prod_{i=0}^{\infty} B^i$ (endowed with the convolution product) of sequences

$$1 + b_1 + b_2 + \cdots$$

with $b_i \in B^i$ and constant term 1. Observe that just as with formal power series, $B^\wedge$ admits inverses. Usually, $B = H^2\wedge(X; \mathbb{Q})$ or $H^{2\ast}(X; \mathbb{Q})$. Let $Q$ be a power series with associated multiplicative sequence $\{K_i\}$.

Proposition 6.5. Let $c$ denote by $B$ a homomorphism.

From the above remarks it is clear that the first $l$ summands will coincide with those of

$$Q(b_1) \cdots Q(b_l)$$

In particular, using $B^\wedge = \mathbb{Q}[X]$ we obtain

$$Q(X) = KQ(1 + X)$$

Proposition 6.5. Let $a \cdot b = c$ in $B^\wedge$. Then $KQ(a \cdot b) = KQ(c)$, i.e. $KQ$ is a group homomorphism.

Proof. Recall that the elementary symmetric polynomials are defined by the relation

$$\prod_{i=1}^{n} (t - X^i) = \sum_{i=0}^{n} (-1)^i \sigma_i(X_1, \ldots, X_i)t^i$$

From $\prod_{i=1}^{n} (t - X^i) \prod_{j=1}^{m} (t - Y^j) = (\sum (-1)^i \sigma_i(X) t^i)(\sum (-1)^j \sigma_j(Y) t^j)$ it follows that for two sets of independent variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ we have

$$\sigma_i(X, Y) = \sum_{p+q=i} \sigma_p(X) \sigma_q(Y)$$

Using the notation $\sigma_i^X = \sigma_i(X)$, $\sigma_i^Y = \sigma_i(Y)$ and the observation that $Q(X_1) \cdots Q(X_n)$ may be used to compute any $K_p$ for $p \leq n$, we conclude

$$\sum_{p+q=n} K_p(\sigma_1^X, \ldots, \sigma_p^X)K_q(\sigma_1^Y, \ldots, \sigma_q^Y)$$

$$= \sum_{p+q=n} \{Q(X_1) \cdots Q(X_n)\}^{(p)} \{Q(Y_1) \cdots Q(Y_m)\}^{(q)}$$

$$= \{Q(X_1) \cdots Q(X_n)Q(Y_1) \cdots Q(Y_m)\}^{(n)} = K_n(\sigma_1^X + \sigma_1^Y, \sigma_2^X + \sigma_2^Y, \ldots)$$

Replacing $\sigma_i^X \to a_i$ and $\sigma_i^Y \to b_i$ in accordance with the definition in (6.3) it follows for $c = \sum_{p+q} a Db_q$ that

$$KQ(a)KQ(b) = KQ(c)$$

---

13 Given $1 + b_1 + b_2 + \cdots$ the ansatz $(1 + a_1 + a_2 + \cdots) \cdot (1 + b_1 + b_2 + \cdots) = 1$ yields the system of equations $0 = a_1 + b_1$, $0 = a_2 + a_1 b_1 + b_2$, $0 = a_3 + a_2 b_1 + a_1 b_2 + b_3$, $\ldots$ which may clearly be solved inductively for the $a_i$. 

A GENERALIZED MIRACULOUS CANCELLATION FORMULA 57
6.2. Genera and Characteristic Classes. We will distinguish between the real and complex case.

**Definition 6.6. (real case)** Let $Q \in \mathbb{Q}[X]$ be a formal power series with associated group homomorphism $K_Q$ for $B = H^{4*}(X; \mathbb{Q})$ by proposition 6.5. The total $Q$-class of a real vector bundle $E \to X$ is then defined as

$$Q(E) := K_Q(p(E)) \in H^{4*}(X; \mathbb{Q})$$

for the total (rational) Pontrjagin class $p(E)$ of $E \to X$. Over a closed oriented manifold $X$ as base space (with fundamental homology class $[X]$), we may define the $Q$-genus of the vector bundle as the number

$$\langle Q(E), [X] \rangle = \int_X Q(E) \in \mathbb{R}$$

Here $(\cdot, \cdot)$ denotes the Kronecker product.

**Definition 6.7. (complex case)** For a complex vector bundle $E \to X$ define similarly the total $Q$-class by $Q^C(E) = K_Q(c(E))$ for the total (rational) Chern class $c(E)$. The $Q$-genus is again defined by $\langle Q^C(E), [X] \rangle$ if the base space $X$ is a closed oriented manifold.

**Remark 6.8.** For a real vector bundle of even rank $E \to X$ we have by (3.5) that $p(E) = \sigma_k(x_1^2, \ldots, x_n^2)$ for the new formal Chern roots $\pm x_k$, $k = 1, \ldots, n$ of the complexification $E \otimes \mathbb{C}$. It follows that

$$Q^R(E) = K_Q^R(p(E)) = 1 + K_1(\sigma_1(x_1^2)) + K_2(\sigma_2(x_1^2, x_2^2)) + \cdots$$

$$= 1 + \{Q^R(x_1^2)\}_1^1 + \{Q^R(x_1^2)Q^R(x_2^2)\}_2^2 + \cdots \tag{6.1}$$

We will define “the same” $Q$-classes for both real and complex bundles (say via power series $Q^R$, $Q^C$). Since we may view a complex bundle as a real vector bundle of even rank, in order to remain consistent we need

$$Q^C(X) = Q^R(X^2) \tag{6.4}$$

Using this convention, we may write $Q(E)$ for both $Q^R(E)$ and $Q^C(E)$. For a real vector bundle of even rank (having new formal Chern roots $\pm x_k$, $k = 1, \ldots, n$) we thus have

$$Q(E) = 1 + \{Q^C(x_1)\}_1^1 + \{Q^C(x_1)Q^C(x_2)\}_2^2 + \cdots$$

**Example 6.9.** The multiplicative sequences associated to the following power series\footnote{$X/\tanh(X)$ and $X^2/\sinh(X/2)$ are even, so $\hat{i}$ and $\hat{a}$ really are power series in $X$} are of particular importance for real vector bundles

$$l^R(X) = \frac{\sqrt{X}}{\tanh \sqrt{X}} = 1 + \frac{1}{3} x - \frac{1}{45} x^2 + \cdots$$

$$\hat{a}^R(X) = \frac{\sqrt{X^2/2}}{\sinh(\sqrt{X}/2)} = 1 - \frac{1}{24} x + \frac{7}{2^7 3^2 5} x^2 + \cdots$$
the associated total cohomology classes are usually denoted by \( L = K_1(p(E)) \) and \( A = K_\hat{a}(p(E)) \). For complex vector bundles in accordance with (6.4) set

\[
\begin{align*}
\hat{l}^c(X) &= \frac{X}{\tanh(X)} \\
\hat{a}^c(X) &= \frac{X/2}{\sinh(X/2)}
\end{align*}
\]

No confusion will therefore arise in writing \( L \) and \( \hat{A} \) for the corresponding classes \( K_{ic}(c(E)), K_{\hat{a}c}(c(E)) \) as well.

**Example 6.10.** In theorem 8.1 we will need the multiplicative sequence coming from the power series

\[
csh^R(X) := \cosh(\sqrt{X}/2)
\]

Denote the total (real or complex, \( csh^C(X) := csh^R(X^2) \)) \( csh \)-class by \( \hat{C}sh \).

**Remark 6.11.** Let \( Q \in \mathbb{Q}[X] \) be some power series, and let \( k \in \mathbb{Q} \). Define \( \hat{Q}(X) = k^{-1}Q(kX) \). Assume \( n = \dim(X) \). As remarked above, we may use \( Q(c_1(E)) \cdots Q(c_n(E)) \) to compute \( K_Q \) up to the relevant degree. By (6.2) the \( n \)-homogeneous part (which is all that matters) of \( Q(c_1(E)) \cdots Q(c_n(E)) \) and \( \hat{Q}(c_1(E)) \cdots \hat{Q}(c_n(E)) = k^{-n}Q(kc_1(E)) \cdots Q(kc_n(E)) \) coincide. This is why the \( L \)-genus may also be obtained by \( (k = \frac{1}{2}) \)

\[
L(E) = \int_X l(c_1(E)) \cdots l(c_n(E)) = \int_X \prod \frac{x_i}{\tanh(x_i/2)} = \int X \hat{L}(E)
\]

with \( \hat{L} \) as in (2.22).

### 6.3. A Formula for the Virtual Index.

Let \( j : U \subset X \) be a closed oriented codimension 2 submanifold of a manifold \( X \) that is also assumed to be compact and oriented. Write \( u \) for the fundamental cohomology class of \( U \) in \( X \). Then \( j^*TX = \nu \oplus TU \) for the normal bundle \( \nu \). Therefore (rational Pontrjagin classes satisfy the Whitney sum axiom)

\[
j^*p(TX) = p(\nu)p(TU) = j^*(1 + u^2)p(TU)
\]

But \( 1 + u^2 \) is invertible, so

\[
p(TU) = j^*\left( \frac{p(TX)}{1 + u^2} \right)
\]

**Proposition 6.12.** Let \( Q \) be the total \( Q \)-class of some power series \( Q \), and let \( f(X) := \frac{X}{Q(X)} \). Let \( u \in H^2(X) \) denote the fundamental cohomology class of \( U \subset X \), considered as a rational class. Then

\[
\langle Q(TU), [U] \rangle = \langle Q(TX)f(u), [X] \rangle
\]

**Proof.** From the multiplicativity of \( K_Q \) we have

\[
Q(TU) = K_Q(p(TU)) = j^*\left( \frac{K_Q(p(TX))}{K_Q(1 + u^2)} \right) = j^*\left( \frac{Q(TX)}{Q(u^2)} \right)
\]
It follows from (5.4) that
\[
\langle Q(TU), [U]\rangle = \langle j^* \left( \frac{Q(TX)}{Q(u^2)} \right), [U]\rangle = \left( \frac{Q(TX)}{Q(u^2)} \right)^g [U], [X]\rangle
\]

\[\square\]

**Corollary 6.13.** Let \( u, v \in H^2(X; \mathbb{Z}) \). Assume \( u \) is realized by \( U \subset X \) as above, and \( V \) realizes the restriction \( v |_U \in H^2(U; \mathbb{Z}) \). Then
\[
\langle Q(TV), [V]\rangle = \langle Q(TU)f(v), [U]\rangle = \langle Q(TX)f(u)f(v), [X]\rangle
\]

(6.6)

**Example 6.14.** Using the Hirzebruch Signature Theorem, we have
\[
\text{sgn}(U) = \int_X \hat{L}(TX) \tanh(u)
\]

6.4. Statement of the Hirzebruch Signature Theorem.

**Definition 6.15.** Let \( M \) be a smooth \( 4\mathfrak{n} \)-dimensional smooth compact oriented manifold. The signature \( \text{sgn}(M) \) of \( M \) is defined as the signature of the symmetric bilinear form
\[
\mathfrak{m} : H^{2\mathfrak{n}}(M) \times H^{2\mathfrak{n}}(M) \rightarrow \mathbb{Z}, \ (a, b) \mapsto \langle a \cup b, [M]\rangle
\]

The proof of the following theorem can be found in [Hir66] or also [MS74].

**Theorem 6.16.** (Hirzebruch) For the signature we have
\[
\text{sgn}(M) = \int_M \hat{L}(M)
\]
7. The Jacobi Theta Functions

7.1. Definition and Convergence. In this section, we will use the shorthands
\[ q := e^{\pi i \tau}, \quad w := e^{\pi iz} \]
(7.1)
The Jacobi theta function is defined for \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \) by
\[ \vartheta(z, \tau) := \sum_{n \in \mathbb{Z}} q^{n^2} w^{2n} = \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau + 2\pi inz} \]
(7.2)
For any \( \varepsilon > 0 \) and \( \text{Im} \, \tau \geq \varepsilon, |\text{Im} \, z| \leq 1/\varepsilon \) we have the estimate
\[ |e^{\pi in^2 \tau + 2\pi inz}| = e^{-\pi n^2 \text{Im}(\tau) - 2\pi n \text{Im}(z)} \leq \left(e^{-\pi n\varepsilon + 2\pi/\varepsilon}\right)^n \]
Splitting the series (7.2) into the parts \( n \geq 0 \) and \( n < 0 \) we see that the respective
geometric series on the right hand side of the inequality converge. Hence \( \vartheta \) is
normally convergent on compact subsets of \( \mathbb{C} \times \mathbb{H} \). From the definition, \( \vartheta \) has the
properties
\[ \vartheta(z, \tau + 1) = \vartheta(z, \tau) \]
(7.3)
\[ \vartheta(z + \tau, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau + 2\pi inz + 2\pi in\tau} = \sum_{n \in \mathbb{Z}} e^{\pi i(n^2 + 2n + 1) - \pi \tau + 2\pi i(n + 1)z - 2\pi iz} = e^{-\pi \tau - 2\pi iz} \vartheta(z, \tau) \]
(7.4)
where the last equality is obtained by summing over \( n + 1 \) instead of \( n \) (by absolute
convergence). Finally, we show that \( \vartheta(z, \tau) \) has zeros at
\[ z = \left(n + \frac{1}{2}\right) \tau + \left(m + \frac{1}{2}\right), \quad n, m \in \mathbb{Z} \]
By (7.3) and (7.4) it suffices to show \( \vartheta\left(\frac{n+1}{2}, \tau\right) = 0. \) But this is immediate from
\[ \vartheta\left(\frac{n+1}{2}, \tau\right) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{\pi in\tau + \pi n} = \sum(-1)^n q^{n(n+1)} = \sum(-1)^{-m-1} q^{(-m-1)(-m)} = - \sum(-1)^{-m} q^{(m+1)m} = - \vartheta\left(\frac{\tau+1}{2}, \tau\right) \]
where we have changed summation \( n = -m - 1. \)

7.2. Product Expansion.

Theorem 7.1. \( \vartheta(z, \tau) = \prod_{n=1}^{\infty} (1 - q^{2n}) (1 + q^{2n-1}w) (1 + q^{2n-1}w^{-2}) = \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}) (1 + e^{(2n-1)\pi i\tau} e^{2\pi iz}) (1 + e^{(2n-1)\pi i\tau} e^{-2\pi iz}). \)
Proof. Denote the right hand product by \( f(z, \tau). \) The convergence of \( \sum q^n \) for
\( |q| < 1 \) implies absolute and local uniform convergence of the product. Clearly,
\[ f \text{ has single zeros precisely at } z = (n + \frac{1}{2}) \tau + (m + \frac{1}{2}), \ n, m \in \mathbb{Z}. \] Moreover, \( f(z + 1, \tau) = f(z, \tau) \) and also
\[
\frac{f(z + \tau, \tau)}{f(z, \tau)} = \prod_{n=0}^{\infty} \frac{1 + q^{2n+1}w^2}{1 + q^{2n-1}w^2} = \prod_{n=0}^{\infty} \frac{1 + q^{2n+1}w^2}{1 + q^{2n-1}w^2} = \frac{1}{1 + qw^2} = \frac{1 + q^{-1}w^{-2}}{1} = (qw^2)^{-1} = e^{-\pi i \tau - 2\pi iz}.
\]
Thus \( f \) transforms in the same way \( \vartheta \) does. Consider now the quotient \( g(z) := \vartheta(z, \tau)/f(z, \tau) \) which is an entire function of \( z \) for fixed \( \tau \). It satisfies
\[ g(z + 1) = g(z), \ g(z + \tau) = g(z) \]
so \( g \) is a double periodic function with periods \( 1, \tau \in \mathbb{H} \) and is in particular already determined by its values on a compact “period parallelogram” \([0, 1] + [0, 1] \tau \). It follows that, for every fixed \( \tau \), \( g \) is a bounded entire function and, as such it reduces to some constant \( \varphi(q) = \prod_{n=1}^{\infty} \frac{1 + q^{2n+1}w^2}{1 + q^{2n-1}w^2} \). \( \varphi \) is a map from \( \{|q| < 1\} \to \mathbb{C} \) which is holomorphic in \( q \) as is seen directly from the explicit expression for \( \varphi \) just given as a series and product depending on \( q \) (and not on \( \tau \)). We have \( \varphi(0) = 1 \). To prove the theorem, it remains to show that \( \varphi \) is the constant function. But this follows from the identity theorem of complex analysis since
\[
\varphi(q) = \frac{\vartheta(1/4, \tau)}{f(1/4, \tau)} = \prod_{n \in \mathbb{Z}} \frac{1+q^{n}w^{2}}{(1-q^{2n})^{2}(1+q^{2n-1})} = \prod_{n \in \mathbb{Z}} \frac{1+q^{n}w^{2}}{(1-q^{2n})^{2}(1+q^{2n-1})} = \frac{\vartheta(1/2, 4\tau)}{f(1/2, 4\tau)} = \varphi(q^{4})
\]
so by induction \( \varphi(q) = \varphi(q^{4k}) \) for any \( |q| < 1 \).

7.3. Transformation Formula.

**Proposition 7.2.** Suppose \( f : \mathbb{H} \to \mathbb{C} \) is a meromorphic function which is periodic with period 1. Then there exists a meromorphic function \( f \) on \( \mathbb{D} \setminus \{0 < |z| < 1\} \) such that
\[ \hat{f}(e^{2\pi i \tau}) = f(\tau), \ \tau \in \mathbb{H} \]
In particular, \( f \) has a Fourier expansion
\[ f(\tau) = \sum_{n=-\infty}^{\infty} a_{n} e^{2\pi i n \tau} \]
with coefficients that may be computed for any \( s > 0 \) by
\[ a_{n} = \int_{\mathbb{D}} f(\tau) e^{-2\pi i n \tau} d\tau \]
Proof. $f$ being periodic implies that $f$ factors
\[ e^{2\pi i z} \downarrow \mathbb{C} \]
\[ \downarrow \mathbb{D} \setminus 0 \]
Locally however we have $\hat{f}(z) = f(\log(\frac{z}{2\pi i}))$ which shows that $\hat{f}$ is holomorphic on $\mathbb{D} \setminus 0$. The statement on the Fourier expansion follows by considering a Laurent expansion for $\hat{f}$ around 0, and the integral formula for the coefficients $a_n = \frac{1}{2\pi i} \int_{|z|=r} \hat{f}(z) z^{-n-1} dz = \int_{-\frac{i\log(r)}{2\pi i}}^{\frac{\log(r)}{2\pi i}} f(\tau)e^{-2\pi i n \tau} d\tau$ for any $0 < r < 1$.

Remark 7.3. Entirely analogously it can be shown that if $f$ is defined on $\Im(\tau) > y$ for some $y \in \mathbb{R}$, we have a Fourier expansion on $\Im(\tau) > y$. The integration to compute the coefficients may then be carried out for any $s > y$. For functions defined on $\mathbb{C}$ this applies of course for any $y \in \mathbb{R}$.

Theorem 7.4. We have the following transformation formula
\[ \vartheta(z, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^2} \vartheta(\tau z, \tau) \]

Proof. For fixed $\tau \in \mathbb{H}$ the function $\varphi(z) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+z)^2} = e^{\pi i z^2} \vartheta(\tau z, \tau)$ clearly converges compactly uniform on $\mathbb{C}$ and has period 1. Assume first $\tau = iy \in i\mathbb{R}$ is purely imaginary. From Proposition 7.2 write
\[ \varphi(z) = \sum_{n \in \mathbb{Z}} e^{-\pi y(n+z)^2} = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k z} \]
where for any $s \in \mathbb{R}$ we have (note that $e^{-2\pi i k w} = e^{-2\pi i k(n+w)}$)
\[ c_k = \int_{0+is}^{1+is} \sum_{n \in \mathbb{Z}} e^{-\pi y(n+w)^2} e^{-2\pi i k w} dw \]
\[ = \frac{1}{\sqrt{y}} \int_{-\infty+is}^{+\infty+is} 1 \sum_{n \in \mathbb{Z}} e^{-\pi y(u+n)^2} e^{-2\pi i k u} du \]
\[ = e^{-\pi k^2/y} \frac{1}{\sqrt{y}} \int_{-\infty+is}^{+\infty+is} e^{-\pi v^2} dv \]
Now choose $s = -k/y$. Using $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$ we obtain $c_k = \sqrt{\frac{\tau}{i}} e^{-\pi k^2/\tau}$. Thus
\[ e^{\pi i z^2} \vartheta(\tau z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+z)^2} = \sum_{k \in \mathbb{Z}} e^{-\pi i k^2/\tau + 2\pi i k z} = \sqrt{\frac{\tau}{i}} \vartheta(z, -\frac{1}{\tau}) \]
for any purely imaginary $\tau$. The identity theorem of complex analysis now implies the theorem in the general case.

\[ ^{15}\text{Every point of } \mathbb{D} \setminus 0 \text{ lies in the domain of some logarithm} \]
7.4. The Four Jacobi Theta Functions. The above defined theta function $\vartheta$ is just one of four theta functions defined by Jacobi. It is also denoted by $\vartheta_1 = \vartheta$.

Since there are several slightly different definitions of the Jacobi theta functions in use, caution must be exercised in comparing the following formulas with the mathematical literature. We will follow [Cha85]. The following is immediate from theorem 7.1,

$$
\vartheta_0(z, \tau) := \vartheta \left( z + \frac{1+\tau}{2}, \tau \right) q^{1/4} e^{\pi i z} \frac{1}{2} \tag{7.5}
$$

$$
= \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)\pi i z} = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin \left( (2n + 1)\pi z \right)
$$

$$
= 2q^{1/4} \sin(\pi z) \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 - q^{2n-1} e^{2\pi i z} \right) \left( 1 - q^{2n-1} e^{-2\pi i z} \right)
$$

$$
\vartheta_1(z, \tau) := \vartheta \left( z + \frac{\tau}{2}, \tau \right) q^{1/4} e^{\pi i z} \tag{7.6}
$$

$$
= \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} e^{(2n+1)\pi i z} = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos \left( (2n + 1)\pi z \right)
$$

$$
= 2q^{1/4} \cos(\pi z) \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 + q^{2n-1} e^{2\pi i z} \right) \left( 1 + q^{2n-1} e^{-2\pi i z} \right)
$$

$$
\vartheta_2(z, \tau) := \vartheta \left( z + \frac{1}{2}, \tau \right) \tag{7.7}
$$

$$
= \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} e^{2n\pi i z} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2n\pi z)
$$

$$
= \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 - q^{2n-1} e^{2\pi i z} \right) \left( 1 - q^{2n-1} e^{-2\pi i z} \right)
$$

$$
\vartheta_3(z, \tau) := \vartheta(z, \tau) \tag{7.8}
$$

$$
= \sum_{n=-\infty}^{+\infty} q^{n^2} e^{2n\pi i z} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2\pi n z)
$$

$$
= \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 + q^{2n-1} e^{2\pi i z} \right) \left( 1 + q^{2n-1} e^{-2\pi i z} \right)
$$

with the notation $q = e^{\pi i \tau}$ in half the nome. Theorem 7.4 may be rewritten as

$$
\vartheta_0(z, -\frac{1}{\tau}) = -i \sqrt{\frac{\tau}{i}} e^{\pi i \tau z} \vartheta_0(\tau z, \tau) \tag{7.9}
$$

$$
\vartheta_1(z, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z} \vartheta_2(\tau z, \tau) \tag{7.10}
$$

$$
\vartheta_2(z, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z} \vartheta_1(\tau z, \tau) \tag{7.11}
$$

$$
\vartheta_3(z, -\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z} \vartheta_3(\tau z, \tau) \tag{7.12}
$$
Directly from the definition we have also
\[ \vartheta_0(z, \tau + 1) = e^{\pi i/4} \vartheta_0(z, \tau) \]  
(7.13)
\[ \vartheta_1(z, \tau + 1) = e^{\pi i/4} \vartheta_1(z, \tau) \]  
(7.14)
\[ \vartheta_2(z, \tau + 1) = \vartheta_3(z, \tau) \]  
(7.15)
\[ \vartheta_3(z, \tau + 1) = \vartheta_2(z, \tau) \]  
(7.16)

Set \[ \vartheta_i(\tau) := \vartheta_i(0, \tau) \] for \( i = 0, 1, 2, 3 \). The following can be proven
\[ \vartheta'_{0}(\tau) = \pi \vartheta_1(\tau) \vartheta_2(\tau) \vartheta_3(\tau) \]  
(7.17)
We will use \( \vartheta'_0 \) only as a shorthand for \( \pi \vartheta_1(\tau) \vartheta_2(\tau) \vartheta_3(\tau) \). The transformation formulas for \( \vartheta'_0 \) then read
\[ \vartheta'_0(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} \vartheta'_0(\tau) \]  
(7.18)
\[ \vartheta'_0(\tau + 1) = e^{\pi i/4} \vartheta'_0(\tau) \]  
(7.19)

While for the product expansion we have
\[ \vartheta'_0(\tau) = 2\pi q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})^3 \]

7.5. Modular Forms. The full modular group is defined as \( \Gamma := SL_2(\mathbb{Z}) \). This group acts on \( \mathbb{H} \) by
\[ A \tau := \frac{a \tau + b}{c \tau + d} \]  
(7.20)

Two important elements of \( \Gamma \) are
\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ translation } \tau \mapsto \tau + 1 \]
\[ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ inversion } \tau \mapsto -1/\tau \]

It can be shown that \( \Gamma \) is generated by these two elements. We however will be particularly interested in the theta subgroup \( \Gamma_0 \) generated by \( T^2 \) and \( S \). The entire group \( \Gamma \) acts (7.20) transitively on \( \mathbb{Q} \cup \{\infty\} \). On the other hand, the action of the subgroup \( \Gamma_0 \) on \( \mathbb{Q} \cup \{\infty\} \) has two distinct orbits - one going through 0 and the other through 1. These orbits are called the cusps of \( \Gamma_0 \). The congruence subgroup \( \Gamma_0 \) has index 3 in \( \Gamma = \Gamma_0 \cup \Gamma_0 \cdot T \cup \Gamma_0 \cdot U \).

Definition 7.5. For any \( k \in \mathbb{Z} \) a right operation of \( \Gamma \) on functions \( f : \mathbb{H} \to \mathbb{C} \) is given by
\[ (f \mid_k A)(\tau) := (c \tau + d)^{-k} f(A \tau) \]
Definition 7.6. A modular form of weight $k$ on $\Gamma$ is a holomorphic function

$$f : \mathbb{H} \to \mathbb{C}$$

such that for any $A \in \Gamma$

$$f |_k A = f$$

Moreover, we assume that $f$ has at most a pole at $i\infty$. This means that we require a Fourier expansion in $q = \exp(2\pi i\tau)$

$$f(\tau) = \sum_{n=-N}^{\infty} a_n q^n$$

which has only finitely many terms of negative order. If there are no terms of negative order, $f$ is called an entire modular form (and also said to be holomorphic at the cusp). We will write $M_k(\Gamma)$ for the complex vector space of entire modular forms of weight $k$.

The following proposition is proven in [KK07].

Proposition 7.7. $M_k(\Gamma) = 0$ for $k < 0$ and $M_k(\Gamma) \cong \mathbb{C}$ for $k = 0$.

There is the notion of a modular form over a congruence subgroup $\Lambda \subset \Gamma$. For simplicity, we will restrict our attention only to the subgroup $\Lambda = \Gamma_0$. Let $U = TS \in \Gamma_0$, so $U\tau = -\frac{1}{U} + 1$.

Definition 7.8. A modular form $f : \mathbb{H} \to \mathbb{C}$ of weight $k$ over $\Gamma_0$ is a holomorphic function with $f |_k A = f$ for any $A \in \Gamma_0$. We now assume that $f$ is meromorphic at both cusps, i.e. we require that both $f$ and $f |_k U$ have Fourier expansions in $q = \exp(i\pi \tau)$ (so not in $\exp(2\pi i\tau)$) which have only finitely many terms of negative order. If $f$ is holomorphic at both cusps it is again called entire and this is denoted by $f \in M_k(\Gamma_0)$.

Remark 7.9. There are no modular forms over $\Gamma_0$ of odd weight. This follows since $-E \in \Gamma_0$ acts as $\tau \mapsto -E\tau = \tau$, so

$$f(\tau) = f(-E\tau) = (-1)^k f(\tau)$$

Example 7.10. $\delta_2(\tau) = -\vartheta_3^2(\tau) + \vartheta_4^2(\tau)$ and $\varepsilon_2(\tau) = \frac{1}{24} \vartheta_3^2(\tau) \vartheta_4^2(\tau)$ are entire modular forms over $\Gamma_0$ of weight 2 and 4 respectively. This is clear from the transformation formulas (7.10), (7.11), (7.14), (7.15), (7.16). By the definition of the theta functions (7.6), (7.7) we have

Proposition 7.11. The functions $\delta_2$ and $\varepsilon_2$ have Fourier expansions

$$\delta_2(\tau) = \sum_{l=0}^{\infty} d_l q^l = 1 - 24q + 24q^2 - 96q^3 + 24q^4 \cdots$$

$$\varepsilon_2(\tau) = \sum_{l=0}^{\infty} e_l q^l = q - 8q^2 + 28q^3 - 64q^4 \cdots$$
where the coefficients may be written explicitly as

\[ d_l = -\delta_4^{\text{odd}}(4l) + (-1)^l \delta_4(l), \quad e_l = \frac{1}{16} \sum_{r=0}^{l} (-1)^r \delta_4(r) \delta_4^{\text{odd}}(4l - 4r) \]

in terms of the number theoretic functions\(^{16}\) (integer = element of \( \mathbb{Z} \))

\[ \delta_4(l) = \text{number of ways } l \text{ may be written as sum of } 4 \text{ integer squares} \]

\[ \delta_4^{\text{odd}}(l) = \text{number of ways } l \text{ may be written as sum of } 4 \text{ odd integer squares} \]

In particular \( d_l, e_l \in \mathbb{Z} \).

**Proof.** By symmetry of the definition, \( \delta_4 \) and \( \delta_4^{\text{odd}} \) are divisible by 4. It follows that \( e_l \in \mathbb{Z} \) once we have established the rest of the proposition. We have \( \vartheta_1(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} = \sum_{n \in \mathbb{Z}} (q^{1/4})^{(2n+1)^2} \) and \( \vartheta_3(\tau) = \sum_{n \in \mathbb{Z}} q^n \). It follows that

\[ \vartheta_1^4(\tau) = \sum_{n=0}^{\infty} \delta_4^{\text{odd}}(n) q^{n/4} = \sum_{n=1}^{\infty} \delta_4^{\text{odd}}(4n) q^n \]

Similarly

\[ \vartheta_2^4(\tau) = \vartheta_3^4(\tau + 1) = \sum_{n=0}^{\infty} \delta_4(n) e^{\pi i n (\tau + 1)} = \sum_{n=0}^{\infty} (-1)^n \delta_4(n) q^n \]

\[ \square \]

To complete the proof that \( \delta_2, \varepsilon_2 \) are entire modular forms over \( \Gamma_0 \) we have to consider the Fourier expansion of \( \delta_2 \mid_U \) and \( \varepsilon_2 \mid_U \). But by (7.14), (7.15), (7.11), (7.12)

\[ (\delta_2 \mid_U)(\tau) = \tau^{-2} \delta_2(-\frac{1}{\tau} + 1) = -\vartheta_2^4(\tau) - \vartheta_3^4(\tau) = -2 - 48q^2 - 48q^4 \cdots \]

\[ (\varepsilon_2 \mid_U)(\tau) = -\frac{1}{16} \vartheta_2^4(\tau) \vartheta_3^4(\tau) = -\frac{1}{16} + q^2 - 7q^4 \cdots \]

The values of \( \delta_2, \varepsilon_2 \) at the cusps are therefore

\[ \delta_2(i\infty) = 1 \quad \delta_2(1) = (\delta_2 \mid_U)(i\infty) = -2 \]  \( \quad \) (7.21)

\[ \varepsilon_2(i\infty) = 0 \quad \varepsilon_2(1) = (\varepsilon_2 \mid_U)(i\infty) = -\frac{1}{16} \]

**Proposition 7.12.** The ring of entire modular forms \( \mathcal{M}(\Gamma_0) := \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_0) \) is generated by the two elements \( \delta_2, \varepsilon_2 \).

\(^{16}\delta_4^{\text{odd}}(n) = 0 \text{ unless } 4 \mid n\)
Proof. Clearly $M_k(\Gamma_0) \supset \mathbb{C}[\delta_2, \varepsilon_2]$. Assume conversely that $f \in M_{2k}(\Gamma_0)$. Then $f - f(i\infty)\delta_2^k$ has a zero at $i\infty$. From the product expansions (7.6), (7.7) of $\vartheta_1, \vartheta_2$ it is clear that $\varepsilon_2$ is non-zero at every point of $H$. By (7.21) we have $\varepsilon_2(1) = -1/16 \neq 0$. Since $\varepsilon_2$ has a zero of multiplicity one at $i\infty$, we obtain an entire modular form
\[
\frac{f - f(i\infty)\delta_2^k}{\varepsilon_2}
\]
of weight $2k - 4$. By induction, it suffices therefore to prove that $M_0(\Gamma_0) = \mathbb{C}$ and $M_k(\Gamma_0) = 0$ for $k < 0$.

Let $f \in M_0(\Gamma_0)$. Since constants belong to $M_0(\Gamma_0)$ we may assume $f(i\infty) = 0$. Then
\[
(\pi(f))(\tau) := f(\tau)f(\tau + 1)f(1 - \frac{1}{\tau})
\]
is clearly invariant under the entire modular group $\Gamma$, so $\pi(f) \in M_0(\Gamma)^{17}$. From proposition 7.7 it follows that $\pi(f)$ is constant, and $\pi(f)(i\infty) = 0$ implies $\pi(f) = 0$. Therefore$^{18}$ $f = 0$. The proof that $M_k(\Gamma_0) = 0$ for $k < 0$ is analogous. \qed

---

$^{17}$Note that $f\left(1 - \frac{1}{\tau + 1}\right) = f\left(\frac{\tau}{\tau + 1}\right) = f\left(-\frac{\tau + 1}{\frac{\tau}{\tau + 1}}\right) = f\left(-1 - \frac{2}{\tau}\right) = f\left(1 - \frac{1}{\tau}\right)$

$^{18}$For any $\tau$ we have either that $f(\tau) = 0$, $f(\tau + 1) = 0$ or $f(1 - \frac{1}{\tau}) = 0$. Consider any convergent $\tau_n \to \tau$ in $H$. Then at least one of the three sequences $f(\tau_n), f(\tau_n + 1)$ or $f(1 - \frac{1}{\tau_n})$ has a subsequence that is constant zero. It follows that $f$ vanishes identically.
8. The Miraculous Cancellation Formula

8.1. $K$-theory with Connection. So far (cf. example 2.34 and remark 2.33) we have shown that a smooth functor on the category of vector spaces induces a corresponding functor on vector bundles with connections, and that natural isomorphisms carry over as well. We have thus established an assignment\(^\text{19}\)

$$\text{Fun}(\text{Vect}) \longrightarrow \text{Fun}(\text{VectBun}^\nabla)$$

of the functor categories, considered as groupoids. Write

$$V^\nabla(M)$$

for the set of isomorphism classes of complex vector bundles with connection, meaning that we additionally require the isomorphisms to be compatible with the connection (i.e. morphisms of $\text{VectBun}^\nabla$). Since the functor $- \oplus -$ may be extended to $\text{VectBun}^\nabla$,

$$[E] + [F] := [E \oplus F]$$

is well-defined (since a functor carries isomorphisms to isomorphisms). Similarly, define a multiplication $[E] \otimes [F] := [E \otimes F]$. Then $V^\nabla(M)$ has the structure of a commutative monoid. This comes from the natural isomorphisms

$$V^0 \cong V, \quad (U \oplus V) \oplus W \cong U \oplus (V \oplus W), \quad V \oplus W \cong W \oplus V.$$  

All in all, the Grothendieck group $K^\nabla(M)$ inherits the structure of a commutative ring.

The Chern character, viewed as taking differential forms as values, clearly extends to a group homomorphisms $V^\nabla(M) \to \Omega^{2*}(M)$, is compatible with the multiplication (the new Chern roots of $E \otimes F$ are $x_i + y_j$, so $\text{ch}(E \otimes F) = \sum e^{x_i + y_j} = (\sum e^{x_i})(\sum e^{y_j}) = \text{ch}(E) \text{ch}(F)$) and thus yields a ring homomorphism

$$\text{ch} : K^\nabla(M) \to \Omega^{2*}(M)$$

We can similarly consider real vector bundles. The corresponding Grothendieck group will be denoted by $KO^\nabla(M)$.

8.2. Statement of the Han/Zhang Theorem. In this section we will give a proof of the generalized miraculous cancellation formula that may involve also an extra twist from a coefficient bundle. In contrast to the approach of [HZ04], we will work here with the congruence subgroup $\Gamma_0$ instead of the two groups $\Gamma_0[2], \Gamma_0[2]$. We will then deal only with one modular form having the two relevant characteristic forms as values at the cusps $i\infty, 1$ of $\Gamma_0$. The goal of this section will be to prove the following:

\textbf{Theorem 8.1. (Han/Zhang)} Let $M$ be a 8$k + 4$ dimensional closed Riemannian manifold, $V$ a rank 2$l$ real vector bundle over $M$, and $\xi$ a real oriented rank 2 “coefficient” bundle with Euler class $c$. Suppose that $p_1(V) = p_1(M)$ for the first Pontrjagin classes. Then we have an equality of differential forms

$$\left\{ \frac{\hat{A}(TM)Csh(V)}{\cosh^2(c/2)} \right\}^{(8k+4)} = \sum_{r=0}^{k} 2^{2k-6r+1} \left\{ \hat{A}(TM) \text{ch}(b_r) \cosh(c/2) \right\}^{(8k+4)} \tag{8.1}$$

\text{19}This assignment is not functorial, but it were if we’d consider isomorphism classes of vector bundles instead. For instance, a problem here is that for a vector bundle $E$ of rank $n$ with associated bundle of frames $P$ we have only an isomorphism $E \cong P \times_{GL} \mathbb{R}^n$. 
for \( b_r \in K^\nabla(M) \) which are explicitly determined in the proof.

**Remark 8.2.** Here we have written \( \{\omega\}^{(l)} \) for the degree \( l \) component of a differential form \( \omega \). \( \text{Csh} \) denotes the genus defined in (6.5) by the power series \( \cosh(X) \).

**Theorem 8.3.** (Han/Zhang) Let \( M \) be a \( 8k \) dimensional closed Riemannian manifold, \( V \) a rank \( 2l \) real vector bundle over \( M \), and \( \xi \) a real oriented rank \( 2 \) “coefficient” bundle with Euler class \( c \). Suppose that \( p_1(V) = p_1(M) \) for the first Pontrjagin classes. Then

\[
\left\{ \frac{\hat{A}(TM) \text{Csh}(V)}{\cosh^2(c/2)} \right\}^{(8k)} = \sum_{r=0}^{k} 2^{2k-6r} \left\{ \frac{\hat{A}(TM) \text{ch}(d_r) \cosh(c/2)}{\text{ch}(d_r)} \right\}^{(8k)}
\]

(8.2)

for some \( d_r \in K^\nabla(M) \).

The proof of (8.2) is entirely parallel to that of (8.1). We will prove only (8.1) and indicate the necessary changes in remark 8.6.

**Remark 8.4.** The \( b_r \) and \( d_r \) of theorems 8.1 and 8.3 are linear combinations of the coefficients of a formal power series \( \Theta_2 \in K^\nabla(M)[q] \) defined in (8.7). These coefficients consist of exterior powers of \( T_C \mathcal{M} \), \( V_C \) and \( \xi \) as well as the trivial bundle.

Let \( E \) be a vector bundle of rank \( n \). We will use the notation

\[
\tilde{E} = E - \mathbb{C}^n
\]

where \( \mathbb{C}^n \) denotes the trivial bundle of rank \( n \) over \( M \).

8.3. The Modular Form \( Q_2 \). Owing to example 3.4, let \( \pm 2\pi i x_j, j = 1, \ldots, 4k+2 \) be the new formal Chern roots of the complexified tangent bundle \( T_C \mathcal{M} \), and let \( \pm 2\pi i y_v, v = 1, \ldots, l \) be those of \( V_C \). The Euler class \( c \) of \( \xi \) is by definition the first Chern class of \( \xi \), viewed as a complex line bundle. Write \( c = 2\pi i u \).
Define a parameter-dependant ($\tau \in \mathbb{H}$) differential form\(^{20}\)

\[
Q_2(\tau) = \left\{ \prod_{j=1}^{4k+2} x_j \frac{\vartheta_0'(\tau)}{\vartheta_0(x_j, \tau)} \left( \prod_{v=1}^{l} \frac{\vartheta_3(y_v, \tau)}{\vartheta_3(\tau)} \right) \frac{\vartheta_2^2(\tau)}{\vartheta_2^2(u, \tau)} \frac{\vartheta_1(u, \tau)}{\vartheta_1(\tau)} \right\}^{(8k+4)}
\]

(8.4)

**Lemma 8.5.** Suppose the first Pontrjagin classes $p_1(M) = p_1(V)$ coincide. Then, at every point of $M$, $Q_2$ is an entire modular form over $\Gamma_6$ of weight $4k+2$.

**Proof.** The assumption on the Pontrjagin classes means $\sum_j x_j^2 = \sum_v y_v^2$. The group $\Gamma_6$ is generated by $S$ and $T^2$. By (7.19), (7.13), (7.16), (7.15), (7.14) an application of $T$ to (8.4) interchanges the indices 2 and 3 while the factors $e^{\pi \tau/4}$ all cancel out. It follows that $Q_2$ is invariant under $T^2$. As for $S$,

\[
Q_2(S\tau) = \left\{ \prod_{j=1}^{4k+2} x_j \frac{\vartheta_0'(S\tau)}{\vartheta_0(x_j, S\tau)} \left( \prod_{v=1}^{l} \frac{\vartheta_3(y_v, S\tau)}{\vartheta_3(S\tau)} \right) \frac{\vartheta_2^2(S\tau)}{\vartheta_2^2(u, S\tau)} \frac{\vartheta_1(u, S\tau)}{\vartheta_1(S\tau)} \right\}^{(8k+4)}
\]

Since there is a $\tau$ in front of every degree 2 element $u, x_j, y_v$ of the formal power series, taking the degree $8k+4$ part thus yields a factor $4k+2$ (cf. (6.2)). It remains to show that $Q_2$ is holomorphic at every cusp. This will be carried out in detail below by exhibiting Fourier expansions of $Q_2$ and $Q_1 := Q_2 \mid_{4k+2} U$.

\[
\square
\]

To compute the value at the cusp 1, we have to consider

\[
Q_1(\tau) := (Q_2 \mid_{4k+2} U)(\tau) = \tau^{-4k-2} Q_2(-\frac{1}{\tau} + 1)
\]

\[
= \left\{ \prod_{j=1}^{4k+2} x_j \frac{\vartheta_0'(\tau)}{\vartheta_0(x_j, \tau)} \left( \prod_{v=1}^{l} \frac{\vartheta_1(y_v, \tau)}{\vartheta_1(\tau)} \right) \frac{\vartheta_3(\tau u, \tau)}{\vartheta_3(\tau)} \frac{\vartheta_2(u, \tau)}{\vartheta_2(\tau)} \right\}^{(8k+4)}
\]

(8.5)

The next step of the proof will be to rewrite (8.4) and (8.5) in terms of familiar characteristic forms. From the product expansions of the theta functions and

---

\(^{20}\)It will be clearer in (8.6) that $Q_2$ is a differential form for any fixed $\tau$.\]
corollary 3.10 we obtain
\[ \prod_{j=1}^{4k+2} x_j \frac{\varphi_j'(\tau)}{\varphi_j(x_j, \tau)} = \prod_{j=1}^{4k+2} \frac{x_j}{\sin(\pi x_j)} \prod_{n=1}^{\infty} (1 - q^{2n}) \frac{\pi \prod_{n=1}^{\infty} (1 - q^{2n} e^{2\pi i x_j})}{\prod_{n=1}^{\infty} (1 - q^{2n} e^{-2\pi i x_j})} \]
\[ = \prod_{j=1}^{4k+2} \frac{x_j}{\sin(\pi x_j)} \frac{\pi x_j}{\sin(\pi x_j)} \text{ch} \left( \bigotimes_{n=1}^{\infty} \Lambda_{-q^{2n}}(T_{\mathbb{C}} M) \right)^{-1} \]
\[ \prod_{v=1}^{l} \frac{\varphi_3(y_v, \tau)}{\varphi_3(\tau)} = \text{ch} \left( \bigotimes_{m=1}^{\infty} \Lambda_{q^{2m-1}}(\tilde{V}_{\mathbb{C}}) \right) \]
\[ \frac{\varphi_2^2(\tau)}{\varphi_2(\tau)} = \frac{1}{\cos^2(\pi u)} \text{ch} \left( \bigotimes_{m=1}^{\infty} \Lambda_{q^{2m-1}}(\tilde{\xi}_{\mathbb{C}}) \right) \]
\[ \frac{\varphi_2(u, \tau)}{\varphi_2(\tau)} = \text{ch} \left( \bigotimes_{s=1}^{\infty} \Lambda_{-q^{2s-1}}(\tilde{\xi}_{\mathbb{C}}) \right) \]
\[ \frac{\varphi_1(u, \tau)}{\varphi_1(\tau)} = \cos(\pi u) \text{ch} \left( \bigotimes_{r=1}^{\infty} \Lambda_{q^{2r}}(\tilde{\xi}_{\mathbb{C}}) \right) \]

for \( Q_2 \), while for \( Q_1 \) we have
\[ \prod_{v=1}^{l} \frac{\varphi_1(y_v, \tau)}{\varphi_1(\tau)} = \prod_{v=1}^{l} \cos(\pi y_v) \text{ch} \left( \bigotimes_{m=1}^{\infty} \Lambda_{q^{2m}}(\tilde{V}_{\mathbb{C}}) \right) \]
\[ \frac{\varphi_2^2(\tau)}{\varphi_1^2(\tau)} = \frac{1}{\cos^2(\pi u)} \text{ch} \left( \bigotimes_{m=1}^{\infty} \Lambda_{q^{2m}}(\tilde{\xi}_{\mathbb{C}}) \right)^{-2} \]
\[ \frac{\varphi_2(u, \tau)}{\varphi_2(\tau)} = \text{ch} \left( \bigotimes_{s=1}^{\infty} \Lambda_{q^{2s-1}}(\tilde{\xi}_{\mathbb{C}}) \right) \]
\[ \frac{\varphi_2(u, \tau)}{\varphi_2(\tau)} = \text{ch} \left( \bigotimes_{r=1}^{\infty} \Lambda_{-q^{2r-1}}(\tilde{\xi}_{\mathbb{C}}) \right) \]

It follows that
\[ Q_2(\tau) = \left\{ \hat{A}(TM) \cos(\pi u) \text{ch}(\Theta_2) \right\}^{(8k+4)} \tag{8.6} \]

where
\[ \Theta_2 = \bigotimes_{n=1}^{\infty} \frac{1}{\Lambda_{-q^{2n}}(T_{\mathbb{C}} M)} \Lambda_{q^{2n-1}}(\tilde{V}_{\mathbb{C}} - 2\tilde{\xi}_{\mathbb{C}}) \Lambda_{-q^{2n-1}}(\tilde{\xi}_{\mathbb{C}}) \Lambda_{q^{2n}}(\tilde{\xi}_{\mathbb{C}}) \in K^{\bigotimes}(M) [q] \] \tag{8.7}

\[ = 1 + q \left( \tilde{V} - 3\tilde{\xi} \right) + q^2 \left( \tilde{V}^2 - \tilde{\xi}^2 - 3\tilde{V} \otimes \tilde{\xi} + TM + 5\tilde{\xi} \otimes \tilde{\xi} + (6 - 2l)\tilde{V} + (6l - 17)\tilde{\xi} \right) + \ldots \]

Here, \( \Theta_2 \) has to be viewed as a power series in \( q \) with coefficients in \( K^{\bigotimes}(M) \), so \( \Theta_2 = \sum_{l \geq 0} B_l q^l \) for some \( B_l \in K^{\bigotimes}(M) \). On the other hand, for \( Q_1 \) we have
\[ Q_1(\tau) = \left\{ \hat{A}(TM) \left( \prod_{v=1}^{l} \cos(\pi y_v) \right) \frac{1}{\cos^2(\pi u)} \text{ch}(\Theta_1) \right\}^{(8k+4)} \tag{8.8} \]
A GENERALIZED MIRACULOUS CANCELLATION FORMULA

for

$$\Theta_1 = \bigotimes_{n=1}^{\infty} \Lambda_{q^{2n}}(V_C - 2\xi_C)\Lambda_{q^{2n-1}}(\tilde{\xi}_C)\Lambda_{-q^{2n-1}}(\tilde{\xi}_C)$$  \hspace{1cm} (8.9)$$

This exhibits the Fourier expansions at \( i\infty \) of both \( Q_1 \) and \( Q_2 \) for every point of \( M \). The values at the cusps are

\[
Q_2(0) = Q_2(i\infty) = \left\{ \hat{A}(TM) \cos(\pi u) \right\}^{(8k+4)} 
\]

\[
Q_2(1) = Q_1(i\infty) = \left\{ \hat{A}(TM) \left( \prod_{i=1}^{l} \cos(\pi y_i) \right) \cdot \frac{1}{\cos^2(\pi u)} \right\}^{(8k+4)} 
\]

8.4. Explicit Computation of the Coefficients. From (8.6) and (8.7) it is clear that we have a “Fourier” expansion in \( q \)

\[
Q_2(\tau) = \sum_{l=0}^{\infty} \left\{ \hat{A}(TM) \cos(\pi u) \cdot \text{ch}(B_l) \right\}^{(8k+4)} q^l 
\]

First, expand \( Q_2(\tau) \) in terms of \( \delta_2, \varepsilon_2 \) (the coefficients \( a_i = a_i(x) \) could depend on the point of \( x \in M \), but when we write them down explicitly it will be clear that this is not the case)

\[
Q_2 = a_0 \delta_2^{2k+1} + a_1 \delta_2^{2k-1} \varepsilon_2 + \ldots + a_l \delta_2^{l} \varepsilon_2^{k} 
\]  \hspace{1cm} (8.12)$$

We will show that it is possible to express the \( a_i \) as linear combinations of the \( \left\{ \hat{A}(TM) \cos(\pi u) \cdot \text{ch}(B_l) \right\}^{(8k+4)} \). For this purpose, define (in dependency of \( k \))

\[
\begin{vmatrix} i \\ j \end{vmatrix} := \left\{ \delta_2^{2i+1}(q) \cdot \varepsilon_2^{j-i}(q) \right\}^{(j)} := j \text{th coefficient of } q \text{ expansion of } \delta_2^{2i+1}(q) \cdot \varepsilon_2^{j-i}(q) 
\]

\[
= \sum_{k+l=j \atop k_1 + \ldots + k_{2i+1} = k \atop l_1 + \ldots + l_k = l} d_{k_1} \cdots d_{k_{2i+1}} \cdot e_{l_1} \cdots e_{l_k} 
\]

From (7.21) we have \( \begin{vmatrix} k - l \\ l \end{vmatrix} = 1 \) and \( \begin{vmatrix} i \\ j \end{vmatrix} = 0 \) for \( i + j < k \). See also table 1.

Comparing the coefficients of \( q^l \) on both sides of (8.12), it follows that

\[
\left\{ \hat{A}(TM) \cos(\pi u) \cdot \text{ch}(B_l) \right\}^{(8k+4)} = a_0 \begin{vmatrix} k-l \\ l \end{vmatrix} + a_1 \begin{vmatrix} k-1 \\ l \end{vmatrix} \cdots + a_{l-1} \begin{vmatrix} k-l+1 \\ l \end{vmatrix} + a_l 
\]  \hspace{1cm} (8.13)$$
From this we get an expression for the $a_l$ as linear combination\(^{21}\)

$$a_l = \sum_{i=0}^{l} \left\{ \hat{A}(TM) \cos(\pi u) \text{ch}(B_i) \right\}^{(8k+4)} \left( \sum_{r \in \mathbb{N}, i=1 \prec \ldots \prec i_r = l} (-1)^{r-1} \left| \begin{array}{ccc} k - i_1 & \ldots & k - i_{r-1} \\ i_2 & \ldots & i_r \end{array} \right| \right)$$

(8.14)

where we have set

$$b_l = \sum_{i=0}^{l} \sum_{i_1 \prec \ldots \prec i_r = l} (-1)^{r+l} \left| \begin{array}{ccc} k - i_1 & \ldots & k - i_{r-1} \\ i_2 & \ldots & i_r \end{array} \right| B_i$$

(8.15)

(for examples see the beginning of the next section)

\(^{21}\)The empty product $\left| \begin{array}{ccc} k - i_1 & \ldots & k - i_{r-1} \\ i_2 & \ldots & i_r \end{array} \right|$ is 1 for $r = 1$. Note that while the coefficients are rather difficult to compute, it is a “canonical” coefficient arising from elementary combinatorics. Note also that the sum $\sum_{i=i_1 \prec \ldots \prec i_r = l} (-1)^{r} k - i_s$ contains one summand (namely 1) for $r = 1$ precisely when $i = i_1 = l$. 

---

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>(k=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-32</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>244</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-1024</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-72</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1800</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-17568</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>57096</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>(k=2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>-40</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>528</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>(k=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>-88</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3072</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>(k=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-128</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6868</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>-200704</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1. Examples of low coefficients for $k = 1, 2, 3$
Proof. (8.14) follows from (8.13) by induction, the case \( l = 0 \) being trivial. For \( l > 0 \) from (8.13) and the inductive hypothesis for \( l - 1, l - 2, \ldots \) we have

\[
a_l = \left\{ \hat{A}(TM) \cos(\pi u) \text{ch}(B_l) \right\}^{(8k+4)}
\]

\[
= - \sum_{s=0}^{l-1} \sum_{i=0}^{s} \left\{ \hat{A}(TM) \cos(\pi u) \text{ch}(B_i) \right\}^{(8k+4)} \sum_{r \in \mathbb{N}} (-1)^r \left| \begin{array}{ccc}
-1 & \cdots & -1 \\
-1 & \cdots & -1 \\
1 & \cdots & 1 \\
\end{array} \right|
\]

\[
= \sum_{i=0}^{l-1} \left\{ \hat{A}(TM) \cos(\pi u) \text{ch}(B_i) \right\}^{(8k+4)} \sum_{i=1}^{s} \sum_{r \in \mathbb{N}} (-1)^r \left| \begin{array}{ccc}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
-1 & \cdots & -1 \\
\end{array} \right|
\]

which shows (8.14) for \( l \).

Evaluating (8.12) at the cusp \( q = 1 \), we obtain from (7.21)

\[
\left\{ \hat{A}(TM) \left( \prod_{v=1}^{l} \cos(\pi y_v) \right) \frac{1}{\cos^2(\pi u)} \right\}^{(8k+4)} = \sum_{r=0}^{k} a_r (-1)^{r+1} \delta_{2k-6r+1}^2
\]

This completes the proof of theorem 8.1.

Remark 8.6. In the \( 8k \)-dimensional case, take the \( 8k \)-degree component of all of the differential forms. Then lemma 8.5 shows that \( Q_2 \) is a modular form of weight \( 4k \). In (8.12) we therefore have to consider a linear combination

\[
Q_2 = \sum_{r=0}^{k} a_r \delta_{2k-6r+1}^2
\]

Again, the \( a_r \) may be expressed as a linear combination of elements of the coefficients of \( \Theta_2 \). If we evaluate at the cusp \( q = 1 \), we end up with (8.2).

8.5. The Original AGW Miraculous Cancellation Formula. Ochanine’s Generalization of Rokhlin’s Divisibility. The first \( b_t \) of (8.15) are

\[
b_0 = -B_0 \\
b_1 = B_1 - \left| \begin{array}{c}
k \\
1 \\
\end{array} \right| B_0 \\
b_2 = -B_2 + \left| \begin{array}{c}
k - 1 \\
2 \\
\end{array} \right| B_1 + \left( \left| \begin{array}{c}
k \\
2 \\
\end{array} \right| - \left| \begin{array}{c}
k - 1 \\
2 \\
\end{array} \right| \right) B_0 \\
b_3 = B_3 - \left| \begin{array}{c}
k - 2 \\
3 \\
\end{array} \right| B_2 + \left( \left| \begin{array}{c}
k - 1 \\
2 \\
\end{array} \right| - \left| \begin{array}{c}
k - 1 \\
3 \\
\end{array} \right| \right) B_1 \\
+ \left( \left| \begin{array}{c}
k \\
3 \\
\end{array} \right| + \left| \begin{array}{c}
k - 1 \\
3 \\
\end{array} \right| + \left| \begin{array}{c}
k - 2 \\
3 \\
\end{array} \right| - \left| \begin{array}{c}
k - 1 \\
2 \\
\end{array} \right| - \left| \begin{array}{c}
k - 1 \\
1 \\
\end{array} \right| - \left| \begin{array}{c}
k - 2 \\
3 \\
\end{array} \right| \right) B_0
\]
We will first specialize to the case $V = TM$. The calculation (now $\pm x_j$ (not $\pm 2\pi ix_j$) denote the new Chern roots of $T_M$

$$
\hat{A}(TM)Csh(V) = \prod x_j/2 \prod \sinh(y_j/2) \prod \cosh(y_j/2) = 2^{-4k-2} \prod \tanh(x_j/2)
$$

shows that (8.1) becomes

$$
\left\{ \frac{\hat{L}(TM)}{\cosh(c/2)} \right\}^{(8k+4)} \sum_{r=0}^{k} 2^{2k-6r+1} \left\{ \hat{A}(TM) \text{ch}(b_r) \cosh(c/2) \right\}^{(8k+4)}
$$

Assume $\xi$ is trivial. Then

$$
\left\{ \frac{\hat{L}(TM)}{\cosh(c/2)} \right\}^{(8k+4)} = 8 \sum_{r=0}^{k} 2^{6(k-r)} \left\{ \hat{A}(TM) \text{ch}(b_r) \right\}^{(8k+4)} \tag{8.16}
$$

This is a good point to recall two theorems of Atiyah and Hirzebruch [AH59] that are basic to all our applications of the cancellation formula.

**Theorem 8.7.** (Atiyah / Hirzebruch) Let $M$ be a closed smooth connected oriented manifold of even dimension. Let $c \in H^2(M; \mathbb{Z})$, and assume $M$ is a $c$-manifold (i.e. $c \equiv w_2(M) \bmod 2$) Then for any $E \in K(M)$

$$
\int_M \hat{A}(TM) \text{ch}(E)e^{c/2} \in \mathbb{Z}
$$

Complexification $E \mapsto E_\mathbb{C} = E \otimes \mathbb{C}$ clearly extends to $KO(M) \to K(M)$.

**Corollary 8.8.** Assume $E$ is the complexification $E = F_\mathbb{C}$ of some $F \in KO(M)$, and that $M$ is an $8k+4$ dimensional manifold as in theorem 8.7. Then

$$
\int_M \hat{A}(TM) \text{ch}(E) \cosh(c/2) \in \mathbb{Z}
$$

**Proof.** Each of $\hat{A}(TM), \text{ch}(E), \cosh(c/2)$ only has components in degrees that are multiples of 4. But $e^{c/2}$ and $\cosh(c/2)$ coincide in all such degrees.

**Theorem 8.9.** (Atiyah / Hirzebruch) Let $M$ be a closed smooth connected oriented manifold of dimension $\equiv 4 \bmod 8$. If $M$ is spin, then for any complexification $E = F_\mathbb{C}$

$$
\int_M \hat{A}(TM) \text{ch}(E) \in 2\mathbb{Z}
$$

**Corollary 8.10.** (Ochanine’s Generalization of Rokhlin’s Theorem) Let $M$ be a smooth closed spin manifold of dimension $8k+4$. Then the signature of $M$ is divisible by 16.

**Proof.** Using theorem 8.9 to see that $\int_M \left\{ \hat{A}(TM) \text{ch}(b_r) \right\}^{(8k+4)} \in 2\mathbb{Z}$, this follows immediately from (8.16).
For \( \xi \) trivial and \( V = TM \) the first coefficients of \( \Theta_2 \) are

\[
\Theta_2 = B_0 + B_1 q + B_2 q^2 + \cdots = C + \hat{T}_C M q + \left( \Lambda^2 \hat{T}_C M - (8k + 3) \hat{T}_C M \right) q^2 + \cdots \quad (8.17)
\]

Originally, (8.16) was stated only in the 12-dimensional case (so \( k = 1 \)). By definition (8.15) \( b_0 = -B_0 \) and \( b_1 = B_1 - \begin{vmatrix} 1 \\ 1 \end{vmatrix} \bigg| B_0 \). Table 1 shows \( \bigg| \begin{vmatrix} 1 \\ 1 \end{vmatrix} \bigg| = -72 \) and (8.17) gives \( B_0 = C, B_1 = T_C M - C^{12} \). (8.16) now reads

\[
\{ \hat{L}(TM) \}_{(12)} = 8 \left\{ \hat{A}(TM) \text{ ch}(-C) \right\}_{(12)} + \left\{ \hat{A}(TM) \text{ ch}(T_C M - C^{12} + 72C) \right\}_{(12)}
\]

**Theorem 8.11.** *(Alvarez-Gaumé-Witten)* Let \( M \) be a 12-dimensional closed manifold. Then

\[
\{ \hat{L}(TM) \}_{(12)} = \left\{ 8 \hat{A}(TM) \text{ ch}(T_C M) - 32 \hat{A}(TM) \right\}_{(12)}
\]

Note that Rokhlin’s Theorem is a direct consequence of this formula.

### 8.6. A Cancellation Formula in Dimension 20.\footnote{\textsuperscript{20}}

Now let \( k = 2 \). Then by (8.15) and table 1

\[
\begin{align*}
b_0 &= -B_0 \\
b_1 &= B_1 - \begin{vmatrix} 2 \\ 1 \end{vmatrix} B_0 = B_1 + 120B_0 \\
b_2 &= -B_2 + \begin{vmatrix} 1 \\ 2 \end{vmatrix} B_1 + \begin{vmatrix} 2 \\ 1 \end{vmatrix} B_0 \\
&= -B_2 - 80B_1 - 3720B_0
\end{align*}
\]

The \( B_0 = C, B_1 = T_C M - 20C, B_2 = \Lambda^2 T_C M - 19T_C M + 190C \) are again the coefficients in (8.17). The general cancellation formula (8.16) specializes to

\[
\{ \hat{L}(TM) \}_{(20)} = 8 \left\{ 2^{12} \left\{ \hat{A}(TM) \text{ ch}(b_0) \right\}_{(20)} + 2^6 \left\{ \hat{A}(TM) \text{ ch}(b_1) \right\}_{(20)} + \left\{ \hat{A}(TM) \text{ ch}(b_2) \right\}_{(20)} \right\}
\]

\[
= 8 \left\{ \hat{A}(TM) \left( 3 \text{ ch}(T_C M) - 6 - \text{ ch}(\Lambda^2 T_C M) \right) \right\}_{(20)}
\]
9. Ochanine’s Congruence Formula

Definition. Above in (8.7) we introduced
\[ \Theta_2 = \Theta_2(T\mathcal{C}M, V; \xi) = \bigotimes_{n=1}^{\infty} \frac{1}{\Lambda_{-q^{2n}}(T\mathcal{C}M)} \Lambda_{q^{2n}-1}(\mathcal{C} \xi) \Lambda_{-q^{2n}-1}(\mathcal{C} \xi) \Lambda_{q^{2n}}(\mathcal{C} \xi) \in K^{\nabla}(M)[q] \]
If necessary to avoid confusion, we will also write \( B_t = B_t(T\mathcal{C}M, V; \xi), \ b_t = b_t(T\mathcal{C}M, V, \xi) \). Define now for an arbitrary vector bundle \( E \)
\[ \Theta_2(E) := \Theta_2(E, E; \xi^2) = \bigotimes_{n=1}^{\infty} \frac{\Lambda_{q^{2n}-1}(\mathcal{E})}{\Lambda_{-q^{2n}}(\mathcal{E})} \]
Because \( \Lambda_1 \) is a homomorphism and from the rule \( \mathcal{E} \oplus \mathcal{F} = \mathcal{E} \oplus \mathcal{F} \) it is clear that
\[ \Theta_3(E \oplus F) = \Theta_2(E) \oslash \Theta_2(F) \]
Therefore \( \Theta_2 \) may be extended to \( K^{\nabla}(M) \to K^{\nabla}(M)[q] \), and the same is now also true for the \( q^l \) component \( B_t = B_t(E) \) of \( \Theta_2 \) (and also \( b_t = b_t(E) \)).

Theorem 9.1. Let \( M \) be a \( 8k + 4 \) dimensional closed oriented Riemannian manifold. Assume that \( M \) is Spin* (i.e. in the formulation of Atiyah / Hirzebruch a \( c := c_1(TM) \) manifold) and let \( B \) be a codimension \( 2 \) closed oriented submanifold of \( M \) realizing the cohomology class \( c \) (meaning that \( [B]_M \) is Poincaré dual to \( c \). Such a submanifold may always be constructed) Then
\[ \frac{\text{sgn}(M) - \text{sgn}(B \cdot B)}{8} \equiv \int_M \hat{A}(M) \text{ch}(b_k(T\mathcal{C}M, \xi)) \text{cosh}(c/2) \mod 64 \]  
(9.1)

Proof. Here, \( B \cdot B \) denotes the self-intersection of \( B \) in \( M \). It may be considered as the codimension \( 2 \) closed oriented submanifold of \( B \) realizing the cohomology class \( c |_B \). Using the formula for the virtual index twice (6.6) we obtain
\[ \text{sgn}(B \cdot B) = \int_M \hat{L}(M) \tanh^2(c) \]
Therefore
\[ \frac{\text{sgn}(M) - \text{sgn}(B \cdot B)}{8} = \int_M \frac{\hat{L}(M)}{\cosh^2(c)} \]
From the cancellation formula (8.16) it now follows that
\[ \frac{\text{sgn}(M) - \text{sgn}(B \cdot B)}{8} = 8 \sum_{r=0}^{k} \frac{1}{2^{6(k-r)}} \left\{ \hat{A}(M) \text{ch}(b_r) \text{cosh}(c/2) \right\} \]
But by theorem 8.7 every summand \( r < k \) is divisible by \( 2^6 = 64 \). This completes the proof.

Lemma 9.2. Let \( E \) be a vector bundle, and write \( \mathbb{Z}[E] \) for the subring generated in \( K^{\nabla}(M) \) by all \( \Lambda^p E, \ p \geq 0 \). Then \( \frac{\Lambda^p(E)}{\Lambda^{p+1}(E)} - 1 \in \mathbb{Z}[E][t] \)

Proof. If we write \( \frac{\Lambda^p(E)}{\Lambda^{p+1}(E)} = 1 + A_1 t + A_2 t^2 + \cdots \) then by comparing the degree \( n \) component of \( \Lambda_1 E = (\sum A_i t^i) \Lambda_{-1} E \)
\[ \Lambda^n E = A_n - A_{n-1} E + A_{n-2} A^2 E - \cdots + (-1)^{n-1} A_1 A^{n-1} E + (-1)^n A^n E \]
The result now follows by induction: For \( n = 1 \) we have \( E = A_1 - E \). Let \( n > 1 \).
The inductive hypothesis applies to everything but the first summand of
\[
A_n = (1 - (-1)^n) \Lambda^n E + A_{n-1}E + \ldots + (-1)^n A_1 \Lambda^{n-1} E
\]
Note however that \( 1 - (-1)^n \) is always even.

\[ \square \]

Remark 9.3. The parity of the right-hand side of (9.1) may be interpreted as the Ochanine invariant \( \phi(B) \). This follows from the “analytic interpretation” of Ochanine’s invariant, given by Liu/Zhang [LZ94]. Let \( \xi \) be the normal bundle of \( B \) in \( M \). They established the following result by \( \eta \)-invariant and adiabatic limit methods:
\[
\phi(B) \equiv \int_M \hat{A}(M) \operatorname{ch}\left(\tilde{b}_1(T_C M + C^2 - \xi_C)\right) \cosh(c/2) \mod 2\mathbb{Z}
\]
Here, \( \tilde{b}_1(E) \) are defined as integer linear combinations of the coefficients \( \tilde{B}_1(E) \) of the power series
\[
\tilde{\Theta}_2(E) = \bigotimes_{n=1}^{\infty} \frac{\Lambda_{-q^2 n-1}(\tilde{E})}{\Lambda_{-q^2 n}(E)} = 1 + \tilde{B}_1(E)q + \ldots \in K^\nabla(M)[q]
\]
whose domain of definition may be extended to \( K^\nabla(M) \) similarly as above. By the lemma, \( \Theta_2(T_C M, T_C M; \xi_C) \) differs from \( \tilde{\Theta}_2(T_C M + C^2 - \xi_C) \) only by a factor \( 1 + 2f(q) \) for some power series \( f \) having only exterior powers of \( T_C M, \xi_C \) and the trivial bundle as coefficients.
\[
\tilde{\Theta}_2(T_C M + C^2 - \xi_C) = \Theta_2(T_C M, T_C M; \xi_C) \bigotimes_{n=1}^{\infty} \frac{\Lambda_{-q^2 n-1}(T_C M)}{\Lambda_{-q^2 n}(T_C M)} \otimes \frac{\Lambda_{-q^2 n}(\xi_C)}{\Lambda_{-q^2 n}(\xi_C)} \otimes \frac{\Lambda_{-q^2 n-1}(2\xi_C)}{\Lambda_{-q^2 n-1}(2\xi_C)}
\]
Theorem 8.7 applies to all of the coefficients of \( f(q) \), and it follows that the right hand side (9.1) is congruent mod 2 to \( \phi(B) \).
References


