Almost Kähler 4-manifolds of Constant Holomorphic Sectional Curvature are Kähler

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Joint work with

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Based on discussions with Luigi Vezzoni

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Preliminaries

Definition
An almost Kähler manifold \((M, \omega, g, J)\) is equipped with

\[ \omega \in \Omega^2(M), \quad J: TM \to TM, \quad g \text{ metric} \]

such that

\[ d\omega = 0, \quad J^2 = -1, \quad \omega = g(J\cdot, \cdot). \]
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Definition
The Hermitian connection (or Chern connection) is
\[ \nabla_X Y := D^g_X Y - \frac{1}{2} J(D^g_X J) Y. \]

- \(\nabla g = 0, \nabla J = 0\), but \(\nabla\) may have torsion.
Holomorphic sectional curvature

Hermitian curvature tensor $R^\nabla \in \Lambda^2 \otimes \Lambda^{1,1}$ has fewer symmetries.

The Hermitian holomorphic sectional curvature is

$$H(X) := |X|^{-4} \cdot R^\nabla_{X,JX,X,JX}, \quad X \in TM.$$ 

It is called

1. constant at $p$ if $H(X) = k_p$ for all $X \in T_pM$,
2. globally constant if $H(X) = k$ for all $X \in TM$. 

Problem (Gray–Vanhecke 1979)

Classify all manifolds of globally constant holomorphic sectional curvature within your favourite class of almost Hermitian manifolds.
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1. **constant** at \( p \) if \( H(X) = k_p \) for all \( X \in T_pM \),
2. **globally constant** if \( H(X) = k \) for all \( X \in TM \).

Problem (Gray–Vanhecke 1979)

*Classify all manifolds of globally constant holomorphic sectional curvature within your favourite class of almost Hermitian manifolds.*
Statement of Result

Theorem (U.–Lejmi, 2017)

Let $M$ be a closed almost Kähler 4-manifold of globally constant Hermitian holomorphic sectional curvature $k \geq 0$.

Then $M$ is Kähler–Einstein, holomorphically isometric to:

1. $(k > 0)$ $\mathbb{C}P^2$ with the Fubini–Study metric.
2. $(k = 0)$ a complex torus or a hyperelliptic curve with the Ricci-flat Kähler metric.

Similar result for $k < 0$ under assumption that Ricci is $J$-invariant.
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Remark

The above conclusion is known for Kähler manifolds, so we just need to prove integrability.
Background

Related Work

**Balas–Gauduchon 1985** Any Hermitian metric of constant nonpositive (Hermitian) holomorphic sectional curvature on a compact complex surface is Kähler
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Lejmi–Vezzoni 2017  Left-invariant structures on almost Kähler 4-dimensional Lie algebras.
Proposition

Pointwise constant holomorphic sectional curvature $H = k$ is equivalent to

1. $W^- = 0$

2. $\ast \rho = r$ for two Ricci contractions of $R^\nabla$:

$$\rho_{\alpha\bar{\beta}} = i R^\nabla_{\alpha\bar{\beta}\gamma} \quad \gamma,$$

$$r_{\alpha\bar{\beta}} = i R^\nabla_{\gamma\alpha\bar{\beta}},$$

Moreover,

$$\nu := \frac{\text{Scal}^g}{12} \leq \frac{k}{2}$$

with equality if and only if $M$ is Kähler.
Sketch of Proof for $W^- = 0$

Use

\[
R_{XYZW} = R_{XYZW}^g + g((\nabla_X A_Y - \nabla_Y A_X - A_{[X,Y]} )Z, W) - g([A_X, A_Y] Z, W).
\]

Play off the symmetries of $R^g : \Lambda^2 \rightarrow \Lambda^2$ against the assumption on $R^\nabla$ (which gives it a special form).

\[
R^g = \begin{bmatrix}
\Lambda^+ & \Lambda^-\\
\mathcal{W}^+ + \frac{\text{Scal}^g}{12} g & R^T_0 + \frac{\text{Scal}^g}{12} g\\
R_0 & \mathcal{W}^- + \frac{\text{Scal}^g}{12} g
\end{bmatrix}, \quad \Lambda^2 = \Lambda^+ \oplus \Lambda^-
\]
Sketch of Proof for $W^- = 0$

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$$R^\nabla_{XYZW} = R^g_{XYZW} + g((\nabla_X A_Y - \nabla_Y A_X - A_{[X,Y]} Z, W) - g([A_X, A_Y] Z, W).$$

Play off the symmetries of $R^g : \Lambda^2 \rightarrow \Lambda^2$ against the assumption on $R^\nabla$ (which gives it a special form).

$$R^g = \begin{bmatrix} \mathbb{R} \omega & \Lambda^{(2,0)+(0,2)} & \Lambda^{1,1}_0 \\ d \cdot g & W^+_F & R_F \\ (W^+_F)^T & W^+_0 + \frac{c}{2} g & R^T_0 \\ R^T_F & R^T_{00} & W^- + \frac{\text{Scal}^g}{12} g \end{bmatrix}$$
Sketch of Proof for $W^{-} = 0$

Use

$$R_{XYZW}^\nabla = R_{XYZW}^g$$


$$\alpha \in \Lambda^2 \otimes \Lambda^{2,0+0,2}$$

$$\beta \in \Lambda^{1,1} \otimes R \cdot \omega$$

Play off the symmetries of $R^g : \Lambda^2 \rightarrow \Lambda^2$ against the assumption on $R^\nabla$ (which gives it a special form).

$$R^\nabla = \begin{bmatrix}
\mathbb{R} \omega & \Lambda^{(2,0)+(0,2)} & \Lambda^{1,1}_0 \\
\frac{s c}{2} g & 0 & * \\
?? & 0 & ?? \\
* & 0 & *
\end{bmatrix}$$
Sketch of Proof for $W^- = 0$

Use

$$R_{XYZW}^\nabla = R_{XYZW}^g$$


Play off the symmetries of $R^g : \Lambda^2 \to \Lambda^2$ against the assumption on $R^\nabla$ (which gives it a special form).

$$R^\nabla = \begin{bmatrix}
R_\omega & \Lambda^{(2,0)+(0,2)} & \Lambda_{0}^{1,1} \\
\frac{sc}{2} g & 0 & R_F \\
(W_F^\perp)^T & 0 & R_{00} \\
-R_F^T & 0 & \frac{s_g}{12} g
\end{bmatrix}$$
From the differential Bianchi identity:

**Proposition**

*Let $M$ be a closed almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature $k$. Then*

\[
\int_M |R_{00}|^2 = \int_M |W_F^+|^2 + |W_{00}^+|^2 + 4(5k - 7\nu)(k - 2\nu) \tag{1}
\]

\[
\chi = \frac{-1}{8\pi^2} \int_M |W_{00}^+|^2 + (60\nu^2 - 72k\nu + 18k^2) \tag{2}
\]

\[
\frac{3}{2} \sigma = \frac{1}{8\pi^2} \int_M 2|W_F^+|^2 + |W_{00}^+|^2 + 6(2k - 3\nu)^2 \geq 0 \tag{3}
\]

Recall: $\nu := \frac{\text{Scal}^g}{12} = \frac{k}{2}$ implies Kähler.
Corollary (Signature zero case)

Let $M$ be closed almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature $k$. Suppose $\sigma = 0$.

Then $k = 0$ and $M$ is Kähler, with a Ricci-flat metric.
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Let $M$ be closed almost Kähler 4-manifold of pointwise constant holomorphic sectional curvature $k$. Suppose $\sigma = 0$.

Then $k = 0$ and $M$ is Kähler, with a Ricci-flat metric.

Proof.

1. From $0 = \frac{3}{2} \sigma = \frac{1}{8\pi^2} \int_M 2|W_F^+|^2 + |W_{00}^+|^2 + 6(2k - 3\nu)^2$ we get $W_F^+ = 0, W_{00} = 0, 2k = 3\nu$.

2. Put this into

$$\int_M |R_{00}|^2 = \int_M |W_F^+|^2 + |W_{00}^+|^2 + 4(5k-7\nu)(k-2\nu) = -\frac{4}{9}k^2 \text{Vol}(M)$$

to get $k = \nu = 0$. 

\[ \blacksquare \]
Corollary (‘Reverse’ Bogomolov–Miyaoka–Yau inequality)

If $M$ is closed almost Kähler of globally constant holomorphic sectional curvature $k \geq 0$, then for the Euler characteristic

$$3\sigma \geq \chi.$$ 

Equality holds if and only if $M$ is Kähler–Einstein.
End of the proof

Theorem
$M^4$ closed almost Kähler of constant holomorphic sectional curvature $k \geq 0$. Then $M$ is Kähler.
End of the proof

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$M^4$ closed almost Kähler of constant holomorphic sectional curvature $k \geq 0$. Then $M$ is Kähler.

Proof.

- Suppose that $M$ is not Kähler: $v < \frac{k}{2}$ somewhere.
End of the proof

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\(M^4\) closed almost Kähler of constant holomorphic sectional curvature \(k \geq 0\). Then \(M\) is Kähler.

Proof.

- Suppose that \(M\) is not Kähler: \(v < \frac{k}{2}\) somewhere.

\[
\int_M c_1(TM) \cup \omega = \int_M \frac{sc}{2\pi} = \int_M \frac{3k}{2\pi} + \int_M \frac{k-2v}{2\pi} > 0.
\]

\[\geq 0\]
End of the proof

Theorem

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Proof.

- Suppose that $M$ is not Kähler: $v < \frac{k}{2}$ somewhere.
- $\int_M c_1(TM) \cup \omega = \int_M \frac{s_c}{2\pi} = \int_M \frac{3k}{2\pi} + \int_M \frac{k-2v}{2\pi} \geq 0.$
- SW-theory $\implies M$ symplectom. to ruled surface or $\mathbb{C}P^2$
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- $M = \mathbb{C}P^2$ has $3\sigma = \chi$. 
End of the proof

Theorem

$M^4$ closed almost Kähler of constant holomorphic sectional curvature $k \geq 0$. Then $M$ is Kähler.

Proof.

- Suppose that $M$ is not Kähler: $\nu < \frac{k}{2}$ somewhere.
- $\int_M c_1(TM) \cup \omega = \int_M \frac{sc}{2\pi} = \int_M \frac{3k}{2\pi} + \int_M \frac{k-2\nu}{2\pi} > 0.$

- SW-theory $\Rightarrow$ $M$ symplectom. to ruled surface or $\mathbb{C}P^2$
- $M = \mathbb{C}P^2$ has $3\sigma = \chi$.
- $M$ rational $\Rightarrow$ $\sigma \leq 0 \Rightarrow \sigma = 0.$
End of the proof

**Theorem**

\( M^4 \) closed almost Kähler of constant holomorphic sectional curvature \( k \geq 0 \). Then \( M \) is Kähler.

**Proof.**

- Suppose that \( M \) is not Kähler: \( v < \frac{k}{2} \) somewhere.
- \( \int_M c_1(TM) \cup \omega = \int_M \frac{sc}{2\pi} = \int_M \frac{3k}{2\pi} + \int_M \frac{k-2v}{2\pi} \geq 0 \).
- SW-theory \( \implies \) \( M \) symplectom. to ruled surface or \( \mathbb{C}P^2 \)
- \( M = \mathbb{C}P^2 \) has \( 3\sigma = \chi \).
- \( M \) rational \( \implies \sigma \leq 0 \implies \sigma = 0 \).
- By previous propositions, ‘=’ implies Kähler, contradiction!