Orientation problems for PDEs and instanton moduli spaces

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Overview

1. Background on gauge theory
2. Special holonomy and PDEs
3. Instanton moduli spaces
4. Results and techniques
Overview

1. Background on gauge theory
2. Special holonomy and PDEs
3. Instanton moduli spaces
4. Results and techniques
Gauge theory

Fix a Hermitian vector bundle $\pi: E \to M$ over a manifold.

- $M$ covered by local gauges $\Phi_\alpha: E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^m$, but not global.
- Unique up to local gauge transforms $\gamma: U_\alpha \to U(m)$. In particular,

$$\Phi_\alpha = \gamma_{\alpha\beta} \cdot \Phi_\beta \quad \text{over} \quad U_\alpha \cap U_\beta.$$
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A section $s: M \to E$ is a map with $\pi \circ s = \text{id}_M$. In a local gauge corresponds to vector-valued $f_\alpha = \Phi_\alpha \circ s: U_\alpha \to \mathbb{C}^m$ satisfying $f_\alpha = \gamma_{\alpha\beta} \cdot f_\beta$ over $U_\alpha \cap U_\beta$.

**Remark**

*Can replace $U(m)$ by gauge Lie group $G$; mostly $G = SU(2), U(2), SO(3)$*
Connections and curvature

Definition

A connection $A$ is a family $A_\alpha \in \Omega^1(U_\alpha; u(m))$ satisfying

$$A_\beta = \gamma_{\alpha\beta}^{-1} \cdot A_\alpha \cdot \gamma_{\alpha\beta} + \gamma_{\alpha\beta}^{-1} \cdot d\gamma_{\alpha\beta}.$$
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- Global ‘covariant’ derivative $\nabla_A s := \Phi^{-1}_\alpha(df_\alpha + A_\alpha \cdot f_\alpha)$
- Connections differ by 1-form with values in $u_E = \text{End}^\text{skew}_\mathbb{C}(E)$
- Space of connections $A_E$ modelled on $\Omega^1(M; u_E)$, Fréchet space topology
- Curvature $F_A := dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(M; u_E)$
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**Definition**

A gauge transformation $\Psi \in \mathcal{G}_E$ is a unitary bundle automorphism $\Psi : E \to E$.

- $(\nabla_{\Psi^* A}) s := \Psi^{-1}\nabla_A(\Psi \circ s)$, so $\mathcal{G}_E \curvearrowright \mathcal{A}_E$
- $F_{\Psi^* A} = \Psi^{-1} F_A \Psi$
Fix Hermitian vector bundle $E \rightarrow M$. Low-dimensional ‘elliptic’ examples:

**Example**

For $\dim M = 2$ and $\dim M = 3$, $A \in \mathcal{A}_E$ is a flat connection if $F_A = 0$. 

In 4D, the problem is Fredholm modulo gauge. When $M$ has special holonomy, there are natural analogues of these non-linear PDEs for $\dim M > 4$. 

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Orientations for PDE moduli spaces
Instanton PDEs

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**Example**

\( \dim M = 4 \) with oriented Riemannian structure \( \implies \exists \) Hodge operator on forms with \( *^2|_{\Omega^2} = 1 \implies \Omega^2 = \Omega^+ \oplus \Omega^- \). \( A \in \mathcal{A}_E \) is an ASD-connection if

\[ *F_A = -F_A \iff F_A^+ = 0. \]
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- In 4D, the problem is Fredholm modulo gauge.
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Definition

- $\phi \in \Lambda^3 V^*$ on a 7-dimensional vector space is non-degenerate if

$$\iota_X \phi \wedge \iota_X \phi \wedge \phi \neq 0 \quad \forall X \in V \setminus \{0\}.$$
\(G_2\)-structures

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  \[ g_{\phi}(X, Y) \text{vol}_{g_{\phi}} = \iota_X \phi \wedge \iota_Y \phi \wedge \phi. \]
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- A $G_2$-structure on a manifold $M^7$ is a non-degenerate smooth 3-form $\phi$.

- A $G_2$-structure is torsion-free $\iff d\phi = 0$ and $d(\star_{g_\phi} \phi) = 0$. 

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Orientations for PDE moduli spaces

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**G\textsubscript{2}-structures**

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- A **\( G\textsubscript{2} \)-structure** on a manifold \( M^7 \) is a non-degenerate smooth 3-form \( \phi \).

- A \( G_2 \)-structure is **torsion-free** \( \iff \) \( d\phi = 0 \) and \( d(*_{g_\phi} \phi) = 0 \).

**Example (prototype)**

In coordinates on \( V = \mathbb{R}^7 \) we have

\[
\phi_{\text{std}} = dx^{123} + dx^1 (dx^{45} + dx^{67}) + dx^2 (dx^{46} - dx^{57}) + dx^3 (dx^{47} + dx^{56})
\]

Then \( G_2 := \left\{ A \in \text{GL}(7, \mathbb{R}) \mid A^* \phi_{\text{std}} = \phi_{\text{std}} \right\} \). Alternatively, \( G_2 = \text{Aut}(\mathbb{O}) \).
Example

We have embeddings of Lie groups

\[ \text{SU}(2) \to \text{SU}(3) \to G_2 = \text{Aut}(\mathbb{O}). \]

Hyperkähler surface \((X^4, \omega_1, \omega_2, \omega_3)\) yields torsion-free \(G_2\)-manifold \(T^3 \times X\)

\[ \phi := dx^{123} - dx^1 \wedge \omega_1 - dx^2 \wedge \omega_2 - dx^3 \wedge \omega_3 \]
Relation to other geometries

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2. If \((Z^6, \omega, \Omega)\) is a Calabi–Yau 3-fold, then \(S^1 \times Z\) is a torsion-free \(G_2\)-manifold

\[ \phi := dt \wedge \omega + \text{Re}(\Omega) \]
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\[ \phi := dt \wedge \omega + \Re(\Omega) \]

3. Examples with holonomy all of \(G_2\) are hard to construct.
Donaldson–Segal: Elliptic PDEs in higher dimensions

Definition

Let \((M^7, \phi)\) be a \(G_2\)-manifold and \(E \to M\) a Hermitian vector bundle. Then a connection \(A \in \mathcal{A}_E\) is a \(G_2\)-instanton if

\[ *(F_A \wedge \phi) = -F_A \iff F_A \wedge (*\phi) = 0. \]
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1. Levi-Civita connection of torsion-free \(G_2\)-manifold
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**Example**

1. **Levi-Civita connection** of torsion-free $G_2$-manifold

2. **ASD-connections** $*F_A = -F_A$ on hyperkähler 4-manifold $X$ with $G_2$-structure

   $$\phi := dx^{123} - dx^1 \wedge \omega_1 - dx^2 \wedge \omega_2 - dx^3 \wedge \omega_3 \quad (\omega_i \text{ self-dual}), \quad M = T^3 \times X$$

   
   $$*_M(F_A \wedge \phi) = *_M(F_A \wedge dx^{123}) = *_X F_A = -F_A.$$
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1. Levi-Civita connection of torsion-free \(G_2\)-manifold
2. ASD-connections \(*F_A = -F_A\) on hyperkähler 4-manifold \(X\) with \(G_2\)-structure
   \[ \phi := dx^{123} - dx^1 \wedge \omega_1 - dx^2 \wedge \omega_2 - dx^3 \wedge \omega_3 \text{ (\(\omega_i\) self-dual), } M = T^3 \times X \]
   \[ *_M(F_A \wedge \phi) = *_M (F_A \wedge dx^{123}) = *_X F_A = -F_A. \]
3. Hermitian Yang–Mills connections \(\Lambda F_A = 0, F_A^{0,2} = 0\) on Calabi–Yau 3-fold (connects to algebraic geometry: Donaldson–Thomas program)
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Fix Hermitian vector bundle $E \to M$.

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**Definition (Moduli of instantons)**

- $\mathcal{M}^{\text{flat}}_E := \{A \in \mathcal{A}_E \mid F_A = 0\} / \mathcal{G}_E$ \hspace{1cm} (3D)
- $\mathcal{M}^{\text{ASD}}_E := \{A \in \mathcal{A}_E \mid F_A^+ = 0\} / \mathcal{G}_E$ \hspace{1cm} (4D oriented)
- $\mathcal{M}^{\text{CY}}_E := \{A \in \mathcal{A}_E \mid \Lambda F_A = 0, F_A^{0,2} = 0\} / \mathcal{G}_E$ \hspace{1cm} (CY3)
- $\mathcal{M}^{G^2}_E := \{A \in \mathcal{A}_E \mid F_A \wedge *\phi = 0\} / \mathcal{G}_E$ \hspace{1cm} (7D $G^2$-manifold)

Remark (difficulties: isotropy, non-transversality)

Reducible solutions $\Rightarrow$ quotient stacks

For 4D and generic metrics on $M$, $\mathcal{M}^{\text{ASD}}_E$ is a smooth manifold of expected positive dimension (regular value).

Similar: $\mathcal{M}^{\text{SU}(3)}_E$ in 6D.

In general: derived stacks (for example, $\mathcal{M}^{\text{flat}}_E$, $\mathcal{M}^{G^2}_E$ of virtual dim. zero).
Moduli spaces

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\[ \mathcal{M}_{E}^{\text{CY}} := \left\{ A \in \mathcal{A}_E \mid \Lambda F_A = 0, F^{0,2}_A = 0 \right\} / \mathcal{G}_E \quad (\text{CY3}) \]

\[ \mathcal{M}_{E}^{G_2} := \left\{ A \in \mathcal{A}_E \mid F_A \wedge \ast \phi = 0 \right\} / \mathcal{G}_E \quad (7\text{D } G_2\text{-manifold}) \]

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- $\mathcal{M}^G_2 := \{A \in \mathcal{A}_E \mid F_A \wedge \ast \phi = 0\} \sslash \mathcal{G}_E$ (7D $G_2$-manifold)

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Atiyah–Hitchin deformation complex

Fix torsion-free \( G_2 \)-manifold \((M^7, \phi)\) and a \( G_2 \)-instanton \( F_A \wedge * \phi = 0 \) on \( E \).
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For deformation $A + a$ with $a \in \Omega^1(M; u_E)$:

$$0 = F_{A+a} \wedge \ast \phi = F_A \wedge \ast \phi + d_A a \wedge \ast \phi + a \wedge a \wedge \ast \phi$$

$\Rightarrow$ linearized $G_2$-instanton equation $d_A a \wedge \ast \phi = 0$
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- Infinitesimal gauge transformations $\gamma \in \Omega^0(M; u_E) = \text{Lie } G_E$ act
  
  \[ d_A(a + d_A \gamma) \wedge \ast \phi = d_A a \wedge \ast \phi + \gamma \wedge F_A \wedge \ast \phi = 0 \]
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Exact $G_2$-instanton deformation complex

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(M; u_E) & \overset{d_A}{\longrightarrow} & \Omega^1(M; u_E) & \overset{d_A \wedge *\phi}{\longrightarrow} & \Omega^6(M; u_E) & \overset{d_A}{\longrightarrow} & \Omega^7(M; u_E) & \longrightarrow & 0
\end{array}
\]

gauge transf. connections curvature Bianchi
Determinants and orientations

Infinitesimal theory of $\mathcal{M}_{E}^{G_{2}}$ at $A$:

$$
0 \longrightarrow \Omega^{0}(M; u_{E}) \xrightarrow{d_{A}} \Omega^{1}(M; u_{E}) \xrightarrow{d_{A} \wedge \ast \phi} \Omega^{6}(M; u_{E}) \xrightarrow{d_{A}} \Omega^{7}(M; u_{E}) \longrightarrow 0
$$

- Roll-up is elliptic operator

$$
\Phi_{u(A)} = \begin{pmatrix}
0 & d_{A}^{*} \\
d_{A} & \ast(d_{A} \wedge \ast \phi)
\end{pmatrix} : \Omega^{0} \oplus \Omega^{1} \rightarrow \Omega^{0} \oplus \Omega^{1},
$$

Diracian on $M$, twisted by $u_{E}$ and $A$
Determinants and orientations

Infinitesimal theory of $\mathcal{M}^2_E$ at $A$:

$$0 \longrightarrow \Omega^0(M; u_E) \xrightarrow{d_A} \Omega^1(M; u_E) \xrightarrow{d_A \wedge \ast \phi} \Omega^6(M; u_E) \xrightarrow{d_A} \Omega^7(M; u_E) \longrightarrow 0$$

- Roll-up is elliptic operator
  $$\mathcal{D}_{u(A)} = \begin{pmatrix} 0 & d^*_A \\ d_A & \ast (d_A \wedge \ast \phi) \end{pmatrix} : \Omega^0 \oplus \Omega^1 \rightarrow \Omega^0 \oplus \Omega^1,$$

  - Diracian on $M$, twisted by $u_E$ and $A$

- Zariski tangent space $T_A \mathcal{M}^2_E = \text{Ker} \mathcal{D}_{u(A)}$ of derived stack
Determinants and orientations

Infinitesimal theory of $\mathcal{M}_E^{G_2}$ at $A$:

$$0 \longrightarrow \Omega^0(M; u_E) \xrightarrow{d_A} \Omega^1(M; u_E) \xrightarrow{d_A \wedge * \phi} \Omega^6(M; u_E) \xrightarrow{d_A} \Omega^7(M; u_E) \longrightarrow 0$$

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**Definition**

Let $\{D_y\}_{y \in \mathcal{Y}}$ be a $\mathcal{Y}$-family of $\mathbb{K}$-linear Fredholm operators. The **Quillen determinant line bundle** is

$$\text{det}_\mathbb{K}\{D_y\} := \bigcup_{y \in \mathcal{Y}} \Lambda_{\mathbb{K}}^{\text{top}}(\text{Ker } D_y) \otimes \Lambda_{\mathbb{K}}^{\text{top}}(\text{Coker } D_y)^* \xrightarrow{\tau} \mathcal{Y}.$$
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**Conclusion:** orientation bundle of $\mathcal{M}_E^{G_2}$ is $\text{Det}_R \mathcal{D}_{u(E)} := \text{det}_R \{\mathcal{D}_{u(A)}\}_{A \in \mathcal{M}_E^{G_2}}$.  

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Results

**Theorem (Joyce–U. 2018)**

Let $E \to M$ be an $\text{SU}(m)$-bundle over a closed $G_2$-manifold. A flag structure on $M$ determines a canonical orientation for $\mathcal{M}_E^{G_2}$.

- Flags are differential-topological structures on $M$ akin to spin structures.
Results

**Theorem (Joyce–U. 2018)**

Let $E \to M$ be an $SU(m)$-bundle over a closed $G_2$-manifold. A flag structure on $M$ determines a canonical orientation for $\mathcal{M}_E^{G_2}$.

- Flags are differential-topological structures on $M$ akin to spin structures.

Side results of our general theory:


Let $E \to M$ be a $G$-principal bundle over a Riemannian 3-manifold. An orientation of $H^0(M) \oplus H^1(M) \oplus H^2(M) \oplus H^3(M)$ determines a canonical orientation of the flat connection moduli space $F_A = 0$. 
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**Theorem (Donaldson 1987, Kronheimer, Joyce–U. 2018)**

Let $E \rightarrow M$ be an $G$-principal bundle over a spin 4-manifold. An orientation of $H^0(M) \oplus H^1(M) \oplus H^+(M)$ determines a canonical orientation for $\mathcal{M}_E^{\text{ASD}}$. 
Let $E \rightarrow M$, $E' \rightarrow M'$ be Hermitian vector bundles over closed spin manifolds.

1. Let $\phi$ be a spin diffeomorphism of open subsets

\[ M \supset U \xrightarrow{\phi} U' \subset M'. \]

2. Let $s$ and $s'$ be unitary frames of $E|_{M\setminus K}$ and $E'|_{M'\setminus K'}$ defined outside compact subsets $K \subset U$ and $K' \subset U'$.

3. Let $\Phi: E|_U \to \phi^* E'|_{U'}$ be an isomorphism with $\Phi(s) = \phi^* s'$.

Then we get an excision isomorphism

\[ \text{Det}_R D_{U(E)} \xrightarrow{\text{Det}(\Phi,s,s')} \text{Det}_R D_{U(E')} \]
framing $E \subset U$

framed isomorphism $\phi$

$\text{Det}_R \mathcal{D}_{u(E)} \xrightarrow{\text{Det}(\Phi, s, s')} \text{Det}_R \mathcal{D}_{u(E')} $
Flag structures

Definition

A flag structure on $M^7$ associates signs $F(Y, s)$ to submanifolds $Y^3 \subset M$ with non-vanishing normal sections $s$ such that

$$F(Y_0, s_0) = (-1)^{D(s_0, s_1)} F(Y_1, s_1) \quad \forall [Y_0] = [Y_1].$$

Here $D(s_0, s_1)$ is defined as follows. Let $\partial Z = [Y_1] - [Y_0]$. Then $D(s_0, s_1) = Z \cdot (s_0 - s_1)$ is the intersection number of $Z$ with perturbations of $Y_0$ and $Y_1$ in direction of $s_0$ and $s_1$. A flag structure $F$ is notion of parity for $(Y, s)$. It reduces choices by picking out a normal section, up to parity.

Proposition

The set of flag structures is a (non-empty) torsor over $H_3(M; \mathbb{Z}_2)$. 

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Orientations for PDE moduli spaces
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**Proposition**

*The set of flag structures is a (non-empty) torsor over $H^3(M; \mathbb{Z}_2)$.***
Definition

The pullback of a flag structure $\mathcal{F}$ on $M$ along a diffeomorphism $\phi: M' \to M$ is $(\phi^* \mathcal{F})(Y', s') := (\phi(Y'), d\phi(s'))$. 

Proposition

Let $\phi: M \to M$ be an orientation-preserving isometry with $\phi|_Y = \text{id}_Y$ for an oriented 3-submanifold $Y \subset M$. Then $(\mathcal{F}/\phi^* \mathcal{F})[Y] = \mathcal{F}(Y, s)$ equals the self-intersection of $[Y \times S^1] \cdot [Y \times S^1]$ in the mapping torus $M_{\phi} := M \times \mathbb{R}$. 

Theorem (Atiyah–Patodi–Singer)

Let $\Phi: E \to E$ be a bundle automorphism over a spin diffeomorphism $\phi: M \to M$ of an odd-dimensional manifold. Then $\det R(\Phi) = (-1)^{\delta(\Phi)} \cdot \text{id}$, $\delta(\Phi) := \int_M \hat{\text{A}}(TM_{\phi}) \text{ch}(E^* \Phi \otimes E \Phi)$, for the mapping torus bundle $E_{\Phi} \downarrow M_{\phi}$. 

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for the mapping torus bundle $E_\phi \downarrow M_\phi$. 
Future research

- Donaldson–Segal propose Casson invariants of $G_2$-manifolds by counting all $G_2$-instantons with appropriate signs, which we have found.

- Gradings over $\mathbb{Z}_8$ rather than $\mathbb{Z}_2$ for a Floer homology type theory

- Applications to algebraic geometry: every Calabi–Yau 3-fold $Z$ determines a $G_2$-manifold $M = S^1 \times Z$. By a new dimension shifting technique, use this to construct ‘orientation data’ for Calabi–Yau 3-folds, square roots $\sqrt{\text{Det}_\mathbb{C} \bar{\partial}_u(E)}$, an important problem in algebraic geometry (Kontsevich).

- Non-simply connected gauge groups, e.g. $U(m)$