Orientation Problems in 7-dimensional Gauge Theory

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Talk based on:

1) M. Upmeier, A categorified excision principle for elliptic symbol families (soon)
3) D. Joyce and M. Upmeier, Canonical orientations for moduli spaces of $G_2$-instantons with gauge group SU($m$), arXiv:1811.02405.

January 28, 2019
Outline

Orientation Problems for Twisted Diracians

Determinants, Symbols, and Excision

Canonical Orientations in Seven Dimensions
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Orientation Problems for Twisted Diracians

Determinants, Symbols, and Excision

Canonical Orientations in Seven Dimensions
Twisted Dirac Operators

Setup

1. Compact 7-dimensional spin manifold \((X, g)\)
2. Real spinor bundle \(\mathcal{S} \to X\), connection \(\nabla\mathcal{S}\)
3. Clifford multiplication \(c: TX \times \mathcal{S} \to \mathcal{S}\)
4. Lie group \(G\)
5. \(G\)-principal bundle \(P \to X\)
6. \(\text{Ad} P := P \times_G \mathfrak{g} \to X\)

Definition

Let \(\nabla \text{Ad} P \in \Omega^1(P; g)\) be a connection on \(P\). The twisted Diracian is

\[\mathcal{D} \nabla \text{Ad} P : C^\infty(\mathcal{S} \otimes \mathcal{R} \text{Ad} P) \to C^\infty(\mathcal{S} \otimes \mathcal{R} \text{Ad} P), s \mapsto -\sum_{i=1}^7 c(e_i, \nabla \mathcal{S} \otimes \text{Ad} P e_i)\]
Twisted Dirac Operators

Setup

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2. *Real spinor bundle* \(\mathcal{S} \xrightarrow{\nabla} X\), *connection* \(\nabla\mathcal{S}\)
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Let \(\nabla^P \in \Omega^1(P; \mathfrak{g})\) be a connection on \(P\). The *twisted Diracian* is

\[
\mathcal{D}^{\nabla \text{Ad } P} : C^\infty(\mathcal{S} \otimes_{\mathbb{R}} \text{Ad } P) \to C^\infty(\mathcal{S} \otimes_{\mathbb{R}} \text{Ad } P),
\]

\[
s \mapsto \sum_{i=1}^{7} c(e_i, \nabla^P_{e_i} \mathcal{S} \otimes \text{Ad } P s)
\]
Example in 7D: Manifolds with $G_2$-Structure

Definition

A topological $G_2$-structure on $(X^7, g)$ is a structure of normed algebras on $O := \mathbb{R} \oplus TX$ with two-sided unit $1 = (1, 0)$:

$$c: O \times O \xrightarrow{bilinear} O, \quad \|v \cdot w\| = \|v\| \cdot \|w\|.$$
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Adjoint of $c|_{TX}$ is $\phi \in \Omega^3(X)$; Let $\psi := *\phi \in \Omega^4(X)$.

1. Every manifold with $G_2$-structure is spin $\mathbb{S} := O$.
2. Clifford multiplication is $c$.
3. For a torsion-free $G_2$-structure $\nabla \mathbb{S} = \nabla^\mathbb{R} \oplus \nabla^{LC}$. 

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General 7-dimensional spin manifold, with preferred spinor.
Example in 7D: Continued

Connection $\nabla^P$ induces $d_{\nabla^P} : \Omega^k(X, \text{Ad } P) \longrightarrow \Omega^{k+1}(X, \text{Ad } P)$.

**Proposition**

Assume $\nabla^{LC} \phi = 0$. Then the twisted Diracian $\nabla^P \text{Ad } P$ equals

$$L_{\nabla^P} = \begin{pmatrix} 0 & d^*_P \\ d_P & * (\psi \wedge d_P) \end{pmatrix}$$

$$\Omega^0(X, \text{Ad } P) \oplus \Omega^1(X, \text{Ad } P) \longrightarrow \Omega^0(X, \text{Ad } P) \oplus \Omega^1(X, \text{Ad } P).$$
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Corollary

The tangent space at $\nabla^P$ of the moduli space of $G_2$-instantons $F^{\nabla \text{Ad } P} \wedge \psi = 0$ is described by the Diracian $\mathcal{D}^{\nabla \text{Ad } P}$.

$$\Omega^0(X; \mathfrak{g}_P) \xrightarrow{d_{\nabla^P}} \Omega^1(X; \mathfrak{g}_P) \xrightarrow{d_{\nabla^P} \wedge \psi} \Omega^6(X; \mathfrak{g}_P) \xrightarrow{d_{\nabla^P}} \Omega^7(X; \mathfrak{g}_P)$$
Today’s Problem

Let $\mathcal{A}_P := \{\text{connections } \nabla^P \text{ on } P \to X\}$. By twisting $\mathfrak{D}$ using each $\nabla^P \in \mathcal{A}_P$ get an $\mathcal{A}_P$-family of differential operators on $X$.

Questions

- Equivariant orientability of $\bigcup_{\nabla^P \in \mathcal{A}_P} \text{Det } \mathfrak{D}^{\nabla^P} \bigg\downarrow \mathcal{A}_P$?
  \[\longrightarrow\text{ Can be answered using index theory.}\]

- How do we pick orientations, canonically, fixing perhaps some topological data on $X$?
Today’s Problem

Let \( A_P := \{ \text{connections } \nabla^P \text{ on } P \to X \} \). By twisting \( \mathcal{D} \) using each \( \nabla^P \in A_P \) get an \( A_P \)-family of differential operators on \( X \).

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- Equivariant orientability of \( \bigsqcup_{\nabla^P \in A_P} \text{Det } \mathcal{D}^{\nabla^P} \to A_P \)? Can be answered using index theory.
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Theorem (Joyce–U. 2018)

Let \((X, \phi^3, \psi^4 = *_{\phi^3} \phi^3)\) be a closed \( G_2 \)-manifold. A flag structure \( \mathcal{F} \) on \( X \) determines, for every principal \( SU(n) \)-bundle \( P \to X \), an orientation of the moduli space \( \mathcal{M}_{P}^{\text{irr}} \) of \( G_2 \)-instantons

\[
\{ A \in \mathcal{A}_{P}^{\text{irr}} \mid F_A \wedge \psi = 0 \} / \text{Aut}(P).
\]
Outline

Orientation Problems for Twisted Diracians

Determinants, Symbols, and Excision

Canonical Orientations in Seven Dimensions
Determinant Line Bundle and Orientations

Definition
Let \( \{D_y\}_{y \in Y} \) be a \( Y \)-family of real Fredholm operators. The Quillen determinant line bundle is

\[
\text{Det}\{D_y\} := \bigcup_{y \in Y} \Lambda^{\text{top}}(\text{Ker } D_y) \otimes \Lambda^{\text{top}}(\text{Coker } D_y)^* \downarrow Y.
\]

The orientation cover is

\[
\text{Or}\{D_y\} = (\text{Det}\{D_y\}_{\text{zero section}}) \downarrow_0 Y.
\]

Today's problem
Given \( G \hookrightarrow P \rightarrow X \), trivialize \( \text{Or}\{D_y\}_{\nabla P \in \mathcal{A} P} \) canonically in terms of data on \( X \).
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\text{Or}\{D_y\}_{y \in Y} := (\text{Det}\{D_y\} \setminus \{\text{zero section}\}) / \mathbb{R}_{>0} \searrow Y.
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Represents \( \pi_1 \text{Fred}_\mathbb{R} = \mathbb{Z}_2 \)
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Given \( G \hookrightarrow P \twoheadrightarrow X^7 \), trivialize \( \text{Or}\{\nabla^{\text{Ad } P}\}_{\nabla^P \in \mathcal{A}_P} \downarrow \mathcal{A}_P \), canonically in terms of data on \( X \).
Index Theory

Definition

1. For $D$ Fredholm, $\text{ind } D := \dim_\mathbb{R} \text{Ker } D - \dim_\mathbb{R} \text{Coker } D \in \mathbb{Z}$
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2. For $Y$-family $\{D_y\}_{y \in Y}$, $\text{ind } D \in KO^0(Y)$. Up to isomorphism

$$w_1(\text{ind } D) = [\text{Or}\{D_y\}] \in H^1(Y; \mathbb{Z}_2)$$
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Properties

- Natural in $Y$
- If $\{D_t\}_{t \in [0,1]}$: $D_0 \sim D_1$ through Fred, then

$$\text{ind } D_0 = i_0^* \text{ind } D = i_1^* \text{ind } D = \text{ind } D_1$$

- $\text{ind}(D_1 \oplus D_2) = \text{ind } D_1 + \text{ind } D_2$
- $\text{ind } D^\dagger = - \text{ind } D$
Elliptic Symbol Families

Definition
Family of Elliptic $(\psi)$DOs $D_y : C^\infty(X, E_y) \to C^\infty(X, F_y)$ on $X$ determined, up to convex choice, by elliptic symbol family

$$p_{\xi,y} = \sigma_{\xi,y}(D) : E_y \xrightarrow{\sim} F_y, \quad p_{\lambda \cdot \xi,y} = \lambda^m p_{\xi,y},$$

for all $0 \neq \xi \in T^*X, y \in Y, \lambda > 0$. Here $m \in \mathbb{R}$ is the order.
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Example
$E \xrightarrow{\pi} X \times Y$ vector bundle, $X^7$ spin, $c : TX \otimes \mathcal{S} \to \mathcal{S}$ Clifford multiplication. Let $p_{\xi,y} := c_\xi \otimes \text{id}_{E_y}$ for $y \in Y$. 
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$E \searrow X \times Y$ vector bundle, $X^7$ spin, $c : TX \otimes $ $\to $ $\mathbb{C}$ Clifford multiplication. Let $p_{\xi,y} := c_\xi \otimes \text{id}_{E_y}$ for $y \in Y$.

Further properties

$\triangleright \ \text{ind } p = \text{ind } D$ well-defined, for any $\sigma(D) = p$

$\triangleright \ i : U \hookrightarrow X$ open embedding, $p$ compactly supported on $U$

$\implies \ i_!(\text{ind } p) = \text{ind } i_! p$
Categorical Index Calculus

For $Y$-family of elliptic symbols $p = \{p_{\xi,y}\}_{y \in Y}$ on $X$ have object

$$\text{Or } p := \lim_{\sigma(D) = p} \text{Or } D \searrow Y \text{ in } \text{Cov}_{\mathbb{Z}_2}(Y)$$
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Properties become structure maps in $\text{Cov}^{\text{gr}}_{\mathbb{Z}_2}(Y)$

1. $\{p_t\}: p_0 \simeq p_1 \implies \text{Or } p_0 \to \text{Or } q_1$
2. $\text{Or}(p \oplus q) \to (\text{Or } p) \otimes (\text{Or } q)$, $\text{Or } p^\dagger \to (\text{Or } p)^*$
3. If $\phi: X_- \to X_+$ diffeomorphism with $\phi^* p_+ = p_-$, then

$$\text{Or}(\phi): \text{Or } p_- \longrightarrow \text{Or } p_+$$

4. For $i: U \hookrightarrow X$ open embedding, $p$ compactly supported on $U$

$$i_!: \text{Or}(p) \longrightarrow \text{Or}(i_! p)$$
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All these maps are compatible.
Excision

\[(\phi, \Pi, K)\]

\( p_-, q_- \rightarrow p_+, q_+ \)

\( L_- \quad U_- \quad X_- \)

\( L_+ \quad U_+ \quad X_+ \)

\( \Xi_- : p_- \cong q_- \)

\( \Xi_+ : p_+ \cong q_+ \)

This data induces an excision isomorphism in \( \text{Cov} \, \text{gr} \, \mathbb{Z}_2(\mathcal{Y}) \):

\( \text{Exc}(\phi, \Pi, K) : \text{Or}(p_-)^* \otimes \text{Or}(q_-) \rightarrow \text{Or}(p_+)^* \otimes \text{Or}(q_+)^* \)

compatible with all structure maps.
Excision

Pair of symbol families $p_\pm, q_\pm$ on $X_\pm$, isomorphic outside compact subsets $L_\pm$ of $U_\pm$
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Identification $\Pi, K$ of pairs over diffeomorphism $\phi: U_{-} \to U_{+}$
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Effect of Global Diffeomorphisms on Orientations

Definition
For $P \xrightarrow{\sim} X$ principal $G$-bundle over spin manifold $X$, Clifford multiplication $c : TX \otimes \mathbb{R} \to \mathbb{R}$, define

$$\text{Or}_P := \text{Or}(c \otimes \text{Ad} P)^* \otimes \text{Or}(c \otimes \mathfrak{su}(n)) = \text{Or}(\hat{\varphi}_{\text{Ad} P})^* \otimes \text{Or}(\hat{\varphi}_{\mathfrak{su}(n)})$$
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Theorem (APS)
Let $\Phi : P \rightarrow P$ be an $\text{SU}(n)$-isomorphism over a spin diffeomorphism $\phi : X \rightarrow X$. Then we have

$$\text{Or}(\Phi) = (-1)^{\delta(\Phi)} \cdot \text{id}_{\text{Or}_P}, \quad \delta(\Phi) := \int_{X_{\phi}} \hat{A}(TX_{\phi}) \left( \text{ch}(P^*_\phi \otimes P_\phi) - n^2 \right),$$

where $P_\phi = P \times_{\mathbb{Z}} \mathbb{R} \rightarrow X_{\phi} = X \times_{\mathbb{Z}} \mathbb{R}$ are the mapping tori.
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where $P_\phi = P \times \mathbb{Z} \mathbb{R} \xrightarrow{\phi} X_\phi = X \times \mathbb{Z} \mathbb{R}$ are the mapping tori.

$$\delta(\Phi) \equiv \frac{1}{2} \int_{X_\phi} p_1(TX_\phi)c_2(P_\phi) \equiv \int_{X_\phi} c_2(P_\phi) \cup c_2(P_\phi) \mod 2$$
Flag Structures

Definition
Let $X^7$ be oriented, $Y^3 \subset X^7$ compact oriented submanifold

- A flag on $Y \subset X$ is a non-vanishing normal section $s: Y \rightarrow N_{Y/X}$.
Flag Structures

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- A flag on $Y \subset X$ is a non-vanishing normal section $s: Y \rightarrow N_{Y/X}$.
- For flags $s_0, s_1$ define degree of $s_0$ w.r.t. $s_1$ as

$$d(s_0, s_1) := [Y] \bullet \{ ts_1(y) + (1 - t)s_0(y) \mid t \in [0, 1], y \in Y \}.$$
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  \[d(s_0, s_1) := [Y] \cdot \{ts_1(y) + (1 - t)s_0(y) \mid t \in [0, 1], y \in Y\} \in 2\mathbb{Z} \cdot\]
- $s_0 \sim s_1 : \iff d(s_0, s_1) \in 2\mathbb{Z}$. Let $\text{Flag}(Y \subset X) := \{s\}/\sim$. 

Flag structure for $Y \subset X$ is a choice of base-point $F: \text{Flag}(Y \subset X) \to \mathbb{Z}/2$. 
Flag structure is a flag structure for all $Y^3 \subset X^7$, where $F(Y_1, s_1) = (-1)^D((Y_1, s_1), (Y_2, s_2)) \cdot F(Y_2, s_2)$ if $[Y_1] = [Y_2]$. 
Define a torsor over $H^3(X; \mathbb{Z}/2)$. 

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  $\mathcal{F} : \text{Flag}(Y \subset X) \xrightarrow{\sim} \mathbb{Z}_2$.

- Flag structure is a flag structure for all $Y^3 \subset X$, where
  $\mathcal{F}(Y_1, s_1) = (-1)^D((Y_1, s_1);(Y_2, s_2)) \cdot \mathcal{F}(Y_2, s_2)$ if $[Y_1] = [Y_2]$.
  Define a torsor over $H^3(X; \mathbb{Z}_2)$.
Flags for Trivial Normal Bundle

Example
Let \( Y := Y_0 = Y_1 \) with trivializable normal bundle. For \( s_0, s_1 : Y \to \mathbb{H} \) unit length write \( s_1 = q \cdot s_0 \) with \( q : Y \to S^3 \).

\[
d(s_0, s_1) =
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Let $Y := Y_0 = Y_1$ with trivializable normal bundle. For $s_0, s_1: Y \to \mathbb{H}$ unit length write $s_1 = q \cdot s_0$ with $q: Y \to S^3$. Set $S(y, t) := (1 - t)s_0(y) + ts_1(y)$.

\[ d(s_0, s_1) = \]

![Diagram showing the relationship between $s_0$ and $s_1$]
Flags for Trivial Normal Bundle

Example
Let $Y := Y_0 = Y_1$ with trivializable normal bundle. For $s_0, s_1: Y \to \mathbb{R}$ unit length write $s_1 = q \cdot s_0$ with $q: Y \to S^3$. Set $S(y, t) := (1 - t)s_0(y) + ts_1(y)$.

$$S(y, t) = [(1 - t) + qt] \cdot s_0(y) = 0 \iff t = \frac{1}{2} \text{ and } q(y) = -1.$$ 

$$\implies d(s_0, s_1) = \deg(q: Y \to S^3).$$
Flag Structures: Continued

Definition
Oriented manifolds $X_{\pm}$, orientation-preserving diffeomorphism $\phi: X_- \to X_+$.

Pullback of flag structure $\mathcal{F}_+$ is

$$(\phi^* \mathcal{F}_+) \left( Y_\rightarrow N_{Y_\subset X_-} \right) = \mathcal{F}_+ \left( d\phi \circ s_- \circ \phi^{-1} |_{Y_-} \right).$$
Definition
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Proposition
If $\phi : X \to X$ or. diffeomorphism, $Y^3 \subset X$, $\phi|_Y = id_Y$, then

$$\mathcal{F}/\phi^* \mathcal{F}[Y] = (-1)^{[Y \times S^1] \bullet [Y \times S^1]}$$

in the mapping torus $X^8_\phi = (X \times [0, 1])/(x, 1) \sim (\phi(x), 0)$. 


Statement of Main Theorem: Slide 1/2

A flag structure $\mathcal{F}$ (on $Y^3 \subset X^7$) determines an orientation

$$o^\mathcal{F}(P) \in \text{Or}_P := \text{Or}(\mathcal{D}_{\text{Ad} P}) \otimes \text{Or}(\mathcal{D}_{\text{su}(n)})^*$$

for every $\text{SU}(n)$-bundle $P \rightarrow X$ (with $[Y]$ Poincaré dual to $c_2(P)$).

This association is uniquely determined by the following properties:
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A flag structure $\mathcal{F}$ (on $Y^3 \subset X^7$) determines an orientation

$$o^\mathcal{F}(P) \in \text{Or}_P := \text{Or}(D_{\text{Ad}P}) \otimes \text{Or}(D_{\text{su}(n)})^*$$

for every $\text{SU}(n)$-bundle $P \rightarrow X$ (with $[Y]$ Poincaré dual to $c_2(P)$).

This association is uniquely determined by the following properties:

1. (Normalization.) For $P = \text{SU}(n)$ trivial: $\exists$ canonical base-point $o^{\text{triv}}(P) \in \text{Or}_P$. For every flag structure $\mathcal{F}$:

$$o^\mathcal{F}(P) = o^{\text{triv}}(P).$$
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2. (Stabilization.) Via the isomorphism $\text{stab}: \text{Or}_{P \times_{\text{SU}(n)} \text{SU}(m)} \cong \text{Or}_P \otimes \mathbb{Z}_2 \text{Or}_{\text{SU}(m)} \cong \text{Or}_P$ we have

   $$o^\mathcal{F}(P \times_{\text{SU}(n)} \text{SU}(m)) = o^\mathcal{F}(P).$$
3. (Natural.)

- Let $P_\pm \to X_\pm$ be SU($n$)-bundles, $\mathcal{F}_\pm$ flag structures on $X_\pm$. 
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   ▶ Let \( P_{\pm} \rightarrow X_{\pm} \) be SU\((n)\)-bundles, \( \mathcal{F}_{\pm} \) flag structures on \( X_{\pm} \).
   ▶ Let \( \rho_{\pm} \) be sections of \( P_{\pm} \) outside open subsets \( U_{\pm} \subset X_{\pm} \).
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- Let $\Phi: P_{-}\big|_{U_{-}} \to P_{+}\big|_{U_{+}}$ be an SU($n$)-isomorphism over a spin diffeomorphism $\phi: U_{-} \to U_{+}$ preserving $\rho_{\pm}$. 
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- Let $\Phi: P_-|_{U_-} \to P_+|_{U_+}$ be an SU($n$)-isomorphism over a spin diffeomorphism $\phi: U_- \to U_+$ preserving $\rho_{\pm}$.

Under excision $\text{Ex}(\phi, \Phi): \text{Or}_{P_-} \to \text{Or}_{P_+}$ we then have

$$\text{Ex}(\phi, \Phi)(o^{\mathcal{F}-}(P_-)) = (\mathcal{F}_- / \phi^* \mathcal{F}_+)(c_2(P_-)) \cdot o^{\mathcal{F}+}(P_+).$$
Illustration of Excision Axiom

\[ P_- \xrightarrow{\text{framing } \rho_-} X_- \xrightarrow{\text{framed isomorphism } \phi: U_- \to U_+} P_+ \xrightarrow{\text{framing } \rho_+} X_+ \]
Proof of Uniqueness for SU(2)-Bundles $P \to X^7$

1. Pick a transverse section $s$ of $E := P \times_{\text{SU}(2)} \mathbb{C}^2$ with zero set $Y^3 = s^{-1}(0)$. 
Proof of Uniqueness for SU(2)-Bundles $P \xrightarrow{\phi} X^7$

1. Pick a transverse section $s$ of $E := P \times_{\text{SU}(2)} C^2$ with zero set $Y^3 = s^{-1}(0)$.

2. Then $ds : N_Y \cong E|_Y$, which defines an SU(2)-structure and hence a spin structure on $N_Y$ and $Y$. 
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By excision axiom ($\mathcal{F}_7$ unique flag structure on $S^7$):

\[
\begin{array}{ccc}
\text{Or}_{P' \searrow S^7} & \xrightarrow{\text{Ex}} & \text{Or}_{P \searrow X} \\
\cup & & \cup \\
\text{o}^{\mathcal{F}_7}(P') & \xrightarrow{} & (-1)^{(\mathcal{F}_7/\phi^* \mathcal{F})[Y]} \cdot \text{o}^{\mathcal{F}}(P)
\end{array}
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Since $\pi_6(SU(4)) = \{1\}$, $P' \times_{SU(2)} SU(4)$ is trivializable on $S^7$. 

Since $\pi_6(SU(4)) = \{1\}$, $P' \times_{SU(2)} SU(4)$ is trivializable on $S^7$. By stabilization and normalization axioms:

$$
\text{Or}_{P' \times_{SU(2)} SU(4)} \searrow S^7 \xrightarrow{\text{stab}} \text{Or}_{P' \searrow S^7}
$$

$$
\circ_{\text{triv}} = o^{F_7}(P' \times_{SU(2)} SU(4)) \xleftarrow{\text{stab}} o^{F_7}(P')
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Proof of Uniqueness for SU(2)-Bundles: Continued

Since \( \pi_6 (SU(4)) = \{1\} \), \( P' \times_{SU(2)} SU(4) \) is trivializable on \( S^7 \).

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\[
\begin{align*}
\text{Or}_{P' \times_{SU(2)} SU(4)} \setminus S^7 & \quad \xrightarrow{\text{stab}} \quad \text{Or}_{P' \setminus S^7} \\
\cup & \\
\circ_{\text{triv}} = \circ^F_7 (P' \times_{SU(2)} SU(4)) & \quad \xrightarrow{} \quad \circ^F_7 (P')
\end{align*}
\]

So uniquely determined by axioms:

\[
\circ^F (E) = \text{Ex} \circ \text{stab}((-1)^{(F_7/\phi^*F)[\gamma]} \cdot \circ_{\text{triv}}) \quad (\ast)
\]
Proof of Existence

- Show that (*) is independent of the choices $s, \phi$
  $(i$ and the tubular neighborhoods are unique up to isotopy).

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- Dependence on $\phi$ reduced to a model calculation for the automorphism of $E_{\text{std}}(N_Y) \to \mathbb{S}(N_Y \oplus \mathbb{R})$ induced by a spin automorphism $\psi: N_Y \to N_Y$. We have calculated $\text{Or}(\psi) = (F/\psi^*F)[Y]$. 


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- Let \(s_0, s_1\) be transverse sections of \(E \rightarrow X\). The two corresponding excision isomorphisms can be deformed into each other \(\implies\) equal by discreteness.
Summary

- Orientation covers $\text{Or}(p) \in \text{Cov}_{\mathbb{Z}_2}(Y)$ behave exactly like $\text{ind}(p) \in KO_0(Y)$, one categorical level up. Here $p$ is an elliptic symbol family on $X$ parameterized by $Y$. 

- A flag structure solves the 'orientation problem' for $7$-dimensional twisted Diracians $D$ when $G = SU(n)$. 

- Canonical orientations defined by comparing the 'orientation problem' on $X$ to that on $S^7$ via excision.
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