On fronts in a vanishing viscosity limit

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Joint work with Corrado Mascia, Rome.

Consider hyperbolic and parabolic equations

$$\left.\begin{array}{ll}(H) & u_t + f(u)_x = g(u) \\ (P) & u_t + f(u)_x = \epsilon u_{xx} + g(u)\end{array}\right\} (x,t) \in \mathbb{R}\times[0,\infty),\ u(x,t) \in \mathbb{R};$$

where

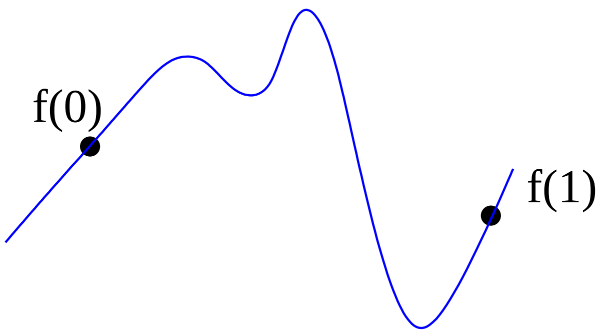
- the reaction term g is smooth, $g(0) = g(1) = 0$ and $g(s) < 0$ for $s \in (0,1)$;

g

"monostable"

- the flux term f is smooth and has a finite number of points of inflection in $(0,1)$;

f



possibly non-convex

Interested in monotone travelling-wave solutions ("fronts")

$$u(x,t) = \phi(x - ct)$$

- ϕ is increasing, and $\phi(x) \longrightarrow 0, 1$ (equilibria of g) as $x \longrightarrow -\infty, +\infty$
- possibly weak/entropy solutions

Question

Given a travelling-front ϕ of the hyperbolic equation of speed c, does there exist a family of travelling fronts ϕ_ϵ of the parabolic equation with speeds c_ϵ such that as $\epsilon \longrightarrow 0$,

$$c_\epsilon \longrightarrow c$$

and the profiles ϕ_ϵ converge in some sense?

Background

- For the hyperbolic equation $u_t + f(u)_x = g(u)$, study is easier when

f is convex

$$f'' > 0 \;\Rightarrow\; f'(u) \text{ is increasing in } u$$

$$\Rightarrow \quad \text{ordering property on propagation speeds of characteristics}$$

$$\frac{dx}{dt} = f'(u(x,t)), \quad \frac{du}{dt} = g(u).$$

- Existing work on the relation between travelling fronts for the hyperbolic and parabolic equations when f is convex

  J. Härterich (2000, 2003) geometrical singular perturbation theory

$$c_\epsilon \longrightarrow c, \quad ||\phi_\epsilon - \phi||_{L^1} \longrightarrow 0$$

- If flux f is not convex, then
  - new ideas are needed to analyse the hyperbolic equation
  - qualitatively different behaviour can arise

    $e.g.$ discontinuities may disappear in finite time

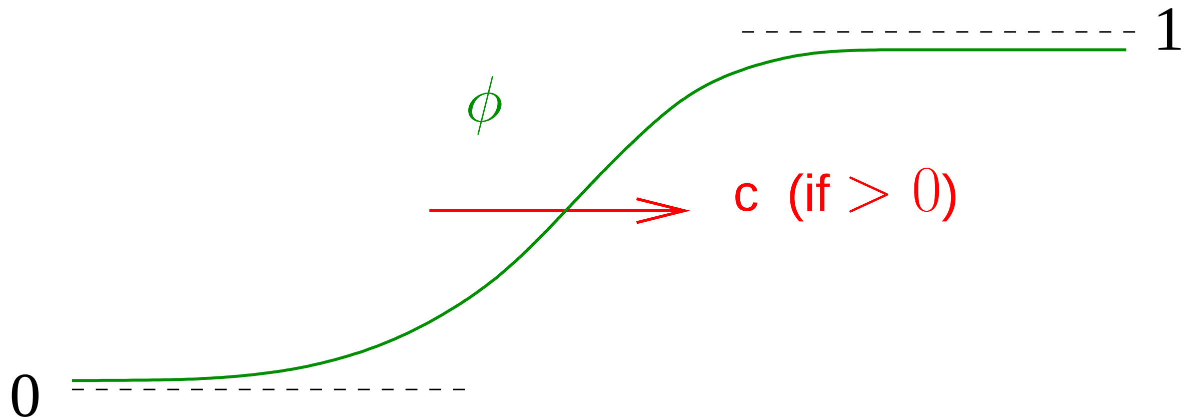
    Sinestrari SIAM J. Math. Anal 1997, Mascia Comm. PDE 1998

- The parabolic equation $u_t + f(u)_x = \epsilon u_{xx} + g(u)$ “does not care” whether or not f is convex

Travelling Fronts

$$u(x,t) = \phi(x - ct)$$

- the profile ϕ is non-decreasing and $\phi(x) \to 0, 1$ as $x \to -\infty, +\infty$
- for the parabolic problem $u_t + f(u)_x = \epsilon u_{xx} + g(u)$
  - $\phi \in C^2(\mathbb{R})$
  - $(f'(\phi) - c)\phi' = \epsilon\phi'' + g(\phi)$



- in fact, $\phi' > 0$ and ϕ is strictly increasing

- for the hyperbolic problem $u_t + f(u)_x = g(u)$

$u(x,t) = \phi(x - ct)$ is an entropy solution, in the sense that

  - $u \in L^\infty(\mathbb{R} \times [0,\infty))$
  - for all non-negative $\psi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ and all $k \in \mathbb{R}$,

$$\int_{\mathbb{R}\times[0,\infty)} \left[|u-k|\frac{\partial\psi}{\partial t} + \text{sgn}(u-k)(f(u)-f(k))\frac{\partial\psi}{\partial x} + \text{sgn}(u-k)g(u)\psi \right] dxdt \;\geq\; 0.$$

- for the hyperbolic problem $u_t + f(u)_x = g(u)$

$u(x,t) = \phi(x - ct)$ is an entropy solution, in the sense that

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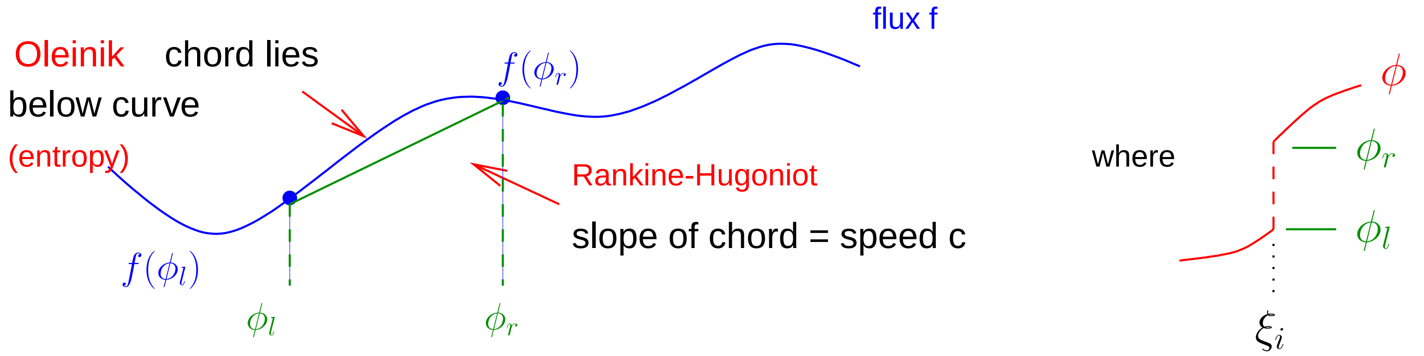
$$\int_{\mathbb{R}\times[0,\infty)} \left[|u-k| \frac{\partial \psi}{\partial t} + \text{sgn}(u-k)(f(u) - f(k)) \frac{\partial \psi}{\partial x} + \text{sgn}(u-k) g(u) \psi \right] dxdt \ \geq \ 0.$$

- here, this is equivalent to

- $\phi \in C^1(\mathbb{R} \setminus \{\xi_1, \ldots, \xi_N\})$ satisfies

$$(f'(\phi) - c)\phi = g(\phi) \quad \text{for } \xi \neq \xi_i$$

- Oleinik and Rankine-Hugoniot jump conditions hold at each $\xi = \xi_i$



Existence Theorems for Travelling Fronts

Hyperbolic Case (Mascia 1999) $u_t + f(u)_x = g(u)$

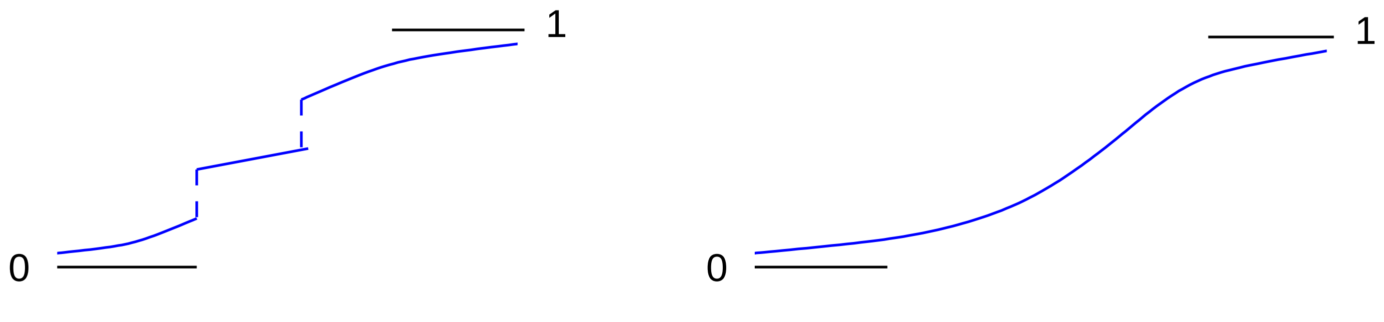
- There exists an entropy travelling-front solution of the hyperbolic equation of speed c if and only if

$$c \geq \sup_{s \in (0,1)} \frac{f(1) - f(s)}{1 - s} =: c^*$$

- Fronts of speed

$$c > \sup_{s \in (0,1)} f'(s) \quad (\geq c^*)$$

are smooth.



General Fronts

Fronts for c sufficiently large

- Given $c \geq c^*$, there is a unique (up to translation) front of speed c

Idea of Proof

- $c > \sup_{s\in(0,1)} f'(s)$ (smooth fronts)
  - choose $\phi(0) \in (0,1)$ - there exists a smooth increasing solution of

$$\phi'(\xi) = \frac{g(\phi(\xi))}{f'(\phi(\xi)) - c}$$

negative

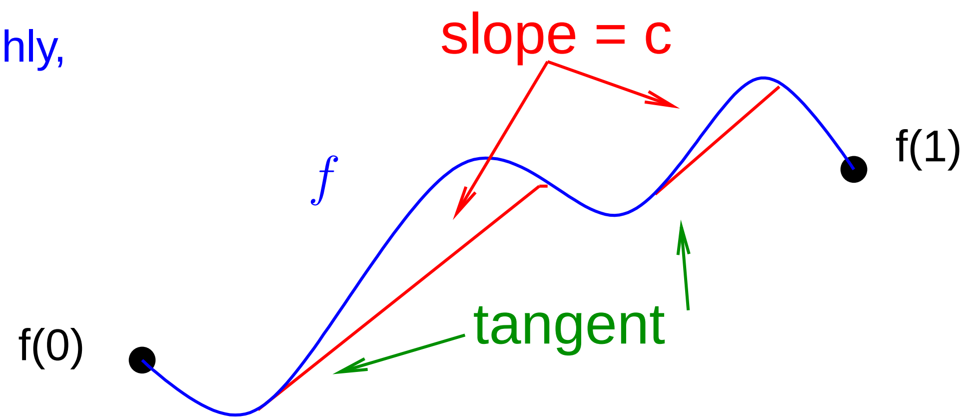
negative and bounded above

locally *ie* for $\xi \in (-\delta, \delta)$

  - can continue to $(-\infty, \infty)$

- $c^* = \sup_{s\in[0,1)} \frac{f(1) - f(s)}{1-s} \le c \le \sup_{s\in(0,1)} f'(s)$

Very roughly,



jump below flux curve

with slope c

Parabolic Case (KPP37, Hadeler and Rothe 75, Volperts 90s, C 03, etc .....)

$u_t + f(u)_x = \epsilon u_{xx} + g(u)$

- For each $\epsilon > 0$, there exists a travelling front solution of the parabolic equation of speed c if and only if

$$c \geq c_\epsilon^*$$

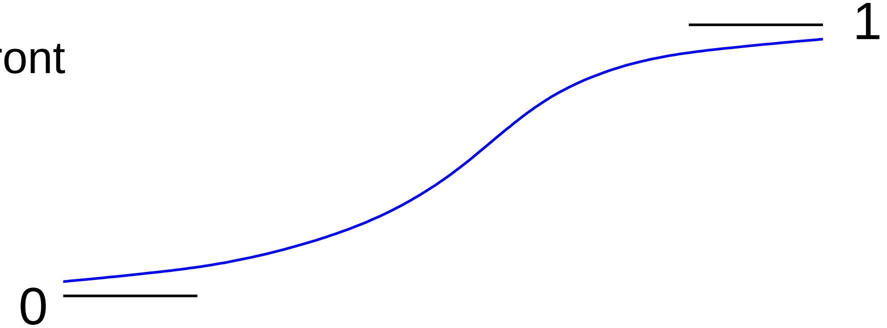
- where

$$c_\epsilon^* = \inf_{\rho \in \mathcal{K}} \sup_{\xi \in \mathbb{R}} \frac{\epsilon \rho''(\xi) + g(\rho(\xi))}{-\rho'(\xi)} + f'(\rho(\xi))$$

and

$$\mathcal{K} = \{\rho \in C^2(\mathbb{R}) : \rho' > 0 \text{ and } \rho(\xi) \to 0, 1 \text{ as } \xi \to -, +\infty\}.$$

Illustration of front



- Given $c \geq c_\epsilon^*$, there is a unique (up to translation) front of speed c

Theorem 1

(i) Properties of minimal speed

- $c_\epsilon^* \geq c^*$ for each $\epsilon \geq 0$

  ($\Rightarrow$ viscous approximation overestimates front speeds)
- $c_{\epsilon_1}^* \leq c_{\epsilon_2}^*$ if $\epsilon_1 \leq \epsilon_2$
- $c_\epsilon^* \downarrow c^*$ as $\epsilon \downarrow 0$

(ii) Convergence of minimal-speed front profiles

let ϕ_ϵ, ϕ denote the profiles of front solutions of (P), (H) with speeds c_ϵ^*, c^* and $\phi_\epsilon(0) = \phi(0) = 1/2$. Then as $\epsilon \to 0$,

$$\phi_\epsilon \to \phi \ \text{ in } \ L^1(\mathbb{R})$$

Theorem 2 (proved by similar methods)

- General convergence of profiles

  -suppose $c \geq c^*$ and $c_\epsilon \to c$

  -let ϕ_ϵ, ϕ denote the profiles of front solutions of (P), (H) with speeds c_ϵ, c and $\phi_\epsilon(0) = \phi(0) = 1/2$. Then as $\epsilon \to 0$,

$$\phi_\epsilon \to \phi \ \text{ in } \ L^1(\mathbb{R})$$

- Convergence of front profiles of a fixed speed c

  - suppose

    (i) $c > c^*$

    (ii) $\epsilon_0 > 0$ be such that $c_\epsilon^* < c$ for $\epsilon \in (0, \epsilon_0)$

  - let ϕ_ϵ, ϕ denote the profiles of the front solutions of (P), (H) with speed c and $\phi_\epsilon(0) = \phi(0) = 1/2$. Then as $\epsilon \to 0$,

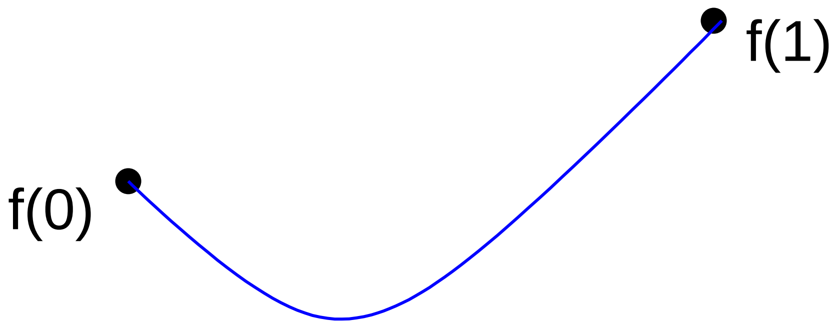
$$\phi_\epsilon \to \phi \ \text{ in } \ L^1(\mathbb{R})$$

Ideas in proof of convergence of minimal speed

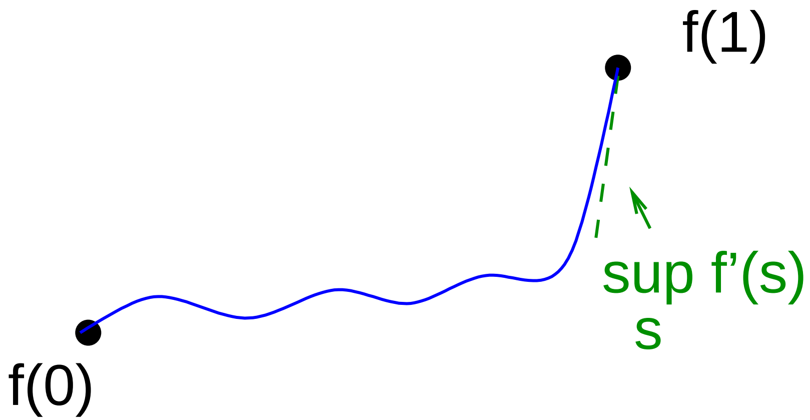
1. Easy Case

$$\sup_{s\in[0,1]} f'(s) = f'(1)$$

e.g. f is convex



or



- **Note**: in general

$$f'(1) = \lim_{s\to 1} \frac{f(s) - f(s)}{s-1} \leq c^* = \sup_{s\in[0,1)} \frac{f(s) - f(1)}{s-1} \leq \sup_{s\in[0,1]} f'(s)$$

- in easy case, have $=$ throughout
- so $c^* = \sup_{s\in(0,1)} f'(s) \Rightarrow$ hyperbolic fronts are smooth for $c > c^*$

Easy Case

Step 1: $\limsup_{\epsilon \to 0} c_\epsilon^* \le c^*$

- use variational formula for c_ϵ^* with a suitable test function $\rho \in \mathcal{K}$ (smooth, increasing from 0 to 1)
- because $c^* = \sup_{s \in (0,1)} f'(s)$ here, can use hyperbolic fronts of speed $c > c^*$ as test functions ρ
- for $\delta > 0$, let ϕ be a hyperbolic front of speed $c^* + \delta$. Then

$$
\begin{aligned}
c_\epsilon^* &= \inf_{\rho \in \mathcal{K}} \sup_{\xi \in \mathbb{R}} \frac{\epsilon \rho'' + g(\rho)}{-\rho'} + f'(\rho) \\
&\le \sup_{\xi \in \mathbb{R}} \frac{\epsilon \phi'' + g(\phi)}{-\phi'} + f'(\phi) \\
&= \sup_{\xi \in \mathbb{R}} -\epsilon \frac{\phi''(\xi)}{\phi'(\xi)} + c^* + \delta \quad \text{since } (f'(\phi) - (c^* + \delta))\phi' = g(\phi) \\
&\to c^* + \delta \text{ as } \epsilon \to 0.
\end{aligned}
$$

Step 2: $c^*_\epsilon \geq c^*$ for each $\epsilon > 0$

- a necessary (but not sufficient) condition for the existence of a parabolic front converging monotonically to 1 as $\xi \to \infty$ is the existence of a real, negative eigenvalue of the linearisation of the parabolic equation at 1

- linearisation $-cy' + f'(1)y' = \epsilon y'' + g'(1)y, \qquad y = \phi - 1$, which gives

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\epsilon^{-1} g'(1) & \epsilon^{-1}(f'(1) - c) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

- eigenvalues satisfy $\epsilon\lambda^2 + (c - f'(1))\lambda + g'(1) = 0$, so

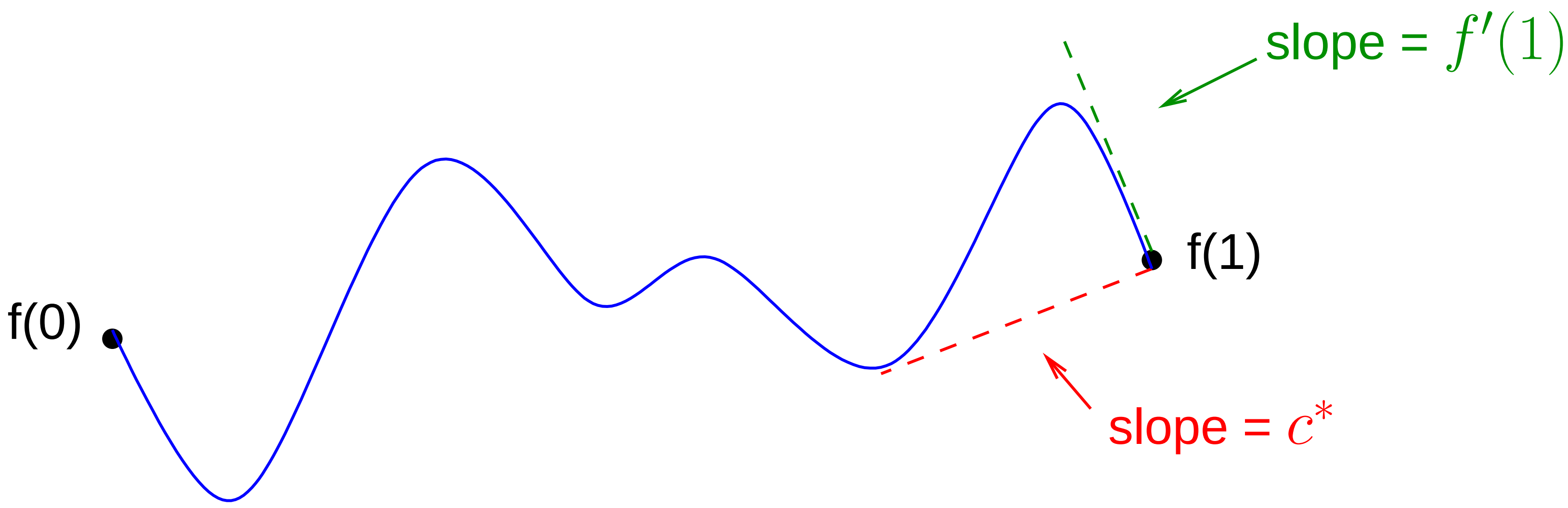
$$\begin{aligned} (c^*_\epsilon - f'(1))^2 &\geq 4\epsilon g'(1) \\ \Rightarrow \quad c^*_\epsilon &\geq f'(1) + 2\sqrt{\epsilon g'(1)} =: c_{L,\epsilon} \text{ (minimal } c \text{ for } \lambda < 0 \text{ to exist)} \\ &= c^* + 2\sqrt{\epsilon g'(1)} \end{aligned}$$

because $c^* = f'(1)$ in easy case

2. General “non-convex” flux

$$c^* = \sup_{s\in[0,\infty)} \frac{f(s)-f(1)}{s-1} < \sup_{s\in[0,1]} f'(s)$$

$\Rightarrow$ hyperbolic fronts of speed close to c^* discontinuous



$$f'(1) < c^* = \sup_{s\in[0,1)} \frac{f(s)-f(1)}{s-1} < \sup_{s\in[0,1]} f'(s)$$

General Case

Step 1: $\limsup_{\epsilon \to 0} c_\epsilon^* \le c^*$

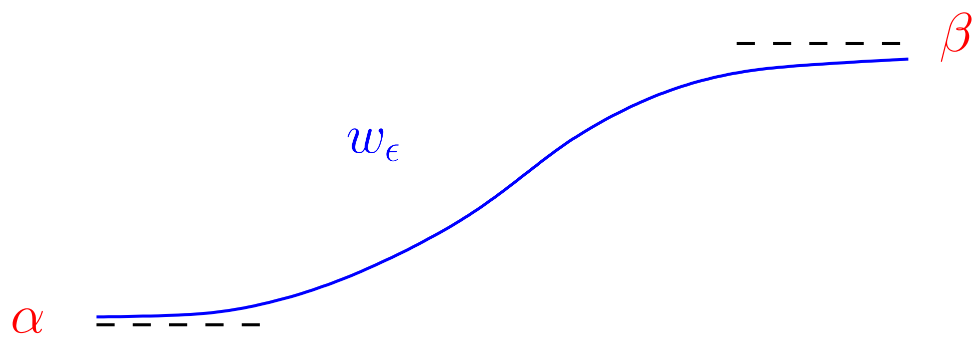
- again use variational formula for c_ϵ^* with suitable $\rho \in \mathcal{K}$
- can't use hyperbolic fronts for c close to c^* because not continuous
- instead use suitable solutions of a reactionless equation

Technical background: Existence of Viscous Shock Profiles

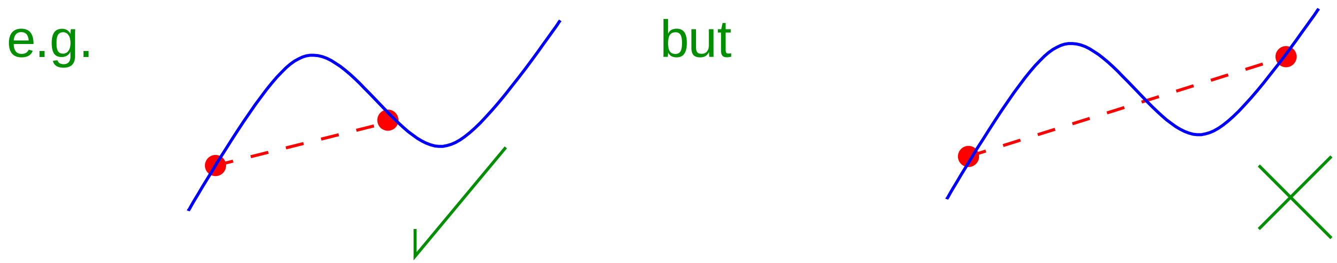
There exists a solution $u(x,t) = w(x-ct)$ of the reactionless equation

$$u_t + f(u)_x = \epsilon u_{xx}$$

with $c \in \mathbb{R}$ and w_ϵ



if and only if the straight line connecting $(\alpha, f(\alpha))$ to $(\beta, f(\beta))$ lies strictly below the graph of f between α and β



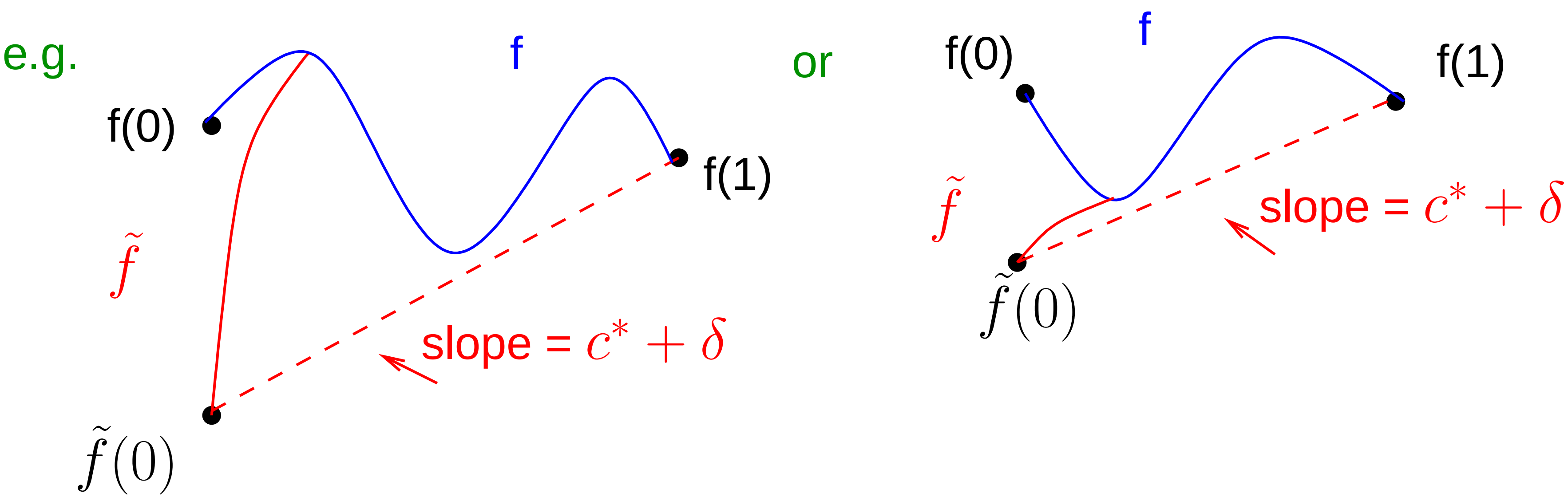
Also 1. speed $c =$ slope $\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ 2. scaling $w_\epsilon(\xi) = w_1(\xi/\epsilon)$

For $\delta > 0$, let ψ_ϵ be a viscous shock profile connecting 0 to 1 and solving the reactionless equation

$$-(c^* + \delta)\psi_\epsilon' + \tilde{f}'(\psi_\epsilon)\psi_\epsilon' = \epsilon\psi_\epsilon''$$

where the modified flux $\tilde{f}$ satisfies

1. $\tilde{f}(1) = \tilde{f}(0) = c^* + \delta$
2. $\tilde{f}'(s) \geq f'(s)$ for all $s \in (0, 1)$



ψ_ϵ exists because chord between $(0, \tilde{f}(0))$ and $(1, \tilde{f}(1))$ lies below graph

Step 1 contd.....

- by scaling, $\psi_\epsilon(\xi) = \psi_1(\xi/\epsilon)$, so

$$
\begin{aligned}
c_\epsilon^* &= \inf_{\rho\in\mathcal{K}} \sup_{\xi\in\mathbb{R}} \frac{\epsilon\rho'' + g(\rho)}{-\rho'} + f'(\rho) \\
&\leq \sup_{\xi\in\mathbb{R}} \frac{\epsilon\psi_\epsilon'' + g(\psi_\epsilon)}{-\psi_\epsilon'} + f'(\psi_\epsilon) \\
&= \sup_{\xi\in\mathbb{R}} c^* + \delta - \frac{g(\psi_\epsilon)}{\psi_\epsilon'} - [\tilde{f}'(\psi_\epsilon) - f'(\psi_\epsilon)]
\end{aligned}
$$

because $(\tilde{f}'(\psi_\epsilon) - (c^* + \delta))\psi_\epsilon' = \epsilon\psi_\epsilon''$

$$
\begin{aligned}
&\leq c^* + \delta + \epsilon \sup_{x\in\mathbb{R}} \left(\frac{-g(\psi_1)}{\psi_1'} \right) \quad \leftarrow \text{finite} \\
&\to c^* + \delta
\end{aligned}
$$

as $\epsilon \to 0$.

Step 2: $c_\epsilon^* \geq c^*$ for each $\epsilon > 0$

- suppose not - that $c_\epsilon^* < c^*$ for some $\epsilon > 0$

- seek $\Theta_\epsilon(x - c^* t)$ such that

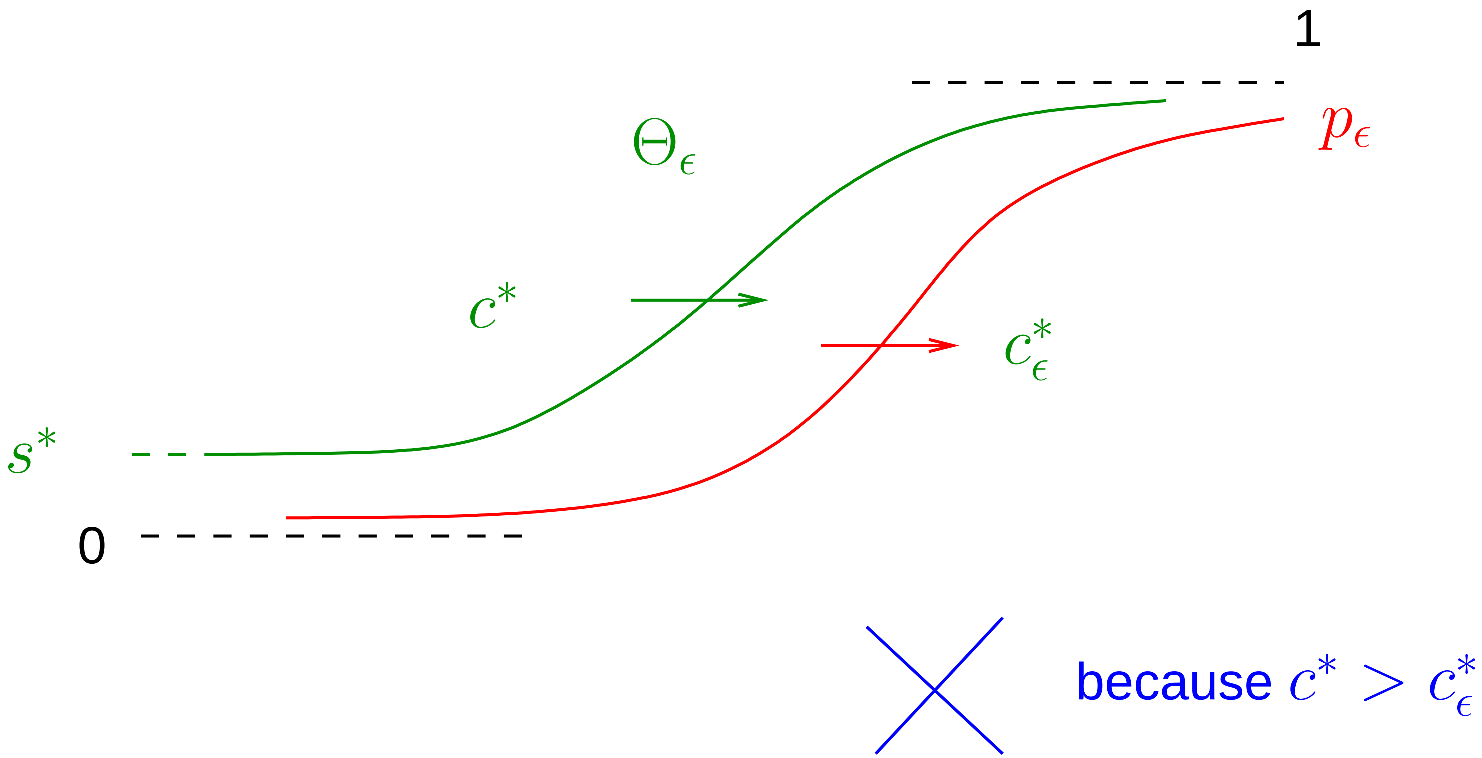
1. $c^* > c_\epsilon^*$
2. $\Theta_\epsilon \geq p_\epsilon$, where p_ϵ is a parabolic front of speed c_ϵ^*
3. $v(x,t) := \Theta_\epsilon(x - c^* t)$ is a supersolution for the parabolic equation:

$$v_t + f(v)_x - \epsilon v_{xx} - g(v) \geq 0$$

- then by the comparison theorem for the parabolic equations,

$$\Theta_\epsilon(x - c^* t) \geq p_\epsilon(x - c_\epsilon^* t) \quad \text{for all } x, t$$

- but this contradicts (1.): that $c^* > c_\epsilon^*$



- take Θ_ϵ as the solution of the reactionless equation

$$(f'(\theta_\epsilon) - c^*)\theta'_\epsilon = \epsilon\theta''_\epsilon$$

- $\Theta_\epsilon(x - c^*t)$ is a supersolution for the parabolic equation because $g(v) < 0$

Ideas in proof of convergence of profiles

Step 1 There exists a sequence $\epsilon_n \downarrow 0$ and a non-decreasing funciton $\phi : \mathbb{R} \longrightarrow [0, 1]$ with $\phi(0) = 1/2$ such that

$$\phi_{\epsilon_n}(\xi) \rightarrow \phi(\xi) \quad \text{for each} \quad \xi \in \mathbb{R},$$

and

$$\phi_{\epsilon_n}(x - c^*_{\epsilon_n} t) \rightarrow \phi(x - c^* t) \;\; \text{for} \;\; a.e. \; (x, t) \in \mathbb{R} \times [0, \infty)$$

Ideas in proof of convergence of profiles

Step 1 There exists a sequence $\epsilon_n \downarrow 0$ and a non-decreasing funciton $\phi : \mathbb{R} \to [0, 1]$ with $\phi(0) = 1/2$ such that

$$\phi_{\epsilon_n}(\xi) \to \phi(\xi) \quad \text{for each} \quad \xi \in \mathbb{R},$$

and

$$\phi_{\epsilon_n}(x - c^*_{\epsilon_n} t) \to \phi(x - c^* t) \ \text{ for } \ a.e. \ (x, t) \in \mathbb{R} \times [0, \infty)$$

because

• $\phi_\epsilon(\xi)$ is increasing in ξ for each ϵ, so Helly's Selection Principle applies

• ϕ is monotone, so $\phi(x - c^* t)$ is continuous at *a.e.* $(x, t) \in \mathbb{R} \times [0, \infty)$

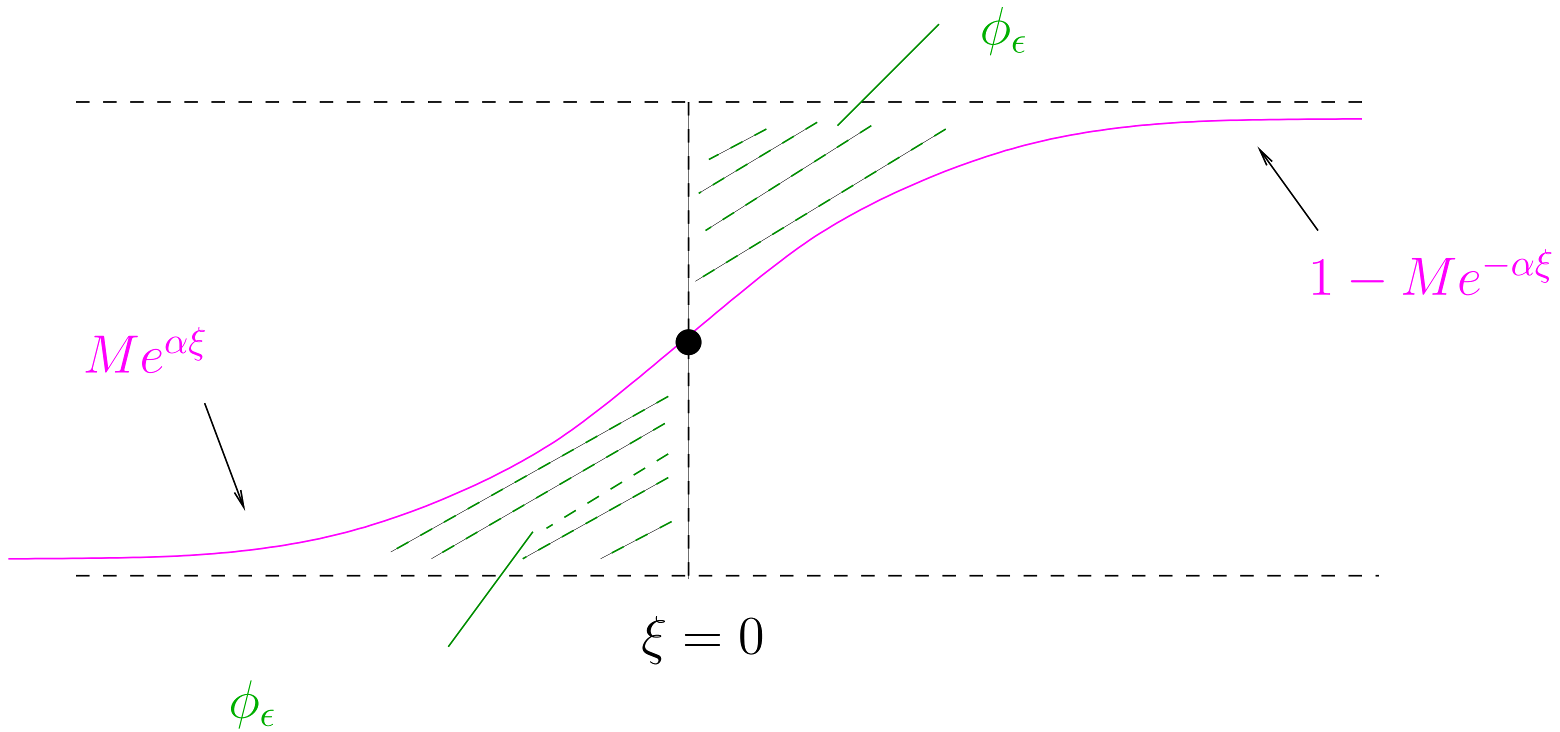
• given $\mu > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon_n \in (0, \epsilon_0)$,

$$\begin{aligned} \phi_{\epsilon_n}(x - c^* t) - \phi(x - c^* t) &\geq \phi_{\epsilon_n}(x - c^*_{\epsilon_n} t) - \phi(x - c^* t) \\ &\geq \phi_{\epsilon_n}(x - (c^* + \mu) t) - \phi(x - c^* t) \end{aligned}$$

$$\Rightarrow \phi_{\epsilon_n}(x - c^* t) \to \phi(x - c^* t) \ \text{ if } \phi \text{ is continuous at } x - c^* t$$

Step 2 (uniform bounds) Given $\epsilon_0 > 0$, there exist $\alpha > 0$ and $M > 0$ such that for each $\epsilon \in (0, \epsilon_0)$,

$$\begin{aligned} 0 \quad &\le \phi_\epsilon(\xi) \le \quad Me^{\alpha\xi} \quad \text{whenever} \quad \xi \le 0 \\ 0 \; &\le 1 - \phi_\epsilon(\xi) \le \; Me^{-\alpha\xi} \quad \text{whenever} \quad \xi \ge 0 \end{aligned}$$



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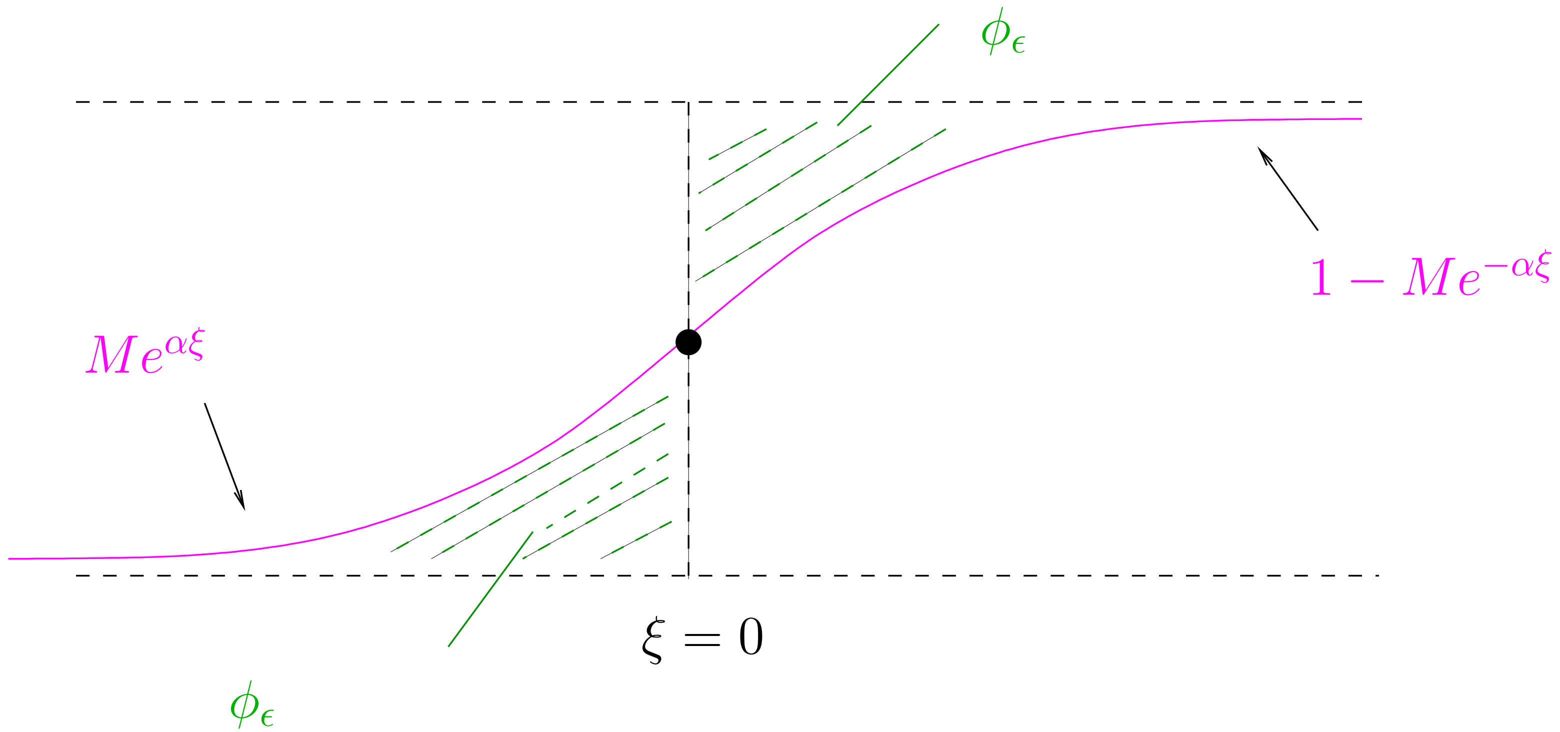

Step 1 + Step 2 $\Rightarrow$ $\|\phi_{\epsilon_n} - \phi\|_{L^1(\mathbb{R})} \to 0$ as $\epsilon_n \downarrow 0$

Step 3 $\phi(x - c^*t)$ is the unique up-to-translation entropy front solution of (H) of speed c^*

• by standard vanishing-viscosity arguments (*cf* Kružkov, Dafermos)

Step 3 $\phi(x - c^* t)$ is the unique up-to-translation entropy front solution of (H) of speed c^*

- by standard vanishing-viscosity arguments (*cf* Kružkov, Dafermos)
- particularly elementary here since ϕ_ϵ is monotone
- for $u_{\epsilon_n}(x,t) = \phi_{\epsilon_n}(x - c^*_{\epsilon_n} t)$, non-negative $\psi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ and $k \in \mathbb{R}$,

$$-\text{sgn}(u_{\epsilon_n} - k)\,\psi\,\left[\frac{\partial u_{\epsilon_n}}{\partial t} + \frac{\partial f(u_{\epsilon_n})}{\partial x} - g(u_{\epsilon_n})\right] \;=\; -\,\epsilon_n \text{sgn}(u_{\epsilon_n} - k)\,\psi\,\frac{\partial^2 u_{\epsilon_n}}{\partial x^2}$$

$\Rightarrow$

$$\int_{\mathbb{R}\times[0,\infty)} |u_{\epsilon_n} - k| \frac{\partial \psi}{\partial t} \;+\; [f(u_{\epsilon_n}) - f(k)]\,\frac{\partial \psi}{\partial x} + \text{sgn}(u_{\epsilon_n} - k)\psi g(u_{\epsilon_n})\,dxdt$$
$$\geq\; -\epsilon_n \int_{\mathbb{R}\times[0,\infty)} |u_{\epsilon_n} - k|\,\frac{\partial^2 \psi}{\partial x^2}\,dxdt.$$

$\Rightarrow$

$$\int_{\mathbb{R}\times[0,\infty)} \left[|u - k|\frac{\partial \psi}{\partial t} + \text{sgn}(u-k)(f(u) - f(k))\frac{\partial \psi}{\partial x} + \text{sgn}(u-k)g(u)\psi\right]\,dxdt \;\geq\; 0$$

where $u(x,t) = \phi(x - c^* t)$

Better convergence for fronts of speed $c > \sup_{s\in[0,1]} f'(s)$

Lemma let ϕ_ϵ, ϕ denote the profiles of the front solutions of (P), (H) with speed $c > \sup_{s\in[0,1]} f'(s)$ and $\phi_\epsilon(0) = \phi(0) = 1/2$. Then as $\epsilon \to 0$,

$$\phi_\epsilon \longrightarrow \phi \ \text{ in } \ W^{1,1}(\mathbb{R}),$$

and for each compact interval $[-M, M] \subset \mathbb{R}$,

$$\phi_\epsilon \longrightarrow \phi \ \text{ in } \ C^1([-M, M]).$$

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and for each compact interval $[-M, M] \subset \mathbb{R}$,

$$\phi_\epsilon \longrightarrow \phi \;\text{ in }\; C^1([-M, M]).$$

because

- uniform bounds on ϕ_ϵ', ϕ_ϵ''
- *e.g.* the maximum of ϕ_ϵ' is attained at a point ξ_0 where $\phi_\epsilon''(\xi_0) = 0$, so

$$(f'(\phi_\epsilon(\xi_0)) - c)\phi_\epsilon'(\xi_0) = g(\phi_\epsilon(\xi_0))$$

$$\Rightarrow \quad \phi_\epsilon'(\xi_0) = \frac{g(\phi_\epsilon(\xi_0))}{f'(\phi_\epsilon(\xi_0)) - c} \;\leq\; A$$

A consequence for parabolic fronts

- a necessary condiiton for existence of parabolic fronts is the existence of a real, negative eigenvalue at 1

$$\Rightarrow c^*_\epsilon \geq c_{L,\epsilon}$$

- $c_{L,\epsilon}$ is minimal c for which linear problem at 1

$$(f'(1)-c)y' = \epsilon y'' + g'(1)y$$

  has an eigenvalue $\lambda < 0$ satisfying

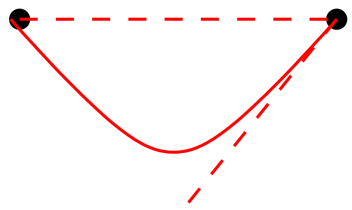
$$(f'(1)-c)\lambda = \epsilon\lambda^2 + g'(1)$$

- would like to know if

$$c^*_\epsilon > c_{L,\epsilon} \quad \text{or} \quad c^*_\epsilon = c_{L,\epsilon}$$

Previous results

- Hadeler and Rothe (1975) $f = 0 \qquad \Rightarrow c^*_\epsilon = c_{L,\epsilon}$



- Berestycki and Nirenberg (1992)

$$f = 0, \quad \sqrt{2}\int_0^1 \sqrt{-g} \geq 2\sqrt{g'(1)} \quad \Rightarrow c^*_\epsilon > c_{L,\epsilon}$$

- Lucia, Muratov and Novaga (2004)

$f = 0,$ various conditions on $g \quad \Rightarrow$ conditions for $c^*_\epsilon > c_{L,\epsilon}$ and $c^*_\epsilon = c_{L,\epsilon}$

- Weinberger, Lewis and Li (2002, 2005)

$f = 0,$ systems $\quad \Rightarrow$ conditions for $c^*_\epsilon = c_{L,\epsilon}$

- Benguria, Depassier and Mendez (2004)

$f \neq 0, \quad g''(u)/\sqrt{g'(1)} + f''(1) < 0$ for $u \in (0,1) \quad \Rightarrow \; c^*_\epsilon = c_{L,\epsilon}$

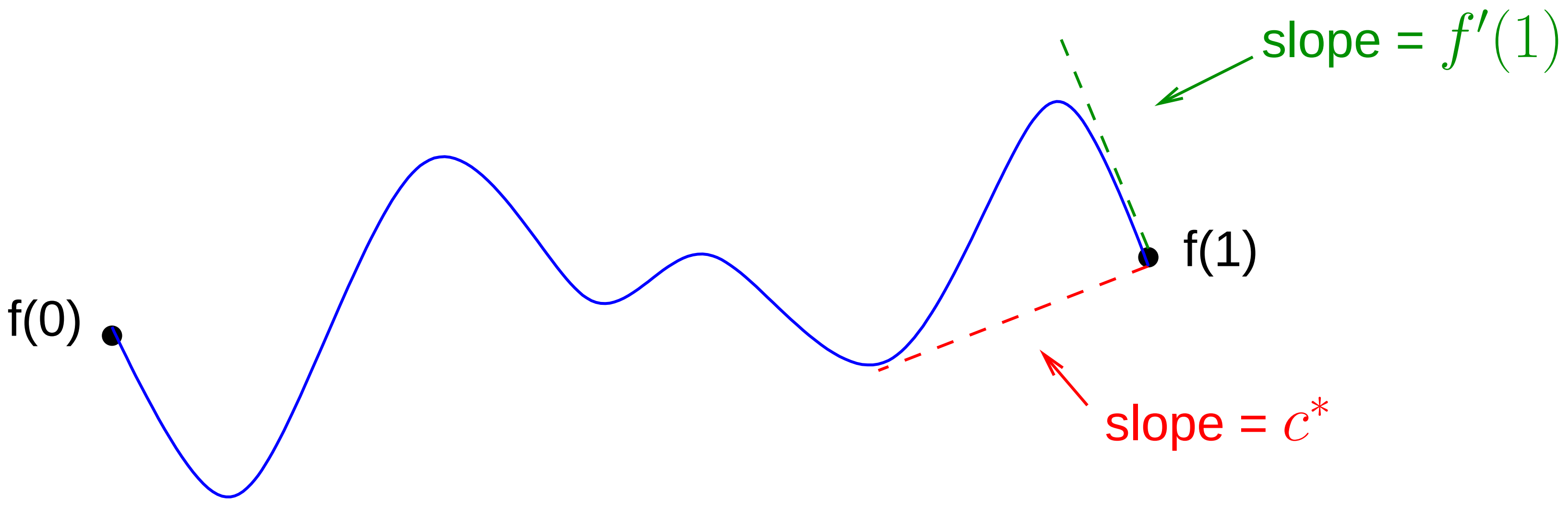
....

Here

- $c_{L,\epsilon} = f'(1) + 2\sqrt{\epsilon g'(1)} \longrightarrow f'(1)$ as $\epsilon \longrightarrow 0$
- $c^*_\epsilon \longrightarrow c^* = \sup_{s\in(0,1)} \frac{f(1)-f(s)}{1-s}$ as $\epsilon \longrightarrow 0$
- thus if

$$f'(1) < \sup_{s\in(0,1)} \frac{f(1)-f(s)}{1-s},$$

e.g.



then $c^*_\epsilon > c_{L,\epsilon}$ for ϵ small

- no condition on reaction g

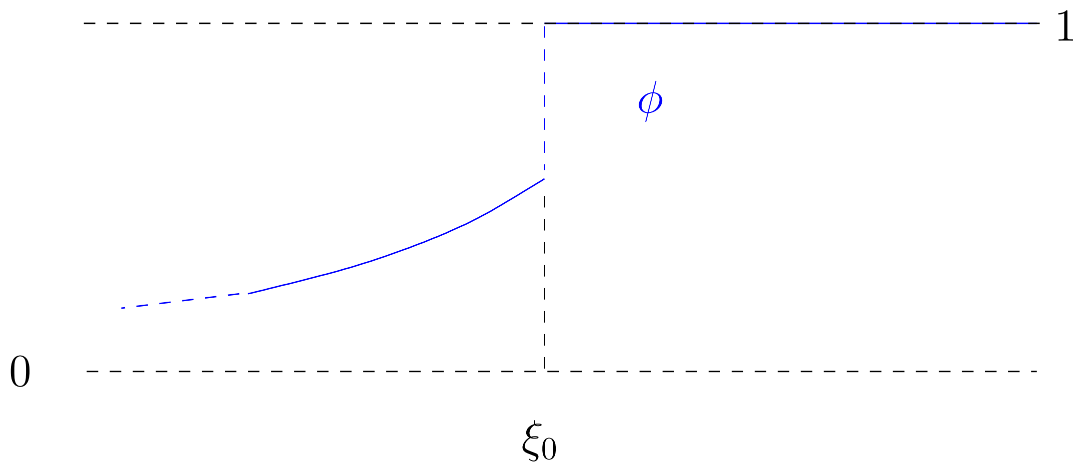
In fact, it can be shown that if

$$f'(1) < \sup_{s \in (0,1)} \frac{f(1) - f(s)}{1 - s},$$

then the minimal speed front solution of (H) is such that

$$\phi(\xi) = 1 \quad \text{for all} \quad \xi \geq \xi_0, \quad \phi(\xi_0^-) < 1$$

for some ξ_0



i.e. unstable state is followed by a shock propagating at the limiting speed c^*, so the pushing of the (P) front to a speed faster than $c_{L,\epsilon}$ is because of a compression effect from the hyperbolic structure of the equation