

# Asymptotic Stability and Resonances in Hamiltonian PDE's

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## An example

$$\begin{aligned}i\partial_t u(t, x) &= (-\Delta + V(x))u + \gamma(t)|u|^\alpha u \quad t \in (0, \infty), \quad x \in \mathbb{R}^n \\ u(0, x) &= u_0 \in H^1\end{aligned}$$

where  $n \geq 2$  and

- $V(x)$  is real valued,  $\widehat{V} \in L^1$  and

$$|V(x)|, |\nabla V(x)| \leq \frac{C}{(1 + |x|)^{3+}};$$

- $-\Delta + V$  as a self-adjoint operator on  $L^2$  has exactly one e-value  $E_0 < 0$  and no zero energy resonance.

The case of two or more e-values can also be treated.

## Assumptions for the nonlinearity I

$$i\partial_t u(t, x) = (-\Delta + V(x))u + \gamma(t)|u|^\alpha u$$

- $\gamma \in C(\mathbb{R}, \mathbb{R})$  bounded;
- $\alpha_0(n) < \alpha < \frac{4}{n-2}$ ,

where

$$\alpha_0(n) = \max \left\{ \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n}, \frac{3}{n} \right\} = \begin{cases} 1.5 & \text{for } n = 2 \\ 1 & \text{for } n = 3 \end{cases} .$$

Note that  $\alpha_0 < \alpha < 4/n$  is part of the subcritical regime.

**Theorem 1.** *If  $\gamma(t) \equiv \gamma \in \mathbb{R}$  and  $\max\{\|u_0\|_{H^1}, \|u_0\|_{L^1}\}$  is sufficiently small then  $u(t)$  approaches a periodic solution (ground state) as  $t \rightarrow \infty$ .*

More precisely there exists  $E, \psi_E(x), \theta(t) \xrightarrow{t \rightarrow \infty} 0$  such that

$$u(t) = e^{-it(E+\theta(t))} \psi_E(x) + r(t)$$

$$\|r(t)\|_{L^p} \leq \frac{C_p}{(1+|t|)^{n(\frac{1}{2}-\frac{1}{p})}} \quad 2 \leq p \leq \alpha + 2.$$

Similar results for supercritical nonlinearities in space dimensions 1 and 3-5 with localized initial data (Soffer-Weinstein, Pillet-Wayne, Buslaev-Perelman, Buslaev-Sulem, Cuccagna) for critical and higher nonlinearities in dimensions 1 to 3 with data in  $H^1$  (Mizumachi, Gustafson-Nakanishi-Tsai). For subcritical nonlinearities in dimensions 2 to 5 see Kirr-Zărnescu and Kirr-Mızrak.

## Assumptions for the nonlinearity II

The theorem can be extended to more general nonlinearities  $g(u)$  :

- $g : \mathbb{C} \mapsto \mathbb{C}$  is  $C^1$  (over the real structure of  $\mathbb{C}$ ),
- $g(e^{i\theta}u) = e^{i\theta}g(u)$ ,  $g(\bar{u}) = \overline{g(u)}$ ,
- $g(0) = 0$  and  $|g'(s)| \leq C(|s|^{\alpha_1} + |s|^{\alpha_2})$  where  $\alpha_0(n) < \alpha_1 \leq \alpha_2 < \frac{4}{n-2}$ ,  $s \in \mathbb{R}$ ,

and to *initial data near large solitary waves*.

**Theorem 2.** (Cuccagna-Kirrr-Pelinovsky, '06) Consider

$$\begin{aligned} i\partial_t u(t, x) &= (-\Delta + V(x))u + (\gamma_0 + \gamma_1 \cos(\omega t))|u|^2 u, \quad x \in \mathbb{R}^3 \\ u(0, x) &= u_0 \in H^1 \cap L^2_3 \end{aligned}$$

where  $V(x)$  is as before and  $\gamma_0, \gamma_1 \in \mathbb{R}$ . If the following resonance condition holds:

$$\Gamma = \frac{\gamma_1^2}{4} \Im \left\langle \psi_0^3, (-\Delta + V - E_0 - \omega - i0)^{-1} P_c \psi_0^3 \right\rangle \neq 0,$$

where  $\psi_0$  is the e-function of  $-\Delta + V$  corresponding to  $E_0$ , then there exists  $\varepsilon > 0$  such that for

$$\max\{\|u_0\|_{H^1}, \|u_0\|_{L^2_3}\} \leq \varepsilon$$

the solution  $u(t)$  converges to zero as  $|t| \rightarrow \infty$  strongly in  $L^p$ ,  $2 < p < 6$ , and weakly in  $H^1$ .

More precisely

$$u(t) = e^{-itE_0} A(t) \psi_0(x) + r(t)$$

where:

$$\begin{aligned} |A(t)| &\leq \frac{C_0 \varepsilon}{(1 + 4\Gamma \varepsilon^4 |t|)^{1/4}}, \\ \|r(t)\|_{L^2_{-3}} &\leq \frac{C_1 \varepsilon}{(1 + |t|)^{3/2}} + \frac{C_2 \varepsilon^3}{(1 + 4\Gamma \varepsilon^4 |t|)^{3/4}}, \\ \|r(t)\|_{L^4} &\leq \frac{C_3 \varepsilon}{(1 + |t|)^{3/4}} + \frac{C_4 \varepsilon^2}{(1 + 4\Gamma \varepsilon^4 |t|)^{1/2}}. \end{aligned}$$

## Assumptions for the nonlinearity III

Conjecture: The theorem can be extended to all dimensions, to more general nonlinearities

$$g(t, u) = \gamma_0 g_0(u) + \sum_{k=1}^{\infty} \gamma_k \cos(\omega_k t + \theta_k) g_k(u), \quad \sum_{k=1}^{\infty} |\gamma_k| = |\gamma| < \infty$$

$g_k$  satisfy uniformly assumptions II,

$$|E_0 + \omega_k| \geq \delta > 0, \quad k \in \mathbb{N}$$

and resonance condition holds at some order for at least one frequency.

Similar results should also be possible for large initial data near a soliton provided  $|\gamma|$  is small.

## Outline

- Generalities on asymptotic stability.
- Existence of solitary waves. Decomposition of dynamics.
- Asymptotic stability of solitary waves.
- Resonances and radiative damping.

## Generalities on asymptotic stability

Consider

$$\frac{du}{dt} = Au + F(t, u)$$

where

$$\begin{aligned} F(t, 0) &= 0 \\ \lim_{\|u\| \rightarrow 0} \frac{\|F(t, u)\|}{\|u\|} &= 0, \text{ uniformly in } t \geq 0 \end{aligned}$$

if there is  $\gamma > 0$  such that  $\|e^{At}\| \leq Ce^{-\gamma t}$ ,  $t \geq 0$  then  $u \equiv 0$  is asymptotically stable.

## Issues in PDE's

- There are elements in the spectrum of  $A$  with zero real part 0. Examples  $A = \Delta$  or  $A = -i\Delta$ . In fact Hamiltonian PDE's are linearly unstable if the spectrum is not purely imaginary.
- The important solutions might not be equilibria, hence the linearization  $A \mapsto A(t)$  and decaying estimates for its propagator are much harder to obtain.

Good news:  $e^{\Delta t}$ ,  $e^{-i\Delta t}$  decay polynomially.

## Existence of solitary waves

**Bound states:**  $u(t, x) = e^{-iEt}\psi_E(x)$  where:

$$F(\psi_E, E) = (-\Delta + V)\psi_E + \gamma|\psi_E|^\alpha\psi_E - E\psi_E = 0. \quad (1)$$

Fréchet derivative (over real structure) at  $\phi \in H^2$  and  $E \in \mathbb{R}$  is:

$$DF_\psi(\phi, E)[\eta] = L_{(\phi, E)}[\eta] = (-\Delta + V - E)\eta + \gamma\frac{\alpha + 2}{2}|\phi|^\alpha\eta + \gamma\frac{\alpha}{2}|\phi|^{\alpha-2}\phi^2\bar{\eta}$$

$L_{(0, E)}$  has nontrivial kernel at  $E_0$  and nontrivial solutions of (1) bifurcate from  $(0, E_0)$  :

$$\begin{aligned} \psi_E &= a\psi_0 + h(a), & \langle \psi_0, h(a) \rangle &= 0 \\ E &= E(|a|) \end{aligned}$$

where  $h : \{a \in \mathbb{C} : |a| < \delta\} \mapsto H^2(\mathbb{R}^2)$  is  $C^2$ .

Elliptic regularity and comparison theorems:  $\psi_E(x) \in H^3$  is exponentially decaying (in absolute value) as  $|x| \rightarrow \infty$ .

## How far can the branch be continued ?

For a real valued bound-state  $\psi_E$

$$L_{(\psi_E, E)} = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

where

$$\begin{aligned} L_-[v] &= (-\Delta + V - E)v + \gamma|\psi_\Omega|^2 v \\ L_+[u] &= (-\Delta + V - E)u + 3\gamma|\psi_\Omega|^2 u \end{aligned}$$

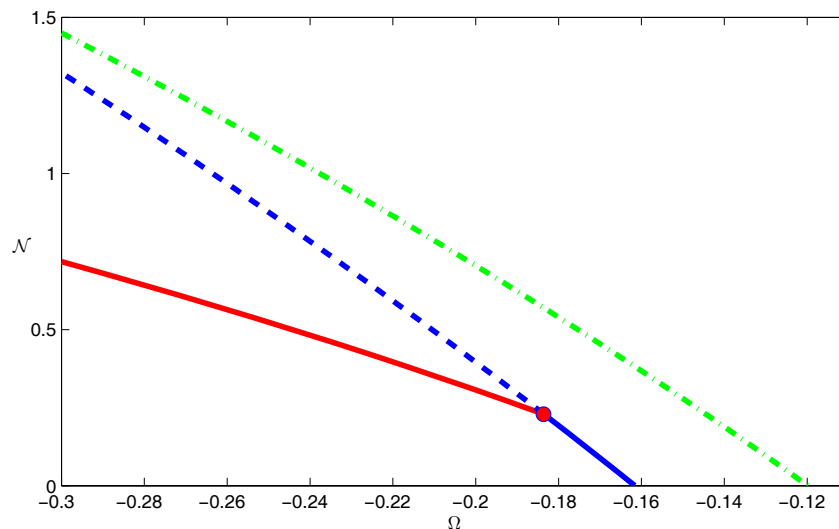
In general, via rotational equivariance:

$$L_{(e^{i\theta}\psi_E, E)} = e^{i\theta} L_{(\psi_E, E)} e^{-i\theta}$$

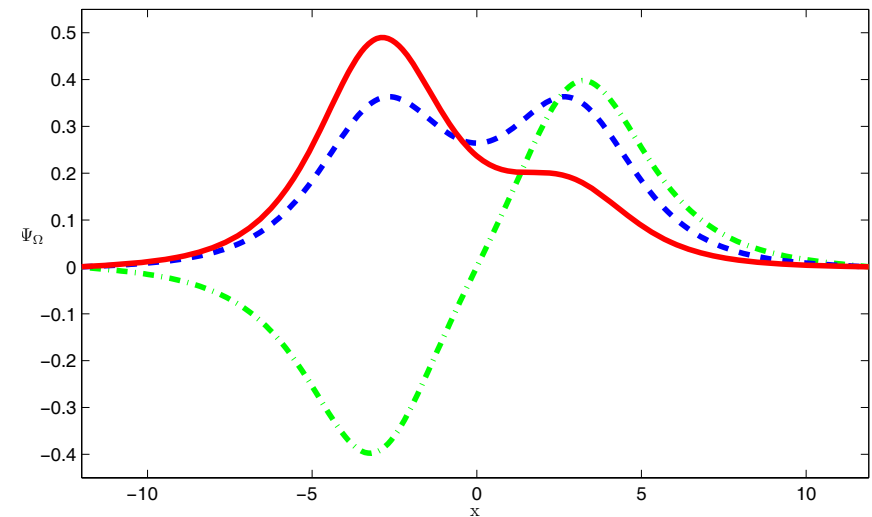
Note that  $\psi_E \in \ker L_-$  because of invariance of the set of solutions under rotations. Hence the  $(\psi_E, E)$  branch can be continued as long as  $\dim \ker L_+ = 0$  and  $\dim \ker L_- = 1$ .

While jumps in  $\dim \ker L_-$  are prevented by the fact that the lowest e-value of a second order strictly elliptic operator has to be simple, jumps in  $\dim \ker L_+$  occur under certain conditions, see Kevrekidis et al, Siam J. Math. Anal '08. In the following bifurcations diagrams  $\Omega = E$  and  $\mathcal{N} = \|\psi_E\|_{L^2}^2$  :

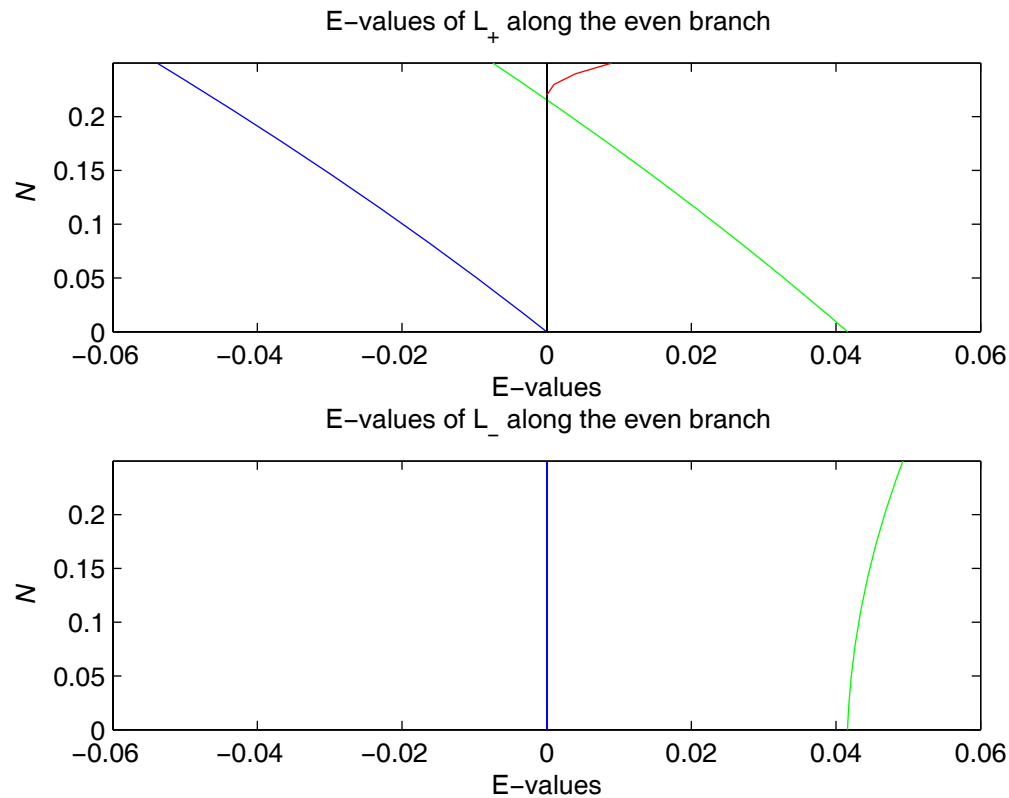
Bifurcation diagram



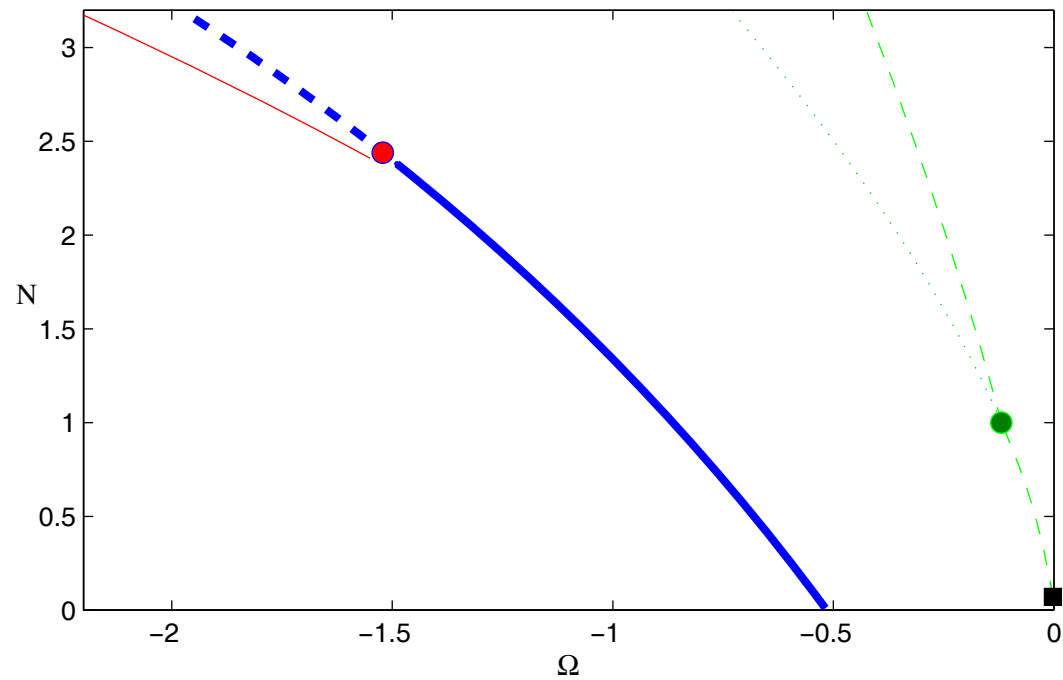
Bound-states at  $\mathcal{N} = 0.5$



Both the existence of the bifurcation point and the transfer of orbital stability are proved via analyzing the e-values of  $L_+$  in perturbative regimes:



In non-perturbative regimes Kevrekidis et al. (in prep.) can rigorously analyze bifurcation diagrams such as the following one (again  $\Omega = E$  and  $N = \|\psi_E\|_{L^2}^2$ ):



## Decomposition of the dynamics

The linearized dynamics at  $\psi_E$  is:

$$\partial_t \eta = -i(L_{\psi_E} + E)\eta$$

use the invariant subspaces of  $-iL_{\psi_E}$  to decompose  $L^2$  and the dynamics:

$$\begin{aligned} L^2 &= \text{span} \{ \partial_{a_1} \psi_E, \partial_{a_2} \psi_E \} \oplus H_a \\ H_a &= \{ -i\partial_{a_2} \psi_E, i\partial_{a_1} \psi_E \}^\perp \end{aligned}$$

If  $u(t)$  is the solution of the time dependent problem then

$$u(t) = a(t)\psi_0 + h(a(t)) + r(t) = \psi_{E(t)} + r(t), \quad r(t) \in H_{a(t)}$$

## The equations on components:

$$\partial_t r = -i(L_{\psi_{E(t)}} + E(t))r - iF_2(\psi_E, r) - (\mathbb{I} - M_u)^{-1} \tilde{F}_2(\psi_E, r)$$

$$D\psi_E|_a[a' + iEa] = (\mathbb{I} - M_u)^{-1} \tilde{F}_2(\psi_E, r)$$

where  $F_2(\psi_E, r)$  is the part of  $|\psi_E + r|^\alpha(\psi_E + r)$  containing second order and higher terms in  $r$  :

$$F_2(\psi_E, r) \sim \gamma[(\alpha + 1)\alpha\psi_E^{\alpha-1}r^2 + |r|^\alpha r]$$

and  $\tilde{F}_2(\psi_E, r)$  is the projection of  $-iF_2$  onto  $\text{span}\{\partial_{a_1}\psi_E, \partial_{a_2}\psi_E\}$ .

For the first equation Duhamel formula gives:

$$r(t) \sim \Omega(t, 0)r(0) - i\gamma \int_0^t \Omega(t, s)[(\alpha + 1)\alpha\psi_E(s)^{\alpha-1}r(s)^2 + |r(s)|^\alpha r(s)] ds$$

## Asymptotic stability

Another Duhamel formula:  $H = -\Delta + V$

$$\begin{aligned} r(t) &\sim e^{-iHt}r(0) - i\gamma \int_0^t e^{-iH(t-s)}(a+1)|\psi_E|^\alpha r(s) ds \\ &\quad - \dots - i\gamma \int_0^t e^{-iH(t-s)}|r|^\alpha r(s) ds \end{aligned}$$

Last term requires control of  $\|r(t)\|_{L^{\alpha+2}}$ . Minimal hypothesis is  $\alpha > \alpha_0$  if one can get:

$$\|r(t)\|_{L^{\alpha+2}} \leq \frac{C}{(1+t)^{n(\frac{1}{2}-\frac{1}{\alpha+2})}}$$

This requires the use of weighted  $L^2$  estimates for the **linear** term. To close the weighted estimates for the nonlinear term one needs  $\alpha > 4/n > \alpha_0(n)$  i.e. supercritical regimes.

The Duhamel formula for the full linear (and time dependent) operator:

$$r(t) \sim \Omega(t, 0)r(0) - i\gamma \int_0^t \Omega(t, s)[(\alpha+1)\alpha\psi_E(s)^{\alpha-1}r(s)^2 + |r(s)|^\alpha r(s)]ds$$

where

$$\begin{aligned} \partial_t \Omega(t, s) &= -i(L_{\psi_{E(t)}} + E(t))\Omega(t, s) \\ \Omega(s, s) &= \mathbb{I} \end{aligned}$$

avoids the weighted estimates at the expense of obtaining estimates of the form:

$$\|\Omega(t, s)\|_{L^{p'} \mapsto L^p} \leq \frac{C}{|t-s|^{n(\frac{1}{2}-\frac{1}{p})}}.$$

## Estimates for the linear propagator

Recall

$$\partial_t \Omega(t, s) = \overbrace{(-\Delta + V)}^H \Omega(t, s) + \gamma \frac{\alpha + 2}{2} |\psi_E|^\alpha \Omega(t, s) + \gamma \frac{\alpha}{2} |\psi_E|^{\alpha-2} \psi_E^2 \overline{\Omega}(t, s).$$

Then the integral form of the equation for the linear propagator is:

$$\Omega(t, s) \sim e^{-iH(t-s)} - i\gamma(\alpha + 1) \int_s^t e^{-iH(t-\tau)} |\psi_{E(\tau)}|^\alpha \Omega(\tau, s) d\tau.$$

Estimates require: *localization, smallness* of the scatterer  $|\psi_{E(\tau)}|^\alpha$  and *near integrable dispersive estimates* for  $H$ .

## Estimates for the linear propagator

- localization: otherwise the scatterer  $|\psi_{E(t)}|^\alpha$  could behave like a long range potential; comes free for linearization at solitary waves.
- smallness: otherwise radiation could be trapped near the scatterer via a parametric resonance phenomena; comes free for small solitary waves and can be obtained by moving to the slowly varying variable in the modulation equations.
- near integrable dispersive estimates for  $H$  : otherwise constructive interference may occur after radiation is scattered from  $|\psi_{E(t)}|^\alpha$ ; present for Schrödinger operators in  $n \geq 2$  space dimensions.

## Weighted estimates for the linear propagator

Contraction principle for the following fixed point problem:

$$\begin{aligned}
 & \langle x \rangle^{-\sigma} \Omega(t, s) \langle x \rangle^{-\sigma} \sim \langle x \rangle^{-\sigma} e^{-iH(t-s)} \langle x \rangle^{-\sigma} \\
 & + \int_s^t \langle x \rangle^{-\sigma} e^{-iH(t-\tau)} \langle x \rangle^{-\sigma} \underbrace{\langle x \rangle^\sigma |\psi_E(\tau)|^\alpha \langle x \rangle^\sigma}_{\text{small}} \\
 & \langle x \rangle^{-\sigma} \Omega(\tau, s) \langle x \rangle^{-\sigma} d\tau,
 \end{aligned}$$

implies non-trapping.

For non-constructive interference use  $T(t, s) = \mathbf{P}_c[\Omega(t, s) - e^{-iH(t-s)}]$  which satisfies:

$$\begin{aligned}
 T(t, s) & \sim -i\gamma(\alpha + 1) \int_s^t e^{-iH(t-\tau)} \mathbf{P}_c |\psi_E|^\alpha e^{-iH(\tau-s)} \mathbf{P}_c \\
 & - i\gamma(\alpha + 1) \int_s^t e^{-iH(t-\tau)} \mathbf{P}_c |\psi_E|^\alpha T(\tau, s) d\tau.
 \end{aligned}$$

## Non-weighted estimates for the linear propagator

In space dimension  $n = 2$  for  $p > 2$  :

$$\begin{aligned} \|T(t, s)\|_{L^{p'} \mapsto L^p} &\sim \left\| \int_s^t e^{-iH(t-\tau)} \mathbf{P}_c |\psi_E|^\alpha e^{-iH(\tau-s)} \mathbf{P}_c d\tau \right\|_{L^{p'} \mapsto L^p} \\ &\leq \int_s^t (t-\tau)^{2/p-1} (\tau-s)^{2/p-1} d\tau \sim (t-s)^{4/p-1} \end{aligned}$$

If the decay estimates are allowed to switch from  $2/p - 1$  for the free Schrödinger operator to  $4/p - 1$  for  $\Omega(t, s)$ , then estimates for the nonlinear terms require supercritical nonlinearities. To fix use  $p = \infty$ . This requires removal of singularities at  $\tau = t, s$ , via a generalized Fourier Multiplier method, which needs Fourier Transform of  $|\psi_E|^\alpha$  in  $L^1$ .

## Resonance phenomena and radiation damping

$$\begin{aligned}i\partial_t u(t, x) &= (-\Delta + V(x))u + (\gamma_0 + \gamma_1 \cos(\omega t))|u|^2 u, \quad x \in \mathbb{R}^3 \\ u(0, x) &= u_0 \in H^1 \cap L^2_3 \text{ small}\end{aligned}$$

Using the decomposition w.r.t. the linearized equation at  $u \equiv 0$  (towards which the dynamics actually converges on long times):

$$u(t) = e^{-iE_0 t} A(t) \psi_0 + u_d(t),$$

we get the coupled system:

$$\begin{aligned}\partial_t u_d(t) &= -i(-\Delta + V)u_d - i\gamma(t)|A|^2(t)A\psi_0^3 + \dots \\ \frac{d|A|}{dt} &= -\Gamma|A|^5 + \dots\end{aligned}$$

$$\begin{aligned}
u_d(t) &= e^{-iHt} \mathbf{P}_c u_d(0) \sim \frac{\varepsilon}{(1 + |t|)^{3/2}} \\
&- i \int_0^t \gamma(s) e^{-iE_0 s} |A|^2 A(s) e^{-iH(t-s)} \mathbf{P}_c \psi_0^3 ds \sim \frac{\varepsilon^3}{(1 + 4\Gamma \varepsilon^4 |t|)^{3/4}} \\
&+ K_A[u_d](t)
\end{aligned}$$

If the **nonlinear term** mixes the time scales one gets

$$u_d(t) = \text{green term} + \text{red term} + \text{correction} \sim \varepsilon^2 (1 + 4\Gamma \varepsilon^4 |t|)^{-3/4}$$

The unknown **correction** dominates the red **resonant term** and the dynamics of  $|A(t)|$ .

To avoid mixing one uses a *superposition principle* for  $K_A$  which is *locally contractive* in Banach spaces of functions with time decay tailored to each **green** and **red** forcing term:

## Contractions in hierarchy of Banach spaces

Let

$$u = f + g + K[u] \equiv K_1[u], \quad f \in X, \quad g \in Y, \quad X \hookrightarrow Y.$$

Assume  $K$  is contractive in both  $X$  and  $Y$  with Lipschitz constants  $0 < L_X, L_Y < 1$ . Solve first

$$v = f + K[v] \tag{2}$$

then approximate

$$u = v + g + corr \tag{3}$$
$$\|corr\|_Y \leq \frac{1}{1 - L_Y} \|K_1[v + g] - v - g\|_Y \leq \frac{L_Y}{1 - L_Y} \|g\|_Y$$

If needed  $v$  can be approximated using (2):

$$\|u - f - g\|_Y \leq \frac{L_X}{1 - L_X} \|f\|_X + \frac{L_Y}{1 - L_Y} \|g\|_Y$$

## Conclusions

- There is a general way to obtain dispersive estimates for small, localized, time-dependent perturbations of sufficiently dispersive operators. *This directly applies to asymptotic stability of solitary waves.*
- There is a general way to keep the effects of different forcing terms separated in equations involving contractive operators. *This is essential in rigorous analysis of resonance phenomena in dispersive equations.*
- Other applications?