Harmonic maps and the classification of stationary black-holes

João Lopes Costa

Lisbon University Institute (ISCTE)
Mathematical Institute and Magdalen College, Oxford

June 2009

A space-time is a time oriented Lorentzian manifolds $(\mathscr{M}, g_{\mu\nu})$

- $g_{\mu\nu}(p)$
  - symmetric non-degenerate bilinear form on $T_p\mathscr{M} \approx \mathbb{R}^{n+1}$
  - signature $(- + + \ldots +)$
- $\mathscr{M} \ni p \mapsto g_{\mu\nu}(p)$ is C^1.

satisfying Einstein equations

The Einstein field equations
(as seen in Uyuni, Bolivia, summer 2005; photographed by Madalena Miranda)

Vacuum

- Energy-momentum tensor

$$T_{\mu\nu} = 0$$

- Einstein equations

$$R_{\mu\nu} = 0$$

in local coordinates $(x^0, x^1, \ldots, x^n)$ read

$$\frac{1}{2} g^{\alpha\beta} (\partial_\mu \partial_\alpha g_{\beta\nu} + \partial_\nu \partial_\alpha g_{\beta\mu} - \partial_\mu \partial_\nu g_{\alpha\beta} - \partial_\alpha \partial_\beta g_{\mu\nu}) + \ldots = 0$$

Vacuum Solutions

- Minkowski ($\mathbb{R}^{1,3}$)

$$\mathscr{M} = \mathbb{R}^4$$

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

- Schwarzschild

$$\mathscr{M} = \mathbb{R} \times (2m, +\infty) \times S^2$$

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)$$

Electro-Vacuum

- Energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right)$$

- $F_{\mu\nu}$ is the electromagnetic 2-form, which satisfies Maxwell's equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\nabla^\mu F_{\mu\nu} = 0$$

Majumdar-Papapetrou

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)$$

$$A = u^{-1}dt$$

Einstein-Maxwell read

$$\frac{\partial u}{\partial t} = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Chruściel and Nadirashvili: for a "regular" MP spacetime

$$\mathscr{M} = \mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_N\})$$

$$u = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}$$

Majumdar-Papapetrou

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)$$

$$A = u^{-1}dt$$

Einstein-Maxwell read

$$\frac{\partial u}{\partial t} = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Chruściel and Nadirashvili: for a “regular” MP spacetime

$$\mathscr{M} = \mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_1, \ldots, \vec{a}_N\})$$

$$u = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}$$

Causal Theory

g has signature $(- + + \ldots +)$

- Causal Vectors
  - X is timelike if
$$g(X,X) < 0$$
  - X is null if
$$g(X,X) = 0 \ \ , \ \text{and } X \neq 0$$
- Spacelike Vectors
$$g(X,X) > 0 \ \ , \ \text{or } X = 0$$

- Time orientation:

$$g(T,T)|_p < 0 \quad ,\forall p \in \mathscr{M}$$

- X is future directed at q if

$$g(T,X)|_q < 0$$

- Chronological future

$$I^+(p) = \{q \in \mathscr{M} \quad | \quad \exists \gamma : [0,1] \to \mathscr{M}$$
$$\gamma(0) = p \ , \ \gamma(1) = q$$
$$\dot{\gamma}(t) \text{ timelike future directed } \forall t\}$$

What is a **stationary** asymptotically flat black hole?

Definitions more technical if non-stationary

Stationarity: there exists a Killing Vector K_0 such that

$$g(K_0, K_0)|_q < 0 \ , \ \text{for some } q \in \mathscr{M} \ .$$

Assume also that K_0 is complete, so that its flow generates an action of $\mathbb{R}$ by isometries

$$\mathbb{R} \times \mathscr{M} \ni (s, q) \mapsto \phi_s(q) \in \mathscr{M} \ .$$

What is a stationary asymptotically flat black hole?

(Stationary) **Asymptotic Flatness:**
Stationary region (where K_0 is timelike) contains an end

$$\mathscr{S}_{\text{ext}} \approx \mathbb{R}^n \setminus B(R) \; .$$

We are able to choose coordinates in $\mathbb{R} \times \mathscr{S}_{\text{ext}} \approx \mathscr{M}_{\text{ext}}$ s.t.

$$g_{\mu\nu} = \eta_{\mu\nu} + O_k(r^{-1})$$

$$\partial_t g_{\mu\nu} = 0$$

$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$

What is a stationary asymptotically flat black hole?

Exterior region:

$$\mathscr{M}_{\text{ext}} := \cup_{t \in \mathbb{R}} \phi_t(\mathscr{S}_{\text{ext}})$$

Domain of outer communications:

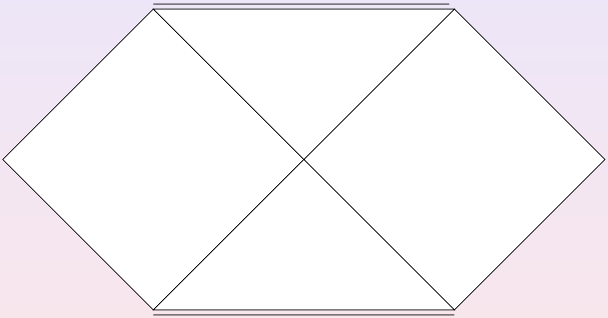
$$\text{d.o.c.} := I^-(\mathscr{M}_{\text{ext}}) \cap I^+(\mathscr{M}_{\text{ext}})$$

Black Hole:

$$\mathscr{B} := \mathscr{M} \setminus I^-(\mathscr{M}_{\text{ext}})$$

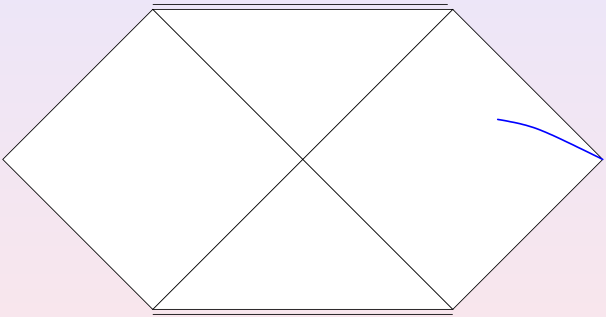
In Schwarzschild:

Our main model our inspiration



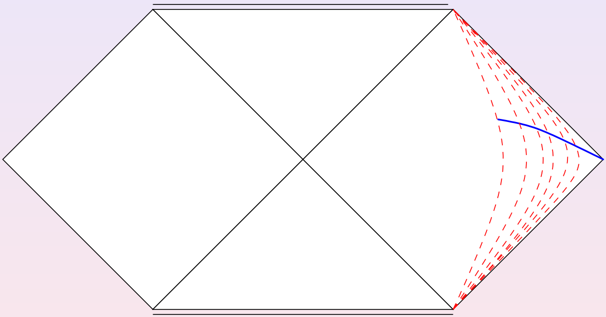
In Schwarzschild: Asymptotic Flat End $\mathscr{I}_{ext}$

Our main model our inspiration



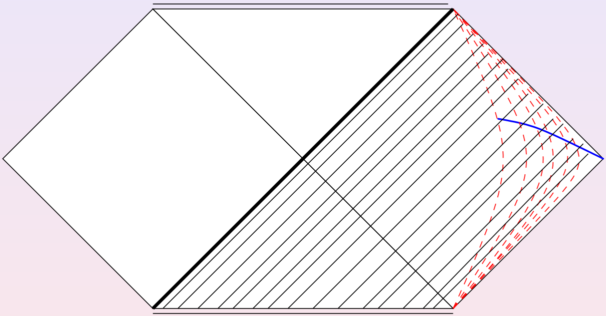
The main model: Exterior region $\mathscr{M}_{\text{ext}}$

Our main model our inspiration



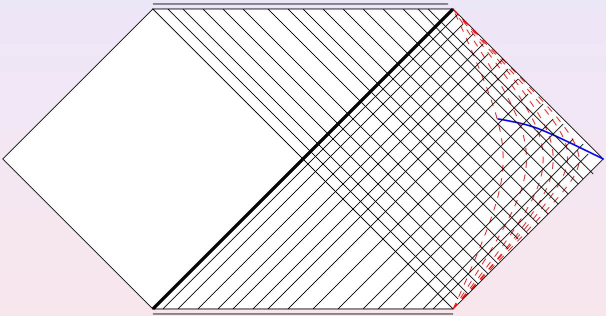
In Schwarzschild: Past of exterior region $I^-(\mathscr{M}_{\text{ext}})$

Our main model our inspiration



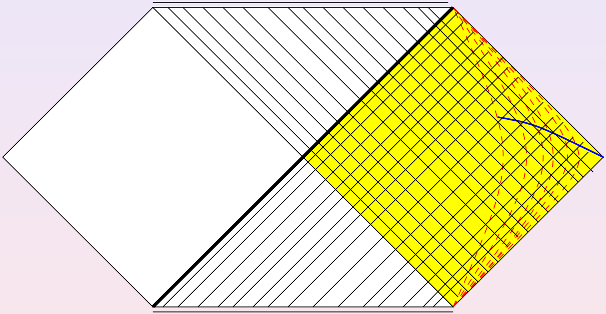
In Schwarzschild: Future of exterior region $I^+(\mathscr{M}_{\mathrm{ext}})$

Our main model our inspiration



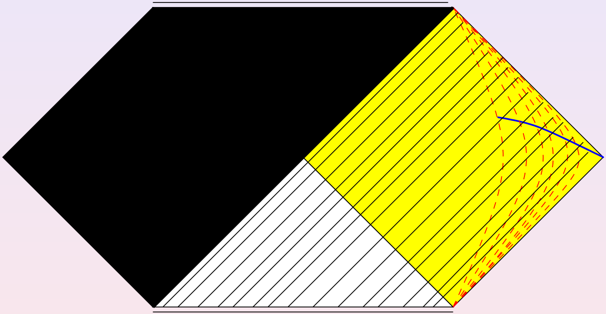
In Schwarzschild: Domain of outer communication d.o.c.

Our main model our inspiration



In Schwarzschild: Black hole region $\mathscr{M} \setminus I^{-}(\mathscr{M}_{\text{ext}})$

Our main model our inspiration



Conjecture

4-dimensional
Stationary
Electro-vacuum

¿ regular? d.o.c.

Conjecture

4-dimensional
Stationary
Electro-vacuum

¿ regular? d.o.c.

=

d.o.c. of
Kerr-Newman, if horizon connected
and
Majundar-Papapetrou, otherwise

Proved

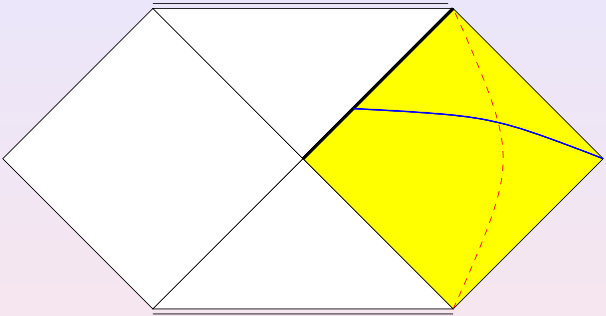
Contributions by Israel, Hawking, Carter, Robinson, Bunting, Masood-ul-Alam, Mazur, Sudarsky, Wald, Chruściel, Weinstein, JLC

4-dimensional
Stationary
Electro-vacuum
Analytic
I^+-regular d.o.c.
with non-degenerate horizons
=
d.o.c. of
a Weinstein solution

Connected, analytic, non-degenerate electro-vacuum

4-dimensional
Stationary
Electro-vacuum
Analytic
I^+-regular d.o.c.
with connected, non-degenerate horizon
=
d.o.c. of
Kerr-Newmann

Schwarzschild as I^+-regular



For such a choice of initial surface we have:

$$\text{d.o.c.} \cup \mathscr{E}^+ \approx \mathbb{R} \times \overline{\mathscr{S}}$$

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)

- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)
  If non-degenerate we have:

$$dK_0^\flat \wedge K_0^\flat = 0$$

  Static Black holes are Schwarzschild ?
  Yes! For all space-time dimensions, if positivity of mass applies.
  Requires analyticity !?

- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)

  Schwarzschild
- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)

  Schwarzschild

- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)
  $\exists$ a second global and complete Killing vector K_1?
  Yes! As long as (M, g) and $\mathscr{E}^+$ are analytic.

  In here lies the infamous need for analyticity.

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)
  Schwarzschild
- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)
  $\exists$ a second global and complete Killing vector K_1?
  Yes! As long as (M, g) and $\mathscr{E}^+$ are analytic.

In here lies the infamous need for analyticity.

$\mathscr{E}^+$ is a priori only a Lipschitz continuous, null hypersurface
If we assume I^+-regularity

$\mathscr{E}^+$ is as regular as the metric allows

in particular

$\mathscr{E}^+$ is analytic if the metric is

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)

  Schwarzschild

- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)
  $\exists$ a second global and complete Killing vector K_1
  Yes! As long as (M, g) and $\mathscr{E}^+$ are analytic.

  In here lies the infamous need for analyticity.

  Alexakis, Ionescu and Klainerman have proven rigidity for:

  C^∞ and I^+-regular black holes

  and

  close to Kerr geometries

$\mathscr{E}^+$ is either

- Non-rotating (K_0 is tangent to the generators of $\mathscr{E}^+$)

  Schwarzschild

- Rotating (K_0 isn't tangent to the generators of $\mathscr{E}^+$)
  $\exists$ a second global and complete Killing vector K_1
  Yes! As long as (M, g) and $\mathscr{E}^+$ are analytic.

  K_0 and K_1 generate action of $\mathbb{R} \times S^1$ by isometries

  The Stationary-Axisymmetric case.

Orthogonal integrability and the Area function
Constructing Weyl coordinates

Assuming (electro-)vacuum

$$dK_0 \wedge K_0 \wedge K_1 = dK_1 \wedge K_0 \wedge K_1 = 0$$

The area function

$$W := -\det \begin{pmatrix} g(K_0, K_0) & g(K_0, K_1) \\ g(K_1, K_0) & g(K_1, K_1) \end{pmatrix}$$

- $W > 0$ in d.o.c. $\setminus$ *axis* and
- W=0 at $\partial(\text{d.o.c.}) \cup$ *axis*

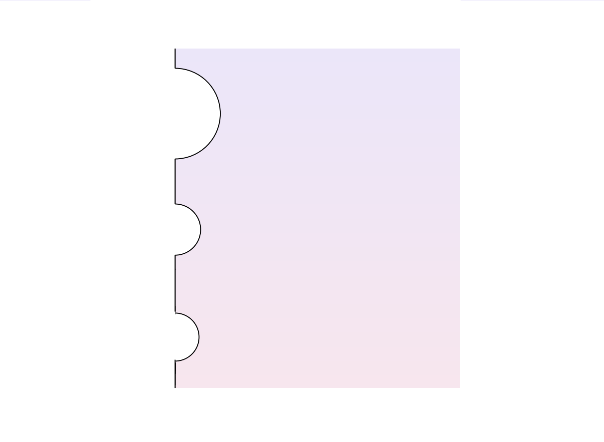
Assuming (electro-)vacuum

$$dK_0 \wedge K_0 \wedge K_1 = dK_1 \wedge K_0 \wedge K_1 = 0$$

The area function

$$W := -\det \begin{pmatrix} g(K_0, K_0) & g(K_0, K_1) \\ g(K_1, K_0) & g(K_1, K_1) \end{pmatrix}$$

- $W > 0$ in $\mathrm{d.o.c.} \setminus$ *axis* and
- W=0 at $\partial(\mathrm{d.o.c.}) \cup$ *axis*



$$\text{d.o.c.} \setminus axis \approx \underbrace{\mathbb{R} \times \mathbb{R}^+}_{Orbit\ Space} \times \underbrace{\mathbb{R} \times S^1}_{Group}$$

- Suppose all horizon components are non-degenerate
- Then the orbit space metric

$$q(Z_1,Z_2) = g(Z_1,Z_2) - h^{\mu\nu} g(Z_1,K_\mu) g(Z_2,K_\nu) \ , \ h_{\mu\nu} = g(K_\mu,K_\nu)$$

may be extended to an axisymmetric AF metric in $\mathbb{R}^2$.

- Construct asymptotically flat isothermal coordinates

$$q = e^{2\hat{u}}(dx^2 + dy^2) \ , \ \hat{u} = O_k(r^{-1})$$

- Construct ρ : Orbit Space $\to [0,+\infty)$ satisfying

$$\Delta_q \rho = 0 \quad , \quad \rho|_{\mathscr{E}^+ \cup \text{axis}} = 0$$

- Then $d\rho \neq 0$ and

$$q = e^{2u}(d\rho^2 + dz^2) \ , \ u = O_k(r^{-1})$$

- Suppose all horizon components are non-degenerate
- Then the orbit space metric

$$q(Z_1, Z_2) = g(Z_1, Z_2) - h^{\mu\nu} g(Z_1, K_\mu) g(Z_2, K_\nu) \ , \ h_{\mu\nu} = g(K_\mu, K_\nu)$$

may be extended to an axisymmetric AF metric in $\mathbb{R}^2$.

- Construct asymptotically flat isothermal coordinates

$$q = e^{2\hat{u}}(dx^2 + dy^2) \ , \ \hat{u} = O_k(r^{-1})$$

- Construct ρ : Orbit Space $\to [0, +\infty)$ satisfying

$$\Delta_q \rho = 0 \quad , \quad \rho|_{\mathscr{E}^+ \cup \text{axis}} \equiv 0$$

- Then $d\rho \neq 0$ and

$$q = e^{2u}(d\rho^2 + dz^2) \ , \ u = O_k(r^{-1})$$

- Field equations and Area Function Theorem imply

$$\Delta_q \sqrt{W} = 0 \quad , \quad \sqrt{W}|_{\mathscr{E}^+ \cup \text{axis}} \equiv 0$$

and maximum principle yields

$$\rho = \sqrt{W}$$

- We conclude that the metric on d.o.c. $\setminus$ axis has a global coordinate representation as

$$g = -\rho^2 e^{2\lambda} dt^2 + e^{-2\lambda}(d\varphi - v dt)^2 + e^{2u}(d\rho^2 + dz^2) \, .$$

$$K_0 = \partial_t \quad , \quad K_1 = \partial_\varphi$$

- Field equations and Area Function Theorem imply

$$\Delta_q \sqrt{W} = 0 \quad , \quad \sqrt{W}|_{\mathscr{E}^+ \cup \text{axis}} \equiv 0$$

and maximum principle yields

$$\rho = \sqrt{W}$$

- We conclude that the metric on d.o.c. $\setminus$ axis has a global coordinate representation as

$$g = -\rho^2 e^{2\lambda} dt^2 + e^{-2\lambda}(d\varphi - v dt)^2 + e^{2u}(d\rho^2 + dz^2) \ .$$

$$K_0 = \partial_t \quad , \quad K_1 = \partial_\varphi$$

Reduction to a harmonic map

Setting

$$\lambda := -\frac{1}{2}\log(g(K_1, K_1)) \qquad \frac{1}{2}d\omega = *(dK_1^\flat \wedge K_1^\flat)$$

in Weyl coordinates the vacuum Einstein equations

$$R_{\mu\nu} = 0$$

are equivalent to

$$(\Delta\lambda - 2e^{4\lambda}|\nabla\omega|^2)^2 + e^{4\lambda}(\Delta\omega + 4\nabla\lambda\cdot\nabla\omega)^2 = 0$$

Reduction to a harmonic map

$\Phi = (\lambda, \omega)$ is, for every $\Omega \subset\subset \mathbb{R}^3 \setminus$ axis, a critical point of

$$\int_\Omega |\nabla \lambda|^2 + e^{4\lambda} |\nabla \omega|^2$$

A harmonic map

$$\Phi = (\phi^1, \dots \phi^n) : (M, g) \to (N, h)$$

is, for every $\Omega \subset\subset M$, a critical point of

$$\int_\Omega h_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j$$

Reduction to a harmonic map

$\Phi = (\lambda, \omega)$ is, for every $\Omega \subset\subset \mathbb{R}^3 \setminus$ axis, a critical point of

$$\int_\Omega |\nabla\lambda|^2 + e^{4\lambda}|\nabla\omega|^2$$

A harmonic map

$$\Phi = (\phi^1, \ldots \phi^n) : (M, g) \to (N, h)$$

is, for every $\Omega \subset\subset M$, a critical point of

$$\int_\Omega h_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j$$

Reduction to a harmonic map

$\Phi = (\lambda, \omega)$ is, for every $\Omega \subset\subset \mathbb{R}^3 \setminus$ axis, a critical point of

$$\int_\Omega |\nabla\lambda|^2 + e^{4\lambda}|\nabla\omega|^2$$

Space-time metric is uniquely determined by the harmonic map

$$\Phi = (\lambda, \omega) : \mathbb{R}^3 \setminus \text{axis} \to \mathbb{H}^2$$

where $\mathbb{H}^2$ is the hyperbolic space with metric

$$b := d\lambda^2 + e^{4\lambda} d\omega^2$$

For any set of axis data:

- $\{\omega_i\}$ values of ω on the connected components of the axis
- I_i intervals representing horizons in the (ρ, z)–plane

there exists a unique harmonic map

$$\Phi : \mathbb{R}^3 \setminus \text{axis} \to \mathbb{H}^2$$

whose distance to the reference (not necessarily harmonic) map Φ_0 with the same axis data satisfies

$$d(\Phi, \Phi_0) \in L^\infty$$
$$d(\Phi, \Phi_0) \to_{r \to +\infty} 0$$

Fundamental ingredient: $\mathbb{H}^2$ has negative sectional curvature.

For any set of axis data:

- $\{\omega_i\}$ values of ω on the connected components of the axis
- I_i intervals representing horizons in the (ρ, z)–plane

there exists a unique harmonic map

$$\Phi : \mathbb{R}^3 \setminus \text{axis} \to \mathbb{H}^2$$

whose distance to the reference (not necessarily harmonic) map Φ_0 with the same axis data satisfies

$$d(\Phi, \Phi_0) \in L^\infty$$
$$d(\Phi, \Phi_0) \to_{r \to +\infty} 0$$

Fundamental ingredient: $\mathbb{H}^2$ has negative sectional curvature.

For any set of axis data:

- $\{\omega_i\}$ values of ω on the connected components of the axis
- I_i intervals representing horizons in the (ρ, z)–plane

there exists a unique harmonic map

$$\Phi : \mathbb{R}^3 \setminus \text{axis} \to \mathbb{H}^2$$

whose distance to the reference (not necessarily harmonic) map Φ_0 with the same axis data satisfies

$$d(\Phi, \Phi_0) \in L^\infty$$
$$d(\Phi, \Phi_0) \to_{r \to +\infty} 0$$

Fundamental requirement: $\mathbb{H}^2$ has negative sectional curvature.

The pointwise distance between the two maps $\Phi_i = (\lambda_i, \omega_i)$ is

$$\cosh(d) - 1 = \tfrac{1}{2}(e^{2(U_1 - U_2)} + e^{2(U_2 - U_1)} - 2) + 2\rho^{-4} e^{2(U_1 + U_2)}(\omega_1 - \omega_2)^2$$

where we have used the rescaling

$$U := \lambda + \log \rho = -\frac{1}{2}\log(g_{\varphi\varphi}/\rho^2)$$

Controlling the distance
Where the axis meets the horizon

$$U = -\frac{1}{2}\log(g_{\varphi\varphi}/\rho^2)$$

and

$$g_{\varphi\varphi} = f(x,y)\,x^2 \quad , C^{-1} \leq f(x,y) \leq C$$

So the key is in understanding the map

$$x + iy \leftrightarrow \rho + i\,z$$

In fact

$$x + iy = a\sqrt{-z + i\rho} + O(\rho^2 + z^2)$$

yielding

$$U = \log(\sqrt{z + \sqrt{\rho^2 + z^2}}) + O(1)$$

Controlling the distance
Where the axis meets the horizon

$$U = -\frac{1}{2}\log(g_{\varphi\varphi}/\rho^2)$$

and

$$g_{\varphi\varphi} = f(x,y)\,x^2 \quad , C^{-1} \leq f(x,y) \leq C$$

So the key is in understanding the map

$$x + iy \leftrightarrow \rho + i\,z$$

In fact

$$x + iy = a\sqrt{-z + i\rho} + O(\rho^2 + z^2)$$

yielding

$$U = \log(\sqrt{z + \sqrt{\rho^2 + z^2}}) + O(1)$$

Also

$$|\omega_i - \mathring{\omega}| \leq C x_i^2$$

consequently

$$\begin{aligned}
\frac{(\omega_1 - \omega_2)^2}{\rho^4 e^{-2U_1 - 2U_2}} &= \frac{(\omega_1 - \omega_2)^2}{g_1(\partial_\varphi, \partial_\varphi) g_2(\partial_\varphi, \partial_\varphi)} \leq 2 \frac{(\omega_1 - \mathring{\omega})^2 + (\omega_2 - \mathring{\omega})^2}{g_1(\partial_\varphi, \partial_\varphi) g_2(\partial_\varphi, \partial_\varphi)} \\
&= 2 \underbrace{\left(\frac{\omega_1 - \mathring{\omega}}{g_1(\partial_\varphi, \partial_\varphi)} \right)^2}_{\leq C^2} \underbrace{\frac{g_1(\partial_\varphi, \partial_\varphi)}{g_2(\partial_\varphi, \partial_\varphi)}}_{= e^{2(U_2 - U_1)}} \\
&+ 2 \underbrace{\left(\frac{\omega_2 - \mathring{\omega}}{g_2(\partial_\varphi, \partial_\varphi)} \right)^2}_{\leq C^2} \underbrace{\frac{g_2(\partial_\varphi, \partial_\varphi)}{g_1(\partial_\varphi, \partial_\varphi)}}_{= e^{2(U_1 - U_2)}} \leq C
\end{aligned}$$

Also

$$|\omega_i - \mathring{\omega}| \leq Cx_i^2$$

consequently

$$\begin{aligned}
\frac{(\omega_1-\omega_2)^2}{\rho^4 e^{-2U_1-2U_2}} &= \frac{(\omega_1-\omega_2)^2}{g_1(\partial_\varphi,\partial_\varphi)g_2(\partial_\varphi,\partial_\varphi)} \leq 2\frac{(\omega_1-\mathring{\omega})^2+(\omega_2-\mathring{\omega})^2}{g_1(\partial_\varphi,\partial_\varphi)g_2(\partial_\varphi,\partial_\varphi)} \\
&= 2\underbrace{\left(\frac{\omega_1-\mathring{\omega}}{g_1(\partial_\varphi,\partial_\varphi)}\right)^2}_{\leq C^2}\underbrace{\frac{g_1(\partial_\varphi,\partial_\varphi)}{g_2(\partial_\varphi,\partial_\varphi)}}_{=e^{2(U_2-U_1)}} \\
&+ 2\underbrace{\left(\frac{\omega_2-\mathring{\omega}}{g_2(\partial_\varphi,\partial_\varphi)}\right)^2}_{\leq C^2}\underbrace{\frac{g_2(\partial_\varphi,\partial_\varphi)}{g_1(\partial_\varphi,\partial_\varphi)}}_{=e^{2(U_1-U_2)}} \leq C
\end{aligned}$$

The coordinate system

$$(x = \rho \cos \varphi, y = \rho \sin \varphi, z)$$

is asymptotically flat and

$$g_{\varphi\varphi} = \rho^2(1 + O(r^{-1}))$$

Using the multipole expansion of Beig and Simon to improve the decay rates provided by asymptotic flatness

$$\omega = 4J - \frac{J}{2}\frac{z}{r}\left(\frac{3\rho^2 - z^2}{r^2} + 9\right) + \rho\, O(\log r/r^2)$$

What is missing?

- Remove analyticity condition
  - Alexakis, Ionescu and Klainerman's 2009 results go a long way
- Remove degeneracy condition
  - We expect our results to solve some of the main issues
- All non-static and non-connected Weinstein solutions are singular?
  - Weinstein and Li: non-regularity of the relevant harmonic maps for slowly rotating black holes
  - Neugebauer and Hennig last month!

What is missing?

- Remove analyticity condition
  - Alexakis, Ionescu and Klainerman's 2009 results go a long way
- Remove degeneracy condition
  - We expect our results to solve some of the main issues
- All non-static and non-connected Weinstein solutions are singular?
  - Weinstein and Li: non-regularity of the relevant harmonic maps for slowly rotating black holes
  - Neugebauer and Hennig last month!

What is missing?

- Remove analyticity condition
  - Alexakis, Ionescu and Klainerman's 2009 results go a long way
- Remove degeneracy condition
  - We expect our results to solve some of the main issues
- All non-static and non-connected Weinstein solutions are singular?
  - Weinstein and Li: non-regularity of the relevant harmonic maps for slowly rotating black holes
  - Neugebauer and Hennig last month!