-OxPDE Lunchtime seminar-

Transonic Shocks
in
Divergent Nozzles

(collaboration with Mikhail Feldman, UW-Madison)

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4 February, 2010

Conservation Laws

$$\partial_t \int_\Omega q(x,t)\;dx - \int_{\partial\Omega} \vec{F}(q,x,t)\cdot \vec{n}_{out}\;dA(x) = 0$$

or $\partial_t q + div\vec{F} = 0$

Steady State

$$\boxed{div\vec{F} = 0}$$

Euler system for steady compressible inviscid flow

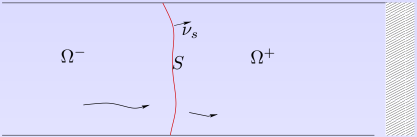
$$div(\rho\vec{u}) = 0, \quad div(\rho\vec{u}\otimes\vec{u} + pI) = \vec{0}$$

$$div(\vec{u}\rho(\frac{1}{2}|\vec{u}|^2 + e + \frac{p}{\rho})) = div(\rho\vec{u}B) = 0$$

Ideal polytropic gas $e = \frac{p}{(\gamma-1)\rho}$ $(\gamma > 1)$, entropy $= c_v \ln\frac{p}{\rho^\gamma} + \kappa$

Sound speed $c = \sqrt{\frac{\gamma p}{\rho}}$ Mach number $M = \frac{|\vec{u}|}{c}$

$(\rho, \vec{u}, p)$ is supersonic if $M > 1$, subsonic if $M < 1$, sonic if $M = 1$.



If $\vec{F} \in C^1(\Omega^\pm) \cap L^1_{loc}(\Omega)$ satisfies $div\vec{F} = 0$ in $\Omega^\pm$, then $\forall \phi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_{\Omega^+\cup\Omega^-} \vec{F}\cdot D\phi dx &= \int_S (\vec{F}^-\cdot\nu^-_{out} + \vec{F}^+\cdot\nu^+_{out})\phi dA \\ &= \int_S (\vec{F}^- - \vec{F}^+)\cdot\nu_s\phi dA\end{aligned}$$

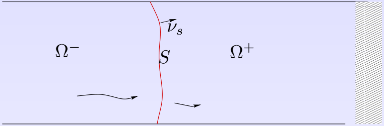
Rankine-Hugoniot conditions

$$[\rho\vec{u}\cdot\nu_s]_S = [\rho(\vec{u}\cdot\nu_s)\vec{u} + p\nu_s]_S = [\rho B\vec{u}\cdot B]_S = 0$$

Shock A jump transition of flow data across a surface with $\vec{u}\cdot\nu_s \neq 0$
$M_1 > 1 \xrightarrow{S} M_2 > 1$: supersonic-supersonic shock
$M_1 > 1 \xrightarrow{S} M_2 < 1$: supersonic-subsonic shock or transonic shock Movie

A shock solution to the steady Euler system

$(\rho, \vec{u}, p)$ weak solution in Ω + C^1 solution in $\Omega^{\pm}$



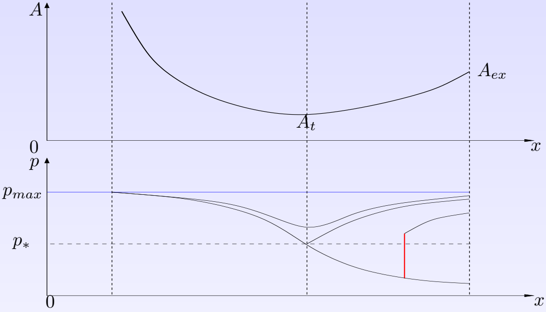
$(\rho, \vec{u}, p) \in L^1_{loc}(\Omega)$ is a (transonic) shock solution if

- $(\rho, \vec{u}, p) \in C^0(\overline{\Omega^{\pm}}) \cap C^1(\Omega^{\pm})$
- $\int_\Omega \rho\vec{u} \cdot D\phi = \int_\Omega (\rho\vec{u} \otimes \vec{u} + pI) \cdot D\phi = \int_\Omega \rho\vec{u}B \cdot D\phi = 0 \; \forall\phi \in C_0^\infty(\Omega)$
- $(0 < \vec{u}^+ \cdot \nu_s < \vec{u}^- \cdot \nu_s$ on S, and $M > 1$ in Ω^-, $M < 1$ in $\Omega^+)$
- $[\rho\vec{u} \cdot \nu_s]_S = [(\rho\vec{u} \otimes \vec{u} + pI) \cdot \nu_s]_S = [\rho B\vec{u} \cdot \nu_s]_S = 0$

$$\Leftrightarrow \begin{cases} div(\rho\vec{u}) = div(\rho\vec{u} \otimes \vec{u} + pI) = div(\rho\vec{u}B) = 0 \text{ in } \Omega^{\pm} \; (n+2 \text{ equations}) \\ [\rho\vec{u} \cdot \nu_s]_S = [(\rho\vec{u} \otimes \vec{u} + pI) \cdot \nu_s]_S = [\rho B\vec{u} \cdot \nu_s]_S = 0, \end{cases}$$

Flow through a de Laval nozzle

Given $(\rho_\infty, \vec{u}_\infty, p_\infty)$ with $M_\infty < 1$, and $p_{ex}(> p_*)$, find $(\rho, \vec{u}, p)$ to describe Euler flow through a de Laval nozzle.



Movie

- subsonic flow($M < 1$) in the convergent part.
- a sonic curve($M = 1$) at the throat, and supersonic flow($M > 1$) behind the sonic curve.
- Given p_{ex}, locate the transonic shock S and find the corresponding subsonic flow($M < 1$) downstream so that $p = p_{ex}$ holds on Γ_{ex}.

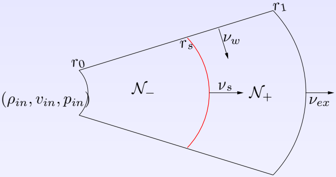
References

✓ R. Courant, K.O. Friedrich, *Superconic flow and shock waves*, Springer-Verlag, 1984

✓ Kuz'min, A.G., Boundary Value Problems for Transonic Flow, John Wiley & Sons, 2002

✓ T.-P. Liu(1982, CMP) stability of transonic shocks in divergent nozzles

Transonic shocks in 2-d divergent nozzle



✓ Constant data H.-R. Yuan(2008 ARWA)

$$(\rho, \vec{u}, p) = (\rho(r), v(r)\hat{r}, p(r)), \quad S = \{r = r_s\}$$

✓ Data smoothly perturbed from constant data, Full Euler system

- L.Liu & H.-R. Yuan(2008 JHPDE): 2-d annulus
- Z.-P. Xin & H.-C. Yin(2005 CPAM, 2009 JDE), S.-X. Chen(2009 CMP): 2-d divergent straight nozzle

✓ Transonic shocks in n-d($n \geq 3$) divergent nozzles
Z.-P. Xin & H.-C. Yin(2009 JDE) 3-d axisymmetric narrow divergent nozzle, Isentropic Euler system

M. Bae & M. Feldman(2009)
n-d($n \geq 2$) divergent nozzle with arbitrary smooth cross-section, Potential flow

Potential flow
Irrotational flow $\nabla \times \vec{u} = \vec{0} \Rightarrow \vec{u} = \nabla\varphi$
Isentropic potential flow $\frac{p}{\rho^\gamma} = \kappa(constant) \Rightarrow \frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma\kappa\rho^\gamma}{(\gamma-1)\rho} = B_0$

$$div(\rho\vec{u}) = 0 \Rightarrow \boxed{div\big((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma}}\nabla\varphi\big) = 0}$$

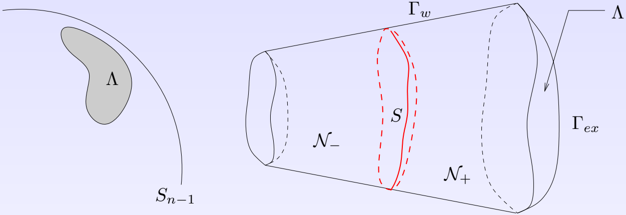
Rankine-Hugoniot condition

$$[\rho\nabla\varphi\cdot\vec{n}]_S = 0, \quad [\varphi]_S = 0 \ \text{(no vortex sheet)}$$

G.-Q. Chen & M. Feldman(2003 JAMS) et al.

Transonic shock problem in divergent nozzles of arbitrary cross-section

Given supersonic flow upstream and p_{ex}, locate the transonic shock S and find the corresponding subsonic flow downstream so that $p = p_{ex}$ holds on Γ_{ex}.



Ill-posedness of isentropic potential flow model

Radial transonic shocks

$$p = p(|\partial_r \varphi|^2)$$
$$\partial_r \varphi(r_1; s) = \partial_r \varphi(r_1; r_0) \ \ \forall \ \ s \in (r_0, r_1)$$

For well-posedness,
the entropy $\frac{p}{\rho^\gamma}$ must jump across a shock.

Non-isentropic potential flow model

$$\begin{cases} div(\rho\nabla\varphi) = 0 \\ \frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho} = B_0 \\ \nabla\varphi\cdot\big(div(\rho\nabla\varphi\otimes\nabla\varphi + pI_n)\big) = 0 \end{cases} \Leftrightarrow \begin{cases} div((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma}}\nabla\varphi) = 0 \\ \frac{1}{2}|\nabla\varphi|^2 + \frac{\gamma p}{(\gamma-1)\rho} = B_0 \\ \nabla\varphi\cdot\nabla\frac{p}{(B_0-\frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0 \end{cases}$$

Rankine-Hugoniot conditions

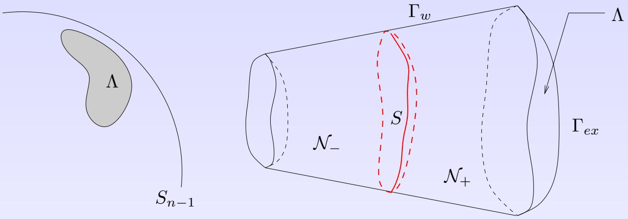
$$[\rho\nabla\varphi\cdot\vec{n}]_S = [(\rho\nabla\varphi\otimes\nabla\varphi + pI_n)\cdot\vec{n}]_S = 0$$

$$\overset{\text{no vortex sheet}}{\Leftrightarrow} [\rho\nabla\varphi\cdot\vec{n}]_S = [\varphi]_S = [\rho(\nabla\varphi\cdot\vec{n})^2 + p]_S = 0$$

Exit boundary condition

$$p = p_{ex}$$

Constant data



Lemma(Full Euler system, H.-R. Yuan, 2008)

$$\Rightarrow \forall p_c \in (p_{min}, p_{max}), \exists!\, r_s \in (r_0, r_1) \ \text{ s.t.}$$

$$(\rho_0, v_0 \hat{r}, p_0)(r; r_s) \ \text{ satisfies } \ p_0(r_1; r_s) = p_c.$$

Definition:Background solutions with the shock at $r = r_s$

$$(\rho_0, \varphi_0, p_0)(r; r_s) = \begin{cases} (\rho_0^-, \varphi_0^-, p_0^-)(r; r_s) & \text{if } r_0 \le r \le r_s \\ (\rho_0^+, \varphi_0^+, p_0^+)(r; r_s) & \text{if } r_s \le r \le r_1 \end{cases}$$

Data perturbed from a constant state

Fix $p_c \in (p_{\min}, p_{max})$. For a small smooth perturbation (ρ_-, φ_-, p_-) of $(\rho_0^-, \varphi_0^-, p_0^-)$ and a given p_{ex}, find a transonic shock solution (ρ, φ, p) in $\mathcal{N}$ so that

$$(\rho, \varphi, p) = (\rho_-, \varphi_-, p_-) \text{ in } \mathcal{N}^-, \quad p = p_{ex} \text{ on } \Gamma_{ex}.$$

Theorem 1(Existence)Bae-Feldman

For any $\alpha \in (0, 1)$, if the perturbation is small with $\|p_{ex} - p_c\|_{1,\alpha,\Lambda}^{(-\alpha,\partial\Lambda)} \leq \sigma$ then $\exists$ a transonic shock solution (ρ, φ, p) s.t.

$$p = p_{ex} \text{ on } \Gamma_{ex},$$

$$\|p - p_0^+\|_{1,\alpha,\mathcal{N}^+}^{(-\alpha,\Gamma_w)} + \|\varphi - \varphi_0^+\|_{2,\alpha,\mathcal{N}^+}^{(-1-\alpha,\Gamma_w)} \leq C\sigma.\text{(stability)}$$

Theorem 2(Uniqueness)Bae-Feldman

For $\alpha \in (\frac{1}{2}, 1)$, the transonic shock solution is unique.

♣The theorems are valid under a small smooth perturbation of $\mathcal{N}$.

Free boundary problem for φ

$$div\Big((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\Big) = 0 \text{ in } \mathcal{N}^+$$

$$\text{(free B.C.)} \quad \varphi = \varphi_- \text{ on } S$$

$$(B.C.)\begin{cases} \nabla\varphi\cdot\nu_s = g(\nabla\varphi_-\cdot\nu_s, p_-) & \text{on } S \\ (B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\cdot\nu_w = 0 & \text{on } \Gamma_w \\ (B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\cdot\nu_{ex} = v_{ex} & \text{on } \Gamma_{ex} \end{cases}$$

Transport equation for $\frac{p}{(B_0-\frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}}$

$$\nabla\varphi\cdot\nabla\frac{p}{(B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0 \text{ in } \mathcal{N}^+ \qquad (*)$$

$$p = p_s(\nabla\varphi_-\cdot\nu_s, p_-) \text{ on } S$$

$$p = p_{ex} \text{ on } \Gamma_{ex}$$

(1) Fix v_{ex} and solve the free boundary problem for φ and solve $(*)$.

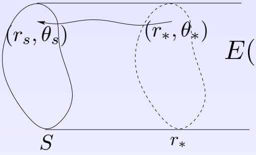
(2) Prove that $\mathcal{P} : v_{ex} \mapsto p_{ex}$ is locally invertible and injective near the background solution.

Difficulty Involvement of $\nabla\varphi \cdot \nabla \frac{p}{(B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0$ with $\varphi \in C^{1,\alpha}(\overline{\mathcal{N}^+}), |D^2\varphi| \sim C[dist(x, \text{corners})]^{-1+\alpha}$

Difficulty Involvement of $\nabla\varphi \cdot \nabla \frac{p}{(B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0$ with $\varphi \in C^{1,\alpha}(\overline{\mathcal{N}^+}), |D^2\varphi| \sim C[dist(x, \text{corners})]^{-1+\alpha}$

Solvability of the transport equation for E $\quad \partial_r E + V \cdot D_\theta E = 0(\clubsuit)$

Method of characteristics: $\boxed{\dot{X} = (1, V)(X), \quad X(0) = (r_*, \theta_*)}$



$$E(r_*, \theta_*) = E(r_s, \theta_s)$$

Proposition

If $V \in C^\alpha(\overline{\mathcal{N}^+})$ and $|DV(x)| \leq C[dist(x, \text{corners})]^{-1+\alpha}$, then $(\clubsuit)$ has a unique solution E. And,

$$E \in C^\alpha(\overline{\mathcal{N}^+}), \quad |DE(x)| \leq C[dist(x, \text{walls})]^{-1+\alpha}$$

Idea:$[dist(X(r), \text{corners})]^{-1+\alpha} \sim r^{-1+\alpha}$ so $dist(X(r), \text{corners})$ is integrable along $X(r)$.

From the proposition, we can define $\mathcal{P} : v_{ex} \mapsto p|_{\Gamma_{ex}}$ where (φ, p) satisfy

$$\begin{cases} div\Big((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\Big) = 0 \text{ in } \mathcal{N}^+ \\ (B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\cdot\nu_{ex} = v_{ex} \text{ on } \Gamma_{ex} + \text{(more BCs)} \end{cases}$$

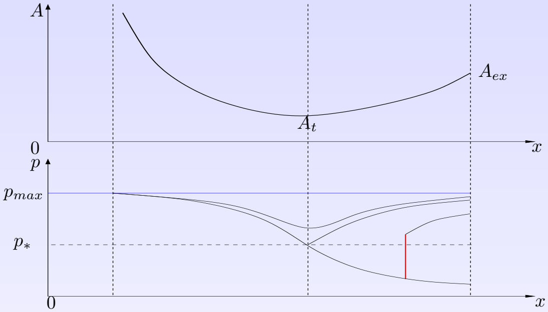
$$\begin{cases} \nabla\varphi\cdot\nabla\frac{p}{(B_0-\frac{1}{2}|\nabla\varphi|^2)^{\frac{\gamma}{\gamma-1}}} = 0 \text{ in } \mathcal{N}^+ \\ p = p_s(\nabla\varphi_-\cdot\nu_s, p_-) \text{ on } S \end{cases}$$

Proposition

(1) $\forall\alpha \in (0,1)$, $\mathcal{P} : v_{ex} \mapsto p_{ex}$ is weakly invertible for $v_{ex}, p_{ex} \in C^{1,\alpha}_{-\alpha,\partial\Lambda}(\Lambda)$.
(2) $\forall\alpha \in (\frac{1}{2}, 1)$, $\mathcal{P} : v_{ex} \mapsto p_{ex}$ is injective and invertible.

Transonic shock solution in $\mathcal{N}$ with fixed p_{ex}
(1)⇒ Existence, (2)⇒Uniqueness

Further questions



- Transonic shock solution in a curved divergent nozzle?
- How to construct subsonic-sonic flow through the convergent part of a de Laval nozzle?

-THANK YOU-