The semantics of non-commutative geometry and quantum mechanics.

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Recall **Morley - Shelah Classification Theory**: For a first-order countable κ-categorical theory T and $\kappa > \aleph_0$, the model M "is characterised by geometric features" - dimensions, homogeneity and so on – *stability theory*.

Geometric dualities

Affine commutative $\mathbb{C}$-algebra	Complex algebraic variety
$R = \mathbb{C}[X_1, \ldots, X_n]/I$	$\mathbf{M}_R$
Commutative unital C^*-algebra	Compact topological space
A	$\mathbf{M}_A$
Affine k-scheme	The geometry of k-definable points, curves etc of an algebraic variety $\mathbf{M}_R$
$R = k[X_1, \ldots, X_n]/I$	
k-scheme of finite type	The geometry of k-definable points, curves etc of a "Zariski geometry" $\mathbf{M}_S$
S	

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Non-example. The models of the theory of arithmetic **do not** provide a semantics of geometric type.

The real 2-torus has co-ordinate ring $A_0 = \langle U, V : U^* = U^{-1}, V^* = V^{-1}, UV = VU \rangle$ in the *comlexified form* $= \mathbb{C}^\times \times \mathbb{C}^\times$.
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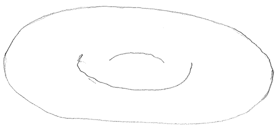

A non-commutative example “at root of unity”

Non-commutative 2-torus at $q = e^{2\pi i \frac{m}{N}}$ has co-ordinate ring $A = A_q =$
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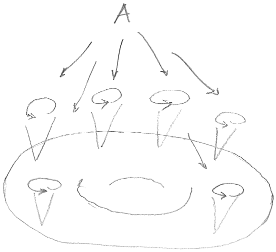
Points have structure of (an orthonormal basis) of a N-dim Hilbert space.



Theorem (2005)

The above dualities can be extended to non-commutative geometry “at roots of unity”.

$$A_{\mathbf{V}} \longleftrightarrow \mathbf{V}_A.$$

$A_{\mathbf{V}}$ – co-ordinate algebra, $\mathbf{V}_A$ – Zariski geometry.

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Complex algebraic variety $\mathbf{M}_R$

Commutative C^*-algebra A

Compact topological space $\mathbf{M}_A$

Affine k-scheme R

The k-definable structure on an algebraic variety $\mathbf{M}_R$

k-scheme of finite type S

The k-definable structure on a Zariski geometry $\mathbf{M}_S$

C^*-algebra A at roots of unity

Zariski geometry $\mathbf{V}_A$

Weyl-Heisenberg algebra
$\langle Q, P : \ QP - PQ = i\hbar \rangle$

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Weyl-Heisenberg algebra $\langle Q, P : \ QP - PQ = i\hbar \rangle$	? *shut up and calculate!*

$QP - PQ = i\hbar$

"The whole of quantum mechanics is in this **canonical commutation relation**".

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An analogy:

$$Y = X^2 + aX + b$$

is (the equation of) a parabola.

$$H = \frac{1}{2}(P^2 + \omega^2 Q^2)$$

is (the Hamiltonian of) a quantum harmonic oscillator.

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$$A_{a,b} = \left\langle U^a, V^b : \; U^a V^b = e^{iab\hbar} V^b U^a \right\rangle,$$

$$a, b \in \mathbb{R}, \; U^a = e^{iaQ}, \; V^b = e^{ibP}.$$

where it is also assumed that U^a and V^b are unitary.

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We may assume that $\frac{\hbar}{2\pi} \in \mathbb{Q}$ and so, when $a, b \in \mathbb{Q}$ the algebra $A_{a,b}$ is at root of unity. We call such algebras **rational Weyl algebras.**

Sheaf of Zariski geometries over the category of rational Weyl algebras

The category $\mathcal{A}_{\text{fin}}$ has objects $A_{a,b}$, rational Weyl algebras, and morphisms = embeddings.

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The duality functor

$$A \;\mapsto\; \mathbf{V}_A$$

can be interpreted as defining a sheaf of Zariski geometries over the category of rational Weyl algebras.

Completions of $\mathcal{A}_{\mathrm{fin}}$ and $\mathcal{V}_{\mathrm{fin}}$.

The completion of $\mathcal{A}_{\mathrm{fin}}$ is $\mathcal{A}$, the category of all Weyl algebras in the Banach algebras topology.

$$B = \lim_{\mathcal{D}} A_i$$

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This construction is called **structural approximation.** Let

$$\mathbf{V}_{{}^*A} = \prod_{\mathcal{D}} \mathbf{V}_{A_i}$$

Apply the **specialisation= surjective morphism**

$$\lim : \mathbf{V}_{{}^*A} \twoheadrightarrow \mathbf{V}_B.$$

The state of space.

We construct a $\mathcal{V}_{\mathrm{fin}}$**-projective** limit object, $\mathbf{V}_{\mathcal{A}} \in \mathcal{V}$:

$$\mathbf{V}_{\mathcal{A}} = \lim \mathbf{V}_{{}^*A}, \quad \mathbf{V}_{{}^*A} = \prod_{\mathcal{D}_{div}} \mathbf{V}_{A_i}$$

$$A \in \mathcal{A}_{\mathrm{fin}} \;\; \Rightarrow \;\; A \subset {}^*A \;\; \& \;\; \mathbf{V}_{\mathcal{A}} \twoheadrightarrow \mathbf{V}_A.$$

This object is closely related to the Hilbert space of quantum mechanics. We call the object **the space of states.**

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The non-standard Zariski geometry $\mathbf{V}_{^*A}$ can be seen as “the huge finite universe” and the state of space is its “observable image”. lim is the standard part map.

Analogy

Rational Weyl algebra	Rational number
Zariski geometry	Interval in rational numbers
Limit object	Interval in real numbers
$\mathbf{V}_{\mathcal{A}}$	$\mathbb{R}$
Zariski-continious maps on $\mathbf{V}_{*A}$	integrable functions on $\mathbb{R}$

Scheme of calculations

- rewrite the formula over $\mathbf{V}_{\mathcal{A}}$ in terms of Zariski-regular pseudo-finite sums and products over $\mathbf{V}_{*A}$
- calculate in $\mathbf{V}_{*A}$ (using e.g. the *Gauss quadratic sums* formula)
- apply the specialisation lim to the result and get the result in terms of the standard reals.

Example. The canonical commutation relation

We define in $\mathbf{V}_{*A}$:

$$Q := \frac{U^a - U^{-a}}{2ia}, \quad P := \frac{V^b - V^{-b}}{2ib}$$

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Calculate that for any state e in $\mathbf{V}_{*A}$

$$(QP - PQ)e = i\hbar e + \epsilon, \quad \lim \epsilon = 0$$

So in $\mathbf{V}_{\mathcal{A}}$

$$QP - PQ = i\hbar I.$$

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The *kernel of the Feynman propagator* is calculated as

$$\langle x_1 | K^t x_2 \rangle = \sqrt{\frac{1}{2\pi i\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1 x_2}{2\hbar \sin t}.$$

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Note that in terms of conventional mathematical physics we have calculated

$$\mathrm{Tr}(K^t) = \sum_{n=0}^{\infty} e^{-it(n+\frac{1}{2})},$$

a non-convergent infinite sum.