Curve and its Jacobian

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1 Introduction and preliminaries

1.1 Let $C(K)$ be the K-points of a smooth projective curve C over its field of definition k^0 of genus $g > 1$ and $J(K)$ its Jacobian, the abelian group of degree 0 cycles on C. We denote k the algebraic closure of k^0 and assume throughout that K is algebraically closed. Sometimes we use these notation with indices.

We consider the structure on the set $C(K)$ defined by the $(4g+2)$-ary relation

$$R(u_1, \ldots, u_g, u_{g+1}, v_1, \ldots, v_g, v_{g+1}, t_1, \ldots, t_g, s_1, \ldots, s_g)$$

interpreted as

$$u_1 + \ldots + u_g + u_{g+1} + s_1, \ldots, s_g \equiv_{\text{linear}} v_1 + \ldots + v_g + v_{g+1} + t_1 + \ldots + t_g,$$

the linear equivalence of divisors. This relation is equivalent to the relation

$$x_1 + \ldots + x_g + x_{g+1} = y_1 + \ldots + y_g$$

for 0-degree divisors of the form $x_i = [u_i - v_i]$, $y_j = [t_j - s_j]$. This is sufficient for defining the relation $z_1 + z_2 = z_3$ for arbitrary 0-cycles $z_k = x_{1,k} + \ldots + x_{g,k} \in J$. Hence for defining the group structure $(J, +)$.

We denote the structure $(C(K), R)$ by $C^J(K)$ or simply C^J. We have observed above that $(J, +)$ is definable (interpretable) in C^J.

Fix a point $c^0 \in C(k)$. Let $j : C(K) \to J(K)$ be the embedding $j : x \mapsto [x - c^0]$. We consider the pair $(J(K), C(K))$ as a structure with the universe $J(K)$, the group operation $+$ and a unary predicate that distingushes the subset $j(C(K))$ in $J(K)$. We write it in a model-theretic way as $(J; C, +)$. Clearly, $(J; C, +)$ is definable in C^J using the parameter c^0 and vice versa.

We study the situation when two such structures over corresponding fields K_1 and K_2 are isomorphic,

$$C_1^J \cong_\alpha C_2^J. \tag{1}$$

By the above this is equivalent to

$$(J_1; C_1, +) \cong_\alpha (J_2; C_2, +) \tag{2}$$

for some choices for c_0 on the corresponding curves.

Examples. 1. Let $K_1 \cong_\alpha K_2$ be two isomorphic fields, C_1 a smooth projective curve over k_1, J_1 its Jacobian and C_2 a curve over k_2 such that $C_2 = \alpha(C_1)$ by the induced bijection. It is easy to see that the image $J_2 = \alpha(J_1)$ is the Jacobian of C_2. In this situation we have (1).

2. Let $K_1 = K_2 = K$ and C_1, C_2, J_1, J_2 be as above and $\psi : J_1 \to J_2$ be an isogeny with trivial kernel such that $\psi(C_1) = C_2$. Then we have (1).

3. Clearly, the composition of α of example 1 and ψ of example 2 gives an example of an isomorphism (1). In particular, when α is of the form Frob^m, some $m \in \mathbb{Z}$, α is an isogeny $J_1 \to J_2$ with trivial kernel.

Remark. If $\mathrm{char}\, K = 0$ then ψ in example 2 has to be an isomorphism of the varieties. This follows from more general facts in 2.2

1.2 In the recent paper [1] F.Bogomolov, M.Korotaev anf Yu.Tschinkel proved the following.

Theorem. Let $K_1 = K_2 = \mathbb{F}_p^{alg}$ be the algebraic closure of a field of p elements, $p > 3$ and (1) holds. Then J_1 and J_2 are isogenous.

They also conjectured that *under the above assumptions C_1 and C_2 are isomorphic as algebraic varieties, modulo Frobenius twisting.*

The method of [1] are very specific to the locally finite field $\mathbb{F}_p^{alg}$ and use the theory of profinite groups of automorphisms.

Our goal is to give a model-theoretic proof of the following

1.3 Theorem. *For any algebraically closed fields K_1 and K_2 any instance of (1) is as in example 3 above:*

There is a field-isomorphism $\beta : K_1 \to K_2$ inducing an isomorphism of pairs and a bijective isogeny ψ such that

$$\alpha = \psi \circ \beta : \ (J_1; C_1, +) \to (J_2; C_2, +).$$

The bijective isogeny ψ is a composition of a bijective morphism of algebraic groups with a Frobenius isomorphism.

In particular, when $K_1 = K_2 = K$ and k^0 is a finite field we have that $\beta(J_1)$ is isogenous to J_1 via a map induced by Frob^m, for some $m \in \mathbb{Z}$, and so J_1 is bijectively isogenous to J_2. Hence the curves C_1 and C_2 are in this sense isomorphic as algebraic varieties, modulo a Frobenius twist.

When the characteristic of the field is 0 the isogeny ψ is an isomorphism of algebraic varieties.

2 Main Construction

2.1 We study here the structure C^J and the related structure $(J; C, +)$, where C is defined over an algebraically closed field k, $C = C(K)$.

We say a set, a map or a structure is definable in a given structure if it is definable using some parameters from the given structure. We say 0-definable if no parameters are used.

Following this terminology C^J is k-defined in K via an interpretation i. We will write this fact as

$$K \sqsupset^i_k C^J.$$

If we extend K to a larger algebraically closed field K^* then we still have,

$$K^* \sqsupset^i_k C^J(K^*),$$

that is the same interpretation i defines K^*-points of the curve and its Jacobian. Moreover, since $K \preccurlyeq K^*$, the extention of the whole structure (including the field K and the interpretation i is elementary.

The main results of this section prove existence of certain definable objects in $C^J(K^*)$ and in the $C^J(K^*)$ together with K^*. By elementary equivalence these results pass to the original setting. So we simply assume in this section, without loss of generality, that transcendence degree of K over k is infinite.

2.2 Lemma. *Let $f : K^n \to K$ be a definable function. Then there is a nonempty open set $U \subseteq K^n$ such that:*

i) If K has characteristic 0, *then there is a rational function r such that $f_{|U} = r$.*

ii) If K has characteristic $p > 0$, then there is a natural number m and a rational function r such that $f_{|U} = \mathrm{Frob}^{-m} \circ r$, where Frob *is the Frobenious automorphism $x \mapsto x^p$.*

This is a well-known corollary of quantifier-elimination for algebraically closed fields. See [2], Theorem 1.11.□

Corollary. *Let G_1 and G_2 be algebraic groups over K and $h : G_1 \to G_2$ a definable isomorphism. Then there is $m \geq 0$ and a bijective morphism of algebraic groups $h' : G_1 \to \mathrm{Frob}^m(G_2)$ such that $h = \mathrm{Frob}^{-m} \circ h'$.*

In case when G_1 and G_2 are abelian varieties h is an isogeny.

Proof. By the Lemma above applied to each co-ordinate there are affine open subsets $V_1 \subseteq G_1$ and $V_2 \subseteq G_2$ and the restriction $h_V : V_1 \to V_2$ such that

$$h_V = \mathrm{Frob}^{-m} \circ r_V, \quad r_V \text{ regular on } V_1,$$

for some $m \geq 0$. Let $G = \mathrm{Frob}^m(G_2)$. Then $r_V : V_1 \to G$ is a regular injective map defined on V_1 satisfying

$$r_V(x_1 \cdot x_2) = r_V(x_1) \cdot r_V(x_2), \;\; r_V(x_1^{-1}) = r_V(x_1)^{-1}, \text{ for generic pair } x_1, x_2 \in V_1.$$

By Weil's group chunk argument r_V can be uniquely extended to a morphism $h' : G_1 \to G$ of algebraic groups that has to be bijective since it is bijective on an open set.

In case of abelian varieties h' and Frob^{-m} are isogenies. □

3 The E.Rabinovich theorem and its corollaries

This technically hard theorem was proved by E.Rabinovich in 1989 as a partial answer to the "restricted trichotomy conjecture" by the present author. The conjecture states that *assuming a strongly minimal structure M is definable in an algebraically closed field K and M is not locally modular, a field isomorphic to K is definable in M.* Note that the case when M is locally modular is well-understood and nicely classifiable.

Theorem (E.D.Rabinovich [6]) *Assume M is a strongly minimal structure definable in an algebraically closed field K in such a way that the universe of M is a rational curve over K. Assume also that M is not locally modular, then a field F isomorphic to K is definable in M.*

So, the theorem sets the restricted trichotomy conjecture under the extra assumption of rationality of the curve on which M is defined. This assumption can be weakened: it is enough to assume that a *rational curve M' is definable in M*. Indeed, then since the combinatorial geometry of M is isomorphic to that of any strongly minimal set in M, we have that the structure induced on M' from M is not locally modular. Finally note that a field F definable in M' will be, by transitivity, definable in M.

3.1 Lemma. In C^J a set isomorphic (as algebraic variety) to the projective line $\mathbf{P}^1$ is definable.

Proof. Fix a class $[D_0]$ of a very ample effective divisors. The divisors of $[D_0]$ form a complete linear system, equivalently it corresponds to an embedding $C \subset \mathbf{P}^n$ with the property that divisors $D \in [D_0]$ correspond to intersections $D = C \cap H$, for hyperplanes $H \subset \mathbf{P}^n$, and vice versa.

For $a_1, \dots, a_m \in C$, denote $L_{a_1,\dots,a_m}$ the set of hyperplanes in $\mathbf{P}^n$ passing through $a_1, \dots, a_m$. For $m = 0$, $L_{a_1,\dots,a_m}$ is the set of all hyperplanes L^n, which is isomorphic to $\mathbf{P}^n$. By definition $L_{a_1,\dots,a_m}$ is a linear subspace of L^n and $\dim L_{a_1,\dots,a_m,a_{m+1}} < L_{a_1,\dots,a_m}$, provided a_{m+1} is not in the linear subspace of $\mathbf{P}^n$ spanned by $a_1, \dots, a_m$, The latter condition is satisfied for some $a_{m+1} \in C$ as long as the linear subspace of $\mathbf{P}^n$ spanned by $a_1, \dots, a_m$ is not the whole $\mathbf{P}^n$ (note that $C \nsubseteq H$ for a hyperplane $H \subset \mathbf{P}^n$, by assupmtions). It follows that for some distinct $a_1, \dots, a_{n-1} \in C$, $\dim L_{a_1,\dots,a_{n-1}} = 1$, that is is isomorphic to $\mathbf{P}^1$ as algebraic varieties. It remains to observe that $L_{a_1,\dots,a_{n-1}}$ is definable in C^J as

$$\{D \in [D_0] : \ a_1, \dots, a_{n-1} \in D\}.$$

□

3.2 Theorem. *There exists an algebraically closed field F, $C(k)$-defined in the structure C^J along with a non-constant $C(k)$-definable map $h : C \to F$. In other words,*

$$K \quad \sqsupset_k \ C^J \quad \sqsupset^h_{C(k)} \ F.$$

Moreover, there is an isomorphism of fields $\phi : K \to F$, definable in the field K. ϕ is determined uniquely up to Frobenius automorphisms of K.

The map h in terms of K can be identified as a rational map defined over k.

Proof. We claim that C^J is not locally modular. This is immediate from the fact that in any group definable in a locally modular structure any definable subset is a coset of a definable subgroup, see [3] for the most general statement of this type. The subset $C \subset J$ contradicts such a condition.

Now the Rabinovich theorem along with Lemma 3.1 tell us that a strongly minimal (and algebraically closed) field F is definable in C^J.

Next, we claim that there is a non-constant rational map $h : C \to F$. Since F is not orthogonal to C in the structure C^J, there is a finite-to-finite correspondence $S \subset C \times F$ between C and F. Given a generic $x \in C$ we get $y_1, \ldots, y_m \in F$ corresponding to x via S. Let $s_1, \ldots, s_m$ be the symmetric functions of m variables, and let $h_i(x) := s_i(y_1, \ldots, y_m)$ for x and the y_k satisfying $\bigwedge_k S(x, y_k)$. At least one of the functions has to be non-constant since S is a correspondence.

Note that the restriction of the structure to k-points is an elementary substructure, since $k \preccurlyeq K$. It follows we can choose F and h definable using parameters in $C(k)$ only.

The isomorphism ϕ is by [4] (but is also implicit in the proof of Rabinovich's theorem). The last statement follows from 2.2.□

Remark. It is not hard to prove that F and h are 0-definable in C^J.

The Rabinovich theorem preceded the work [8] which introduced and classified Zariski geometries. In [8] the key ideas of [6] had been generalised and adapted to the context of abstract Zariski geometries. It is therefore natural to expect that Rabinovich's theorem or even a full proof of the restrited trichotomy conjecture can be obtained as a corollary to the classification theorem of [8].

4 Representing the curve in F.

Below we continue to work in the structure C^J or rather, equivalently, in $(J; C, +)$. We use the notion of dimension and of generic points in the model-theretic sense, where $\dim S$ is understood as the Morley rank of a definable set S, but note that these notions coincide with those of algebraic geometry, since the universe C of the structure C^J is of dimension 1 and irreducible, in both senses.

4.1 Proposition. *There exists n and a rational map with finite fibres $f : J \to F^n$, $C(k)$-definable in C^J.*

Moreover, the following three conditions are equivalent

- *one can choose f to be generically injective on J (injective on an open subset J^0 of J)*
- *one can choose f to be generically injective on the shift $y^0 + C$ of the curve C, for some $y^0 \in J(k)$,*

- *the h in 3.2 can be chosen generically injective on $\mathbb{C}$.*

Proof. Once a point $c_0 \in C(k)$ is fixed, we may identify a point $x \in C$ with a point $x - c_0 \in J$. Then a generic element y of J can be represented as $y = x_1 + \ldots + x_g$ for some generic g-tuple $x_1, \ldots, x_g \in C$, $g = \text{genus}\, C$. Moreover, this representation is unique up to the permutation of the $x_1, \ldots, x_g$.

We now have a well-defined map $x_1 + \ldots + x_g \mapsto \{h(x_1), \ldots, h(x_g)\}$, from an open subset of J to $F^{(g)}$, the set of g-element subsets of F. On the other hand, there is an injective map

$$F^{(g)} \to F^g;\ \{z_1, \ldots, z_g\} \mapsto \langle s_1(z_1, \ldots, z_g), \ldots, s_g(z_1, \ldots, z_g)\rangle,$$

where s_i are the symmetric functions of g variables. The composition of these two maps is the required map associated with h.

Clearly, f is injective if h is. Now assume that f is injective on an open subset J^0 of J. By dimensional consideration, for some $y^0 \in J$ of the form $y^0 = x_1 + \ldots + x_{g-1}$ the curve $y^0 + C$, except maybe a finite number of points, is a subset of J^0. Since k is algebraically closed we may choose $y^0 \in J(k)$. Hence, setting $h(x) := f(y^0 + x)$ we get a generically injective map $f : C \to F^n$. $\square$

4.2 Let $f : J \to F^l$ be a non-constant rational map over k defined in C^J. Pick up a generic $y \in J$ and define

$$\text{fibre}_y f := \{y' \in J : f(y) = f(y')\}.$$

Let n and $f^0 : J \to F^n$ over k be such that the Zariski closed subset $\text{fibre}_y f^0$ is of minimal dimension and of number of absolutely irreducible components, when we make choices among the maps defined in C^J.

Lemma. *For every generic pair $y, t \in J$, and every $y' \in J$*

$$f^0(y) = f^0(y') \Rightarrow f^0(y + t) = f^0(y' + t).$$

Proof. Choose an arbitrary $a \in J(k)$ and consider the map $f^0_a : J \to F^n$, $y \mapsto f^0(y + a)$ along with the map $\langle f^0, f^0_a\rangle : J \to F^{2n}$, $y \mapsto \langle f^0(y), f^0_a(y)\rangle$. Clearly,

$$\text{fibre}_y\langle f^0, f^0_a\rangle = \text{fibre}_y f^0 \cap \text{fibre}_y f^0_a.$$

By minimality we get $\text{fibre}_y f^0 = \text{fibre}_y f^0_a$. Hence, the statement holds for $t = a$, for all $a \in J(k)$. Since k is algebraically closed Lemma follows.$\square$

Corollary. *There is a k-definable equivalence relation on an open subset of J,*

$$y \sim y' \Leftrightarrow f^0(y) = f^0(y').$$

The equivalence classes are exactly the fibres of f^0.

Moreover, for any $f : J \to F^l$ defined in C^J, on an open subset of J depending on f,

$$y \sim y' \Rightarrow f(y) = f(y').$$

4.3 Lemma. *The set*

$$A = \{a \in J : f^0(y+a) = f^0(y) \text{ for all } y \text{ in an open subset of } J\}$$

is a finite subgroup of $J(k)$.

The generic fibre is a coset of A,

$$\text{fibre}_y f^0 = y + A.$$

A *is trivial iff* f^0 *is generically injective.*

Proof. Let $Z = f^0(J) \subset F^n$ and consider the map $h : Z \times J \to K^n$ defined on an open subset of $Z \times J$ as follows:

$$h(z,t) = w \Leftrightarrow \exists y \in J\, f^0(y) = z \,\& f^0(y+t) = w.$$

By Lemma 4.2 this is well-defined.

Let $t, t' \in J$, t generic over k, $t \sim t'$ and fix $z_0 \in J$ generic over $k(t,t')$. We have by Corollary to Lemma 4.2 $h(z_0,t) = h(z_0,t')$.

It follows that for $y_0 \in J$, generic over $k(t,t')$, we have $f^0(y_0+t) = f^0(y_0+t')$. This can be rewritten as

$$f^0(y+a) = f^0(y)$$

where $a = t - t'$, $y = y_0 + t'$. By genericity this holds for all y in an open subset of J. Hence a belongs to the subgroup A.

Since the fibres of f^0 are finite, A is finite and so $A \subset J(k)$. By construction, every t' in the equivalence class of t is of the form $t + a$, $a \in A$, so the f^0-fibre containing t is of the form $t + A$. Hence all generic fibres are of this form. □

4.4 *A is trivial, f^0 is generically injective on J and there is an $h_0 : C \to F^n$ generically injective on C.*

Proof. Suppose towards a contradiction that A is not trivial. By 4.3 f^0 is not generically injective and so, by 4.1, f^0 is not generically injective on $y^0 + C \subset J^0$ (up to finitely many points) where J^0 is the domain of f^0 and $y^0 \in J(k)$. So, for some generic $x \in C$ there is $x' \in C$, $x' \neq x$, $f^0(y^0+x) = f^0(y^0+x')$. It follows by 4.3 that for some non-zero $a \in A$, $x' = x + a \in C$. Since x is generic, the latter holds for any $x \in C$, i.e. $a + C = C$. But this is not possible unless $a = 0$, by Lemma 2.1 of [1]. □

4.5 Model-theoretic generalisation of Weil's group chunk theorem.

Consider again the definable injection

$$J^0 \quad \to_{f^0} \quad F^n$$

of an open definable subset J^0 into the affine space F^n, and denote $G^0 = f^0(J^0)$. The map f^0 transfers the definable subsets and relations on J^0 to ones on G^0. Note that $(J^0,+)$, where $+$ is a partial operation (relation) on J^0 is a Weil's group chunk. It follows that its image $(G^0,+)$ is a *definable group chunk.* This means that $+$ is a definable partial operation such that $-z$ and z_1+z_2 is defined for any generic z and any generic pair z_1, z_2 in G^0, and for any generic triple $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$.

There have been various generalisations of Weil's group chunk theorem to the definable context, the most general one by E.Hrushovski, see a detailed exposition of this in [5]. We need the following corollary of these results.

Fact. *There is a group G definable in C^J, and a definable embedding $f^1 : G^0 \to G$ such that for a generic pair $z_1, z_2 \in J^0$, $f^1(z_1+z_2) = f^1(z_1)+f^1(z_2)$ and G^0 generates G. Moreover, the embedding*

$$f^1 \circ f^0 : J^0 \to G$$

can be extended to a definable isomorphism

$$\psi : J \to G.$$

5 Proof of the Main Theorem

5.1 We now come back to the assumption (1) and represent it, using the results of section 2 schematically as follows (dropping the assumption of infinite transcendence degree K/k).

$$\begin{array}{llllll} K_1 & \sqsupset_k^{i_1} & C_1^J & \to_{f^0} & F_1^n; & \phi_1 : K_1 \to F_1 \\ & & \updownarrow \alpha & & \updownarrow \bar{\alpha} & \\ K_2 & \sqsupset_k^{i_2} & C_2^J & \to_{f^0} & F_2^n; & \phi_2 : K_2 \to F_2 \end{array}$$

Here, i_1 and i_2 are two realisations (definitions) of the curves and the Jacobians in K_1, K_2 and f^0 is the definable injective map of section 2 which is defined by the same formula in both structures C_1^J and C_2^J. Recall that this map is an interpretation of the field in the initial structure, that is f^0 restricted to C_i defines also the field structure on F_i. α is the isomorophism of the pairs given by (1) and $\bar{\alpha}$ the isomorphism between the fields F_1 and F_2 induced by α via f^0, that is such that the right quadrangle of the diagram commutes.

And in addition the picture recalls that there are also definable isomorphisms between fields $\phi_1 : K_1 \to F_1$ and $\phi_2 : K_2 \to F_2$. It implies

5.2 Claim 1. *In (1) the fields are isomorphic,*

$$\check{\alpha} : K_1 \cong K_2, \text{ where } \check{\alpha} = \phi_2^{-1}\bar{\alpha}\phi_1.$$

In particular, when $K_1 = K_2 = K$ we will have the following diagram

$$\begin{array}{llllll} K & \sqsupset_k^{i_1} & C_1^J & \to_{f^0} & F_1^n; & \phi_1 : K \to F_1 \\ \updownarrow \check{\alpha} & & \updownarrow \alpha & & \updownarrow \bar{\alpha} & \\ K & \sqsupset_k^{i_2} & C_2^J & \to_{f^0} & F_2^n; & \phi_2 : K \to F_2 \end{array}$$

Claim 2. *If* $\breve{\alpha} = \mathrm{id}$ *then the structures* C_1^J, C_2^J, F_1 *and* F_2 *are definable in the field* K *and* $\bar{\alpha}$ *is a definable in* K *isomorphism of fields.*

Indeed, the definability of structures is by interpretations i_1, i_2 and f^0, and the definability of $\bar{\alpha}$ follows from the definability of ϕ_1 and ϕ_2 by Claim 1.

5.3 We rewrite the diagram above replacing the affine spaces F_1^n and F_2^n with (definably equivalent to these) groups G_1 and G_2, correspondingly, constructed in 4.5,

$$\begin{array}{lllll} K & \sqsupset_k^{i_1} & C_1^J & \to_{\psi_1} & G_1(F_1); \quad \phi_1 : K \to F_1 \\ \updownarrow \breve{\alpha} & & \updownarrow \alpha & & \updownarrow \hat{\alpha} \\ K & \sqsupset_k^{i_2} & C_2^J & \to_{\psi_2} & G_2(F_2); \quad \phi_2 : K \to F_2 \end{array},$$

with the isomorphism $\hat{\alpha}$ between the definable groups induced by $\bar{\alpha}$.

5.4 Finally, we apply a transformation to the diagram 5.3 by applying a field automorphism $\breve{\alpha}^{-1}$ to the left item of the bottom line. This automorphism clearly induces an isomorphism $\beta : C_2^J \to C_3^J$ onto a new curve and its Jacobian. By denoting $\breve{\beta} := \breve{\alpha}^{-1}$ and setting $\hat{\beta}$ the isomorphism of 4.5 we get,

$$\begin{array}{lllll} K & \sqsupset_k^{i_1} & C_1^J & \to_{\psi_1} & G_1(F_1); \quad \phi_1 : K \to F_1 \\ \updownarrow \breve{\alpha} & & \updownarrow \alpha & & \updownarrow \hat{\alpha} \\ K & \sqsupset_k^{i_2} & C_2^J & \to_{\psi_2} & G_2(F_2); \quad \phi_2 : K \to F_2 \\ \updownarrow \breve{\beta} & & \updownarrow \beta & & \updownarrow \hat{\beta} \\ K & \sqsupset_k^{i_2} & C_3^J & \to_{\psi_3} & G_3(F_3); \quad \phi_3 : K \to F_3 \end{array}$$

Since $\breve{\alpha}\breve{\beta} = \mathrm{id}$, by Claim 2 of 5.2 we have the chain of isomorphisms definable in the field K of definable groups with distinguished curves

$$J_1 \to_{\psi_1} G_1(F_1) \to_{\hat{\beta} \circ \hat{\alpha}} G_3(F_3) \to_{\psi_3^{-1}} J_3.$$

By Corollary 2.2 the definable isomorphism of abelian varieties is an isogeny. Composing the isomorphisms we get a definable isomorphism, so ψ is a bijective isogeny between J_1 and J_3 which also respects the curves

$$\psi : (J_1; C_1, +) \to (J_3; C_3, +).$$

Recall that $J_3 = \beta(J_2)$, where β is induced by an isomorphism of fields. This finishes the proof of the main part of the main theorem.

Consider now the case when the field of definition k^0 is finite. Then $\beta_{|k^0} = \mathrm{Frob}^m$ for some $m \in \mathbb{Z}$. Now we have $J_3 = \mathrm{Frob}^m(J_2)$ and by definition Frob^m is an isogeny between J_2 and J_3.

References

[1] F.A. Bogomolov, M. Korotaev and, Yu. Tschinkel, *A Torelli theorem for curves over finite fields*, Pure Appl. Math. Q., 1, (2010), pp. 245-294.

[2] D. Marker, *Introduction to the Model Theory of Fields*, in D. Marker, M. Messmer, A. Pillay, **Model Theory of Fields** (Berlin: Springer-Verlag, 1996)

[3] E.Hrushovski and A.Pillay, *Weakly normal groups*, Logic Colloquium '85,

[4] B. Poizat, **Groupes Stables**, Nur Al-Mantiq wal-Ma'rifah, (1987).

[5] E. Bouscaren, Model theoretic versions of Weil's theorem for pregroups, in A. Nesin and A. Pillay, **The Model Theory of Groups**, Notre Dame Press (1989).

[6] E.Rabinovich, **Definability of a field in sufficiently rich incidence systems**, Maths Notes 14, Queen Mary and Westfield College, University of London, 1986.

[7] B.Zilber,*Algebraic geometry via model theory*, Cont.Maths.v.131(part 3), 1992, pp.523-537.

[8] E.Hrushovski and B.Zilber, *Zariski Geometries,*

[9] B.Zilber, **Zariski Geometries,** CUP, 2010.