# A Minkowski space-time lattice with Lorentzian invariance

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**Abstract**

We use the notion of structural approximation to represent the Lorentz-invariant Minkowski space-time as a limit of finite cyclic lattices with the action of finite quasi-Lorentz groups

## 1 Introduction

**1.1** A physical theory is an approximation to reality. But what is an approximation? In [1] we discussed this problem from the perspective of model theory. This results in the definition of **structural approximation** which we use here along with a more advanced recent paper [2] which sets a general background for applications in Foundations of Physics.

Unilke conventional approimations based on metrics, structural approximation is designed to preserve the structure even when the structure is given without a metric. E.g. a sequence of finite groups can approximate a continuous group (or more often a "compactified" version of a group).

In the paper we construct a sequence of finite groups acting on finite lattices which approximates the Lorentz group acting on a compactified Minkowski space.

We suggest this construction as a form of discretisation of space-time with Lorentzian symmetry, a problem discussed in various publications, see e.g. [3]. In some sense our mathematical techniques is not dissimilar to ones proposed in [4] and some other publications relying on the p-adic and adelic number system: the pseudo-finite resudue ring K underlying our construction, is quite similar to the ring of adeles.

**1.2 Structural approximation.** A structure $\mathbf{M}$ is a set $M$ with a collection $\Sigma$ of $n$-ary relations $S \subset M^n$ for some $n$, called the language (or the vocabulary) of $\mathbf{M} = (M; \Sigma)$.

Suppose we are given a sequence $\{\mathbf{M}_i : i \in \mathbb{N}\}$, $\mathbf{M}_i = (M_i, \Sigma)$, of structures in language $\Sigma$. One can choose a Fréchet ultrafilter $D$ on $\mathbb{N}$ and construct the ultraproduct

$$^*\mathbf{M} := \prod_{i \in \mathbb{N}} \mathbf{M}_i / D$$

which is a structure in language $\Sigma$ with the key property (the *Loś theorem*): given a first-order sentence $\sigma$ in the language $\Sigma$,

$$\sigma \text{ is true in } {}^*\mathbf{M} \text{ if and only if } \sigma \text{ is true in } \mathbf{M}_i \text{ along } D \qquad (1)$$

$^*\mathbf{M}$ is often referred to as the model-theoretic limit of $\mathbf{M}_i$ along $D$.

In case the $\mathbf{M}_i$'s are finite, $^*\mathbf{M}$ is said to be a **pseudo-finite** structure.

It is convenient to consider the system $\mathcal{T}_n$ of topologies on $^*\mathbf{M}^n$, all $n$, the basic closed sets of which are realisations $S(^*\mathbf{M})$ of the $n$-ary $S \in \Sigma$. Such a system is said to be quasi-compact (or complete) if the projection maps $\mathrm{pr}_{n+1,n} : {}^*\mathbf{M}^{n+1} \to {}^*\mathbf{M}^n$ preserve closed subsets, that is $\mathrm{pr}_{n+1,n}(S)$ closed, for $S$ closed.

This definition makes sense for the $\mathbf{M}_i$ and indeed for any $\mathbf{M}$ in the language $\Sigma$.

A structural approximation of $\mathbf{M}$ by $\{\mathbf{M}_i : i \in I\}$ along $D$ is a surjective map

$$\mathsf{lm} : {}^*\mathbf{M} \twoheadrightarrow \mathbf{M} \qquad (2)$$

which has the property:

$$S \subset {}^*\mathbf{M}^n \text{ closed} \Rightarrow \mathsf{lm}(S) \subset \mathbf{M}^n \text{ closed},$$

As it happens, below, most of the time $\mathbf{M}_i$, $\mathbf{M}$ and $^*\mathbf{M}$ are rings in the language $\{x + y = z,\ x \cdot y = z\}$ or groups in the language $x * y = z$ and *closed* for $S \subset \mathbf{M}^n$ means $S$ is the set of solutions of a system of algebraic equations in $n$-variables with parameters in $\mathbf{M}$, or equivalently, closed in **Zariski topology**.

Note that despite the coarse topology we still are able to use the intuition of infinitesimals: two elements $a, a' \in {}^*\mathbf{M}^n$ are seen to be "infinitesimally close" if $\mathsf{lm}\, a = \mathsf{lm}\, a'$.

Thus, in view of (1) and (2) structural approximation is a formalisation of the statement *a very large structure* $\mathbf{M}_i$ *looks like* $\mathbf{M}$ *from afar.*

**1.3 Approximation and compactness.** It was established in [1] that $\mathbf{M}$ has to be quasi-compact (complete) in order for it to appear in (2) for non-trivial sequences $\mathbf{M}_i$. In particular, the field $\mathbb{C}$ is not quasi-compact but its compactification $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{C}\mathbf{P}^1$ is. Theorem 5.2(i) of [1] proves that for any zero-characteristic pseudo-finite field $\mathrm{E}$ (in particular, for $\mathrm{E} = \mathrm{F}_{\mathfrak{p}}$, $\mathfrak{p}$ non-standard prime number) there exists a structural approximation

$$\mathsf{lm}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}. \tag{3}$$

Warning: such an approximation *can not be explained in terms of non-standard analysis.*

The limit map in (3) is far of being unique but we can pick ones with some specific and useful properties as in [2], which allows to mimic complex analysis in the pseudo-finite field $\mathrm{E}$.

On the other hand, the notion is quite restrictive in another sense: it is proved in Theorem 5.2(ii) of [1] that $\mathbb{C}$ is the only locally compact field for which an approximation by finite fields is possible.

**1.4 Scales and scale-dependendence of approximation.**

The interplay between the domain and the range of the approximation map $\mathsf{lm}_{\mathrm{E}}$ as in (3) brings in some features not encountered in the limit construction with inherent metrics. By its nature field $\mathrm{E}$ is of pseudo-finite characteristic $\mathfrak{p}$ (more generally we also consider pseudo-finite residue rings $\mathrm{E} = {}^*\mathbb{Z}/\mathcal{N}$ ) while $\mathbb{C}$ is characteristic zero field with a natural metric.

It is clear by algebraic considerations that

$$\mathsf{lm}_{\mathrm{E}} : \{1, 2, 3, \ldots\}_{\mathrm{E}} \mapsto \{1, 2, 3, \ldots\}_{\mathbb{C}}$$

where $\{1, 2, 3, \ldots\}_{\mathrm{E}} \subset \mathrm{E}$, $\{1, 2, 3, \ldots\}_{\mathbb{C}} \subset \mathbb{C}$ include all usual (standard) integers. Moreover,

$$\mathsf{lm}_{\mathrm{E}} : \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}_{\mathrm{E}} \mapsto \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}_{\mathbb{C}}$$

and in the limit we can see approximate reals emerging in E. In other words, an observer which has only access to small scale elements of E can think of E as being $\mathbb{R}$. Define

$$\mathrm{E}_{\mathrm{real}} = \{x \in \mathrm{E} : \mathsf{Im}_{\mathrm{E}}(x) \in \mathbb{R}\}.$$

So the remark we made is that small-scale elements of E are in $\mathrm{E}_{\mathrm{real}}$.

However, as we continue along the natural cyclic order $1, 2, 3, \ldots$ of E, inevitably, we will encounter an element $\mathbf{i} \in \mathrm{E}$ such that

$$\mathsf{Im}_{\mathrm{E}} : \mathbf{i} \mapsto \imath = \sqrt{-1}; \quad \mathbf{i} \cdot \mathrm{E}_{\mathrm{real}} \to \imath\,\mathbb{R}$$

and so with other complex numbers.

Again by (3), we will also have a non-empty domain

$$\mathrm{E}_{\infty} = \{x \in \mathrm{E} : \mathsf{Im}_{\mathrm{E}}(x) = \infty\}.$$

So, an observer which has tools to explore the global characteristics of E has to think of E as a Riemann sphere $\bar{\mathbb{C}}$. We say that E **locally** looks like $\mathbb{R}$ while **globally** looks like $\bar{\mathbb{C}}$.

It is clear that $\mathrm{E}_{\mathrm{real}}$, $\mathbf{i}\mathrm{E}_{\mathrm{real}}$ and $\mathrm{E}_{\infty}$ should be considered of "different scales", perhaps in some context related to "low energy – high energy" philosophy. In [2], section 3, we introduce a formal notion which allows us to speak about scales and use it in constructing approximations with prescribed properties.

**1.5** The main result of the paper is a construction of a structural approximation of the Minkowski space $\mathcal{M}$ by finite 4-dim lattices along with an approximation of the Lorentz group $\mathrm{SO}^+(1,3)$ by finite groups acting on the lattices respectively and preserving the Minkowski metric.

Note that unlike other approximations of Lorentz action, we have actual groups $G_i$ acting on the discrete spaces $\mathcal{M}(\mathrm{K}_i)$ so that the groups $G_i$ approximate, locally, the Lorentz group and the space $\mathcal{M}(\mathrm{K}_i)$ approximates locally the Minkowski space $\mathcal{M}(\mathbb{R})$ with Minkowski metric, see 2.9 and (10) therein. Globally, the same lattices approximate the complexified and compactified model of Minkowski space-time $\bar{\mathcal{M}}(\mathbb{C})$ along with a complexified and compactified version of the group which acts on $\bar{\mathcal{M}}(\mathbb{C})$.

Note that, for the reasons explained above, the discrete (pseudo-finite) Minkowski space-time in its limit continuous version presents itself as compactified and complexified, agreeing with Penrose's approach.

## 2 Pseudo-finite rings and groups and their limits

**2.1** As in [2] let ${}^*\mathbb{Z}$ be an $\aleph_0$-saturated model of arithmetic, $\mathcal{N} \in {}^*\mathbb{Z}$ divisible by all standard integers and $\mathrm{K} = \mathrm{K}_{\mathcal{N}} := {}^*\mathbb{Z}/\mathcal{N}$ be the (non-standard) residue ring.[1]

**2.2** It is well-known that $\mathrm{SL}(2,\mathbb{C})$ is a double cover of the Lorentz group $\mathrm{SO}^+(1,3)$ and it acts in agreement with this on the Minkowski space .

More precisely (see e.g. [5]): represent a vector with components $(t,x,y,z) \in \mathbb{R}^4$ (Minkowski space) as a $2\times 2$ matrix

$$X := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

with $X^\dagger = X$ and $\det(X) = t^2 - x^2 - y^2 - z^2$, consider

$$X \mapsto MXM^\dagger \text{ with } M \in \mathrm{SL}(2,\mathbb{C}). \tag{4}$$

This preserves $\det X$ and thus the Minkowski metric, which leads to the proof that (4) is a Lorentz transformation and all Lorentz transformations can be expressed in this way. The fact that $\pm M$ both give the same transformation of $X$ corresponds to the fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of the Lorentz group, that is

$$\mathrm{SO}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \tag{5}$$

We denote

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2)$$

the structure which consists of $\mathbb{R}$-linear Minkowski space $\mathcal{M}(\mathbb{R})$ with metric given by $X \mapsto \det X$ along with the group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ acting on the space as describe in (4).

We note that the isomorphism of groups induces the isomorphism of structures

$$(\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2) \cong \big(\mathcal{M}(\mathbb{R}), \mathrm{SO}^+(1,3)\big) \tag{6}$$

[1]Note that

$$\mathrm{K} \cong \prod_{p|\mathcal{N} \text{ primes}} {}^*\mathbb{Z}/p^{\eta_p}$$

where $\eta_p \in {}^*\mathbb{Z}$ positive, and so, for all standard primes $\eta_p >> 1$. It follows that in the limit ${}^*\mathbb{Z}/p^{\eta_p}$ will be seen as the ring $\mathbb{Z}_p$ of $p$-adic integers (see [1]) and the whole K as the ring $\mathcal{A}_{\mathbb{Z},\mathrm{fin}}$ of finite integral adeles.

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we will treat as a Zariski structure, that is the set with Zariski closed relations $R \subset \bar{\mathbb{C}}^n$ on it.

**2.3** *There is a surjective homomorphism* $\mathsf{Im}_{\mathrm{K}}$ *of Zariski structures*

$$\mathsf{Im}_{\mathrm{K}} : \ \mathrm{K} \to \bar{\mathbb{C}}.$$

*In particular,*

$$\mathsf{Im}_{\mathrm{K}}(x + y) = \mathsf{Im}_{\mathrm{K}}\, x + \mathsf{Im}_{\mathrm{K}}\, y, \ \textit{if} \ \ \mathsf{Im}\, x \neq \infty \ \textit{and} \ \ \mathsf{Im}\, y \neq \infty$$

$$\mathsf{Im}_{\mathrm{K}}(x \cdot y) = \mathsf{Im}_{\mathrm{K}}\, x \cdot \mathsf{Im}_{\mathrm{K}}\, y, \ \textit{if} \ \ \mathsf{Im}_{\mathrm{K}}\, x \neq \infty \ \textit{and} \ \ \mathsf{Im}_{\mathrm{K}}\, y \neq \infty$$

$\mathsf{Im}_{\mathrm{K}}$ *is the composition of two Zariski homomorphisms*

$$\mathrm{pr}_{\mathrm{K,E}} : \mathrm{K} \twoheadrightarrow \mathrm{E} \ \textit{and} \ \mathsf{Im}_{\mathrm{E}} : \mathrm{E} \twoheadrightarrow \bar{\mathbb{C}}$$

*where* E *is a pseudo-finite field.*

*The subsets*

$$\mathrm{K}_{\mathrm{fin}} = \{x \in \mathrm{K} : \ \mathsf{Im}\, x \neq \infty\} \ \textit{and} \ \mathrm{K}_{\mathrm{real}} = \{x \in \mathrm{K} : \ \mathsf{Im}\, x \in \mathbb{R}\}$$

*are subrings of* K.
*For every positive* $n \in \mathbb{N}$

$$nx = 0 \Rightarrow \mathsf{Im}\, x = 0$$

**Proof.** Since $\mathcal{N}$, the order of K, is divisible by every standard prime $q$, there is a ring-homomorphism $\mathrm{pr} : \mathrm{K} \twoheadrightarrow \mathrm{E}_{\mathbf{q}}$, for an infinite non-standard $\mathbf{q}$. It follows that, if a polynomial $P(X)$ over $\mathbb{Z}$ has a zero in K then it has a zero in $\mathrm{E}_{\mathbf{q}}$, a field of characteristic 0, and so in $\mathbb{C}$. Now one constructs $\mathsf{Im}$ by the same back-and-forth procedure as in the proof of Proposition 5.2(i) of [1] using the fact that the cardinality of K is not smaller than that of $\mathbb{C}$.

The statements about $\mathrm{K}_{\mathrm{fin}}$ and $\mathrm{K}_{\mathrm{real}}$ follow from the fact that $\mathsf{Im}$ preserves $+$ and $\cdot$ of K.

Finally, assume that $nx = 0$ for $x \in \mathrm{K}$. Note that since $\mathsf{Im}$ is surjective $\mathsf{Im}\, 0 = 0$ and $\mathsf{Im}\, n \cdot 1 = n$ for $1 \in \mathrm{K}$. Clearly, if $\mathsf{Im}\, x \neq \infty$ then $0 = \mathsf{Im}\, nx = n\mathsf{Im}\, x$ and so $\mathsf{Im}\, x = 0$. But if $\mathsf{Im}\, x = \infty$ then by the law on multiplication $\mathsf{Im}\,(n \cdot 1 \cdot x) = \infty$ which contradicts the fact that $n \cdot 1 \cdot x = 0$.

□

**2.4 Complexification of a ring.** Let $A$ be a commutative unitary ring. Define

$A^{(2)}$ be the unitary ring obtained from the ring $A$ as follows:

$$A^{(2)} := \{(a,b) \in A \times A\};$$

$$(a_1,b_1)+(a_2,b_2) := (a_1+a_2, b_1+b_2), \ (a_1,b_1)\cdot(a_2,b_2) := (a_1a_2-b_1b_2, a_1b_2+a_2b_1).$$

Clearly, $a \mapsto (a,0)$ is an embedding of $A$ into $A^{(2)}$ as a subring $(A,0)$ and

$$(a,b) \mapsto (a,-b) \text{ an automorphism of } A^{(2)}.$$

**2.5** Let $\mathrm{M}(2,A^{(2)})$ be the set of $2\times2$ matrices over $A^{(2)}$ which we treat as an 8-dim $A$-module and let $\mathrm{SL}(2,A^{(2)})$ be the group of matrices of determinant 1.

A **Minkowski** $A$**-lattice** is the $A$-submodule $\mathcal{M}(A)$ of $\mathrm{M}(2,A^{(2)})$ consisting of matrices $X_{t,x,y,z}$ over $A^{(2)}$ of the form

$$X_{t,x,y,z} = X := \begin{pmatrix} (t+z,0) & (x,-y) \\ (x,y) & (t-z,0) \end{pmatrix}, \quad t,x,y,z \in A.$$

We have

$$\det(X) = (t^2 - x^2 - y^2 - z^2, 0) \in A \times \{0\}$$

and this defines Minkowski $A$-metric length of $(t,x,y,z)$.

For the general $A^{(2)}$-matrix

$$Y = \begin{pmatrix} (a_1,a_2) & (b_1,b_2) \\ (c_1,c_2) & (d_1,d_2) \end{pmatrix}$$

define the adjoint matrix

$$Y^\dagger := \begin{pmatrix} (a_1,-a_2) & (c_1,-c_2) \\ (b_1,-b_2) & (d_1,-d_2) \end{pmatrix}$$

Clearly, $X^\dagger = X$ for $X \in \mathcal{M}(A)$. In general

$$(YZ)^\dagger = Z^\dagger Y^\dagger.$$

In particular, $Y$ is self-adjoint ($Y = Y^\dagger$) iff $a_2 = 0 = d_2$ and $b_1 = c_1$, $b_2 = -c_2$.

It follows that for any $M \in \mathrm{SL}(2, A^{(2)})$, $X \in \mathcal{M}(A)$

$$MXM^\dagger \in \mathcal{M}(A) \text{ and } \det X = \det MXM^\dagger \tag{7}$$

Let

$$C = \{M \in \mathcal{M}(A) : MXM^\dagger = X \text{ for all } X \in \mathcal{M}(A).\}$$

Let $M_0 \in C$. In particular, $M_0 M_0^\dagger = \mathrm{I}$. It is equivalent to $M_0^\dagger = M_0^{-1}$ and thus $M_0 X M_0^{-1} = X$ for all $X \in \mathcal{M}(A)$. This readily implies that $M_0$ is diagonal, in the centre of $\mathrm{SL}(2, A^{(2)})$ and so

$$C = \{M = \begin{pmatrix} (a_1, a_2) & 0 \\ 0 & (a_1, a_2) \end{pmatrix}; \quad a_1^2 - a_2^2 = 1 \ \& \ a_1 a_2 = 0\} \tag{8}$$

Thus we have established:

**2.6 Proposition.** *The 2-sorted structure*

$$\big(\mathcal{M}(A), \mathrm{SL}(2, A^{(2)})/C\big)$$

*is interpretable in the ring $A$ along with the group action $X \mapsto MXM^\dagger$ and $A$-Minkowski metric.*

*The action and Minkowski metric are defined by systems of polynomial equations over $\mathbb{Z}$.*

*In particular,* $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$ *is the group of $K$-linear transformations of* $\mathcal{M}(\mathrm{K})$ *preserving Minkowski* K*-valued metric.*

**2.7 Lemma**.

$$\mathrm{SL}(2, \mathbb{C}^{(2)})/C \cong \mathrm{SO}(4, \mathbb{C})$$

where $C$ is the centre of $\mathrm{SL}(2, \mathbb{C}^{(2)})$ and

$$C \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

**Proof.** By the Proposition $\mathrm{SL}(2, \mathbb{C}^{(2)})/C$ is the group of transformations of $\mathcal{M}(\mathbb{C})$ preserving Minkowski $\mathbb{C}$-valued metric, that is the form $x_0^2 + x_1^2 + x_2^2 + x_3^2$. But this is also the definition of group $\mathrm{SO}(4, \mathbb{C})$. The form of $C$ is determined by (8). □

**2.8 Compactification of $\mathbb{C}$-structures**. Consider $\mathcal{M}(\mathbb{C})$ and $\mathrm{SO}(4,\mathbb{C})$ as complex quasi-projective algebraic varieties, in particular we have

$$\mathcal{M}(\mathbb{C}) \times \mathrm{SO}(4,\mathbb{C}) \subset \mathbf{P}$$

where $\mathbf{P}$ is a projective variety (not uniquely determined, to be chosen later). Note that

$$\mathcal{M}(\mathbb{C}) \times \mathrm{SO}(4,\mathbb{C}) \times \mathcal{M}(\mathbb{C}) \hookrightarrow \mathbf{P} \times \mathbf{P}$$

and so the graph of the action of $\mathrm{SO}(4,\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$ is also a quasi-projective subvariety of $\mathbf{P} \times \mathbf{P}$.

Define the **compactification of the structure** $(\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))$,

$$(\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))^{\mathbf{P}} \supseteq (\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))$$

to be the structure defined by the relevant Zariski closed subsets and relations in cartesian powers of $\mathbf{P}$.

**2.9 Theorem.** *There is a Zariski homomorphism of structures*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}), \mathrm{SL}(2, \mathrm{K}^{(2)})/C\Big) \twoheadrightarrow (\mathcal{M}(\mathbb{C}), \mathrm{SO}(4,\mathbb{C}))^{\mathbf{P}} \qquad (9)$$

*Its restriction to the structure over* $\mathrm{K}_{\mathrm{real}}$ *is a Zariski homomorphism*

$$\mathrm{Lm} : \Big(\mathcal{M}(\mathrm{K}_{\mathrm{real}}), \mathrm{SL}(2, \mathrm{K}_{\mathrm{real}}^{(2)})/C\Big) \twoheadrightarrow \Big(\mathcal{M}(\mathbb{R}), \mathrm{SO}^{+}(1,3)\Big) \qquad (10)$$

**Proof.** By 2.3 we have an induced Zariski homomorphism

$$\mathrm{lm}_{\mathrm{K}} : \Big(\mathcal{M}(\mathrm{K}), \mathrm{SL}(2, \mathrm{K}^{(2)})/C\Big) \twoheadrightarrow \Big(\mathcal{M}(\mathbb{C}), \mathrm{SL}(2, \mathbb{C}^{(2)})/C\Big)^{\mathbf{P}}$$

which by 2.7 is the same as (9).

The restriction of limit maps to the structure over $\mathrm{K}_{\mathrm{real}}$ by construction has the form

$$\Big(\mathcal{M}(\mathrm{K}_{\mathrm{real}}), \mathrm{SL}(2, \mathrm{K}_{\mathrm{real}}^{(2)})/C\Big) \twoheadrightarrow (\mathcal{M}(\mathbb{R}), \mathrm{SL}(2,\mathbb{C})/C)$$

which becomes (10) when one takes into account (5). $\square$

**2.10 Commentary.**

The statement in (10) can be interpreted as the statement that *at low scale* the pseudo-finite space looks like the canonical Minkowski space $\mathcal{M}(\mathbb{R})$ with the action of the Lorentz group $\mathrm{SO}^{+}(1,3)$.

# 3 Addendum

**3.1 Relation between the discrete Minkowski space $\mathcal{M}(\mathrm{K})$ and the discrete universe $\mathbb{U}$ of [2].**

Recall that the 1-dimensional universe $\mathbb{U}$ of [2] is defined as the additive group of the residue ring

$$\mathrm{K} = {}^*\mathbb{Z}/\mathcal{N}, \text{ where } \mathcal{N} = (\mathfrak{p}-1)\mathfrak{l},$$

$\mathfrak{p}$ non-standard prime and $\mathfrak{l}$ a highly divisible non-standard integer. Thus $\mathbb{U}$ can also be considered a 1-dimensional K-module, where we can now identify K with the one from previous sections, introduced in 2.2.

Thus, for the Minkowski $K$-space $\mathcal{M}(\mathrm{K})$ one establishes an isomomrphism

$$\mathcal{M}(\mathrm{K}) \cong \mathbb{U}^4$$

as K-modules, and the constructions above define the action of the quasi-Lorentz group $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$ on $\mathbb{U}^4$ along with the Minkowski K-valued metric invariant under $\mathrm{SL}(2, \mathrm{K}^{(2)})/C$.

In [2] we identified in the universe $\mathbb{U}$ and its cartesian powers $\mathbb{U}^n$ subdomains which correspond to the scales of quantum mechanics and statistical mechanics and developed elements of these theories in the model on $\mathbb{U}$ which unified the two theories. The current work demonstrates that the same model can incorporate special relativity.

**3.2 Klein-Gordon wave-functions**

$$\phi(\bar{r}, t) := \exp(i\bar{k} \cdot \bar{r} - i\omega t)$$

where

$$\bar{k} \cdot \bar{r} = \sum_{j=1}^{j=3} k_j r_j.$$

Thus

$$\frac{\partial}{\partial r_j}\phi = ik_j\phi; \quad \frac{\partial}{\partial t}\phi = i\omega\phi$$

and Klein-Gordon is satisfied:

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\phi = (-\hbar^2 c^2 \sum \frac{\partial^2}{\partial r_j^2} + m^2 c^4)\phi$$

In operator terms

$$\mathbf{T} := \hbar \bar{k}; \quad \mathrm{E} = \hbar\omega$$

and Klein-Gordon

$$\mathrm{E}^2\phi = (\mathrm{P}^2 c^2 + m^2 c^4)\phi$$

so

$$(\mathrm{P}^2 c^2 + m^2 c^4) = \mathrm{H}^2$$

Recall that **solutions to the Dirac equation** is a combination of two solutions of the Klein-Gordon equation.

**3.3 Lorentz invariance**: $(\bar{r}, t) \mapsto g(\bar{r}, t)$ implies $\phi \mapsto \phi^g$ such that

$$\phi^g(g(\bar{r}, t)) = \phi(\bar{r}, t).$$

That is, for $X \in \mathcal{M}$

$$\phi^g(gX) = \phi(X) = \exp(iK \cdot X)$$

where

$$K \in \mathcal{M}, \quad K \cdot X = \sum_{j=1}^{j=3} k_j x_j - k_0 x_0, \text{ Minkowski scalar product.}$$

It follows

$$\phi^g(Y) = \exp(iK \cdot g^{-1} \cdot Y)$$

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