Modular curves and their pseudo-analytic cover

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1 Introduction

1.1 Motivation and origins. The original aim of the paper is the extension of the project *categoricity of non-elementary theories of analytic covers* from "abelian" cases, such as $\exp : \mathbb{C} \to \mathbb{G}_m$ or $\exp_\Lambda : \mathbb{C} \to \mathrm{E}_\Lambda$ (elliptic curve, to hyberbolic curves and possibly wider, see e.g. the survey part in [1] for some history and references.

The first obstruction for this project is that an adequate formalism (that is the language) in which such an analytic cover can be properly presented is not easy to determine. Some attempts in this direction were by M.Gavrilovich [20], as well as later attempts by A.Harris [3]. The one which we found satisfactory and applied here is based on the formalism close to the one applied in the recent [4].

More specifically, we formalise e.g. the case $\exp : \mathbb{C} \to \mathbb{G}_m$ as a structure with two sorts $\mathbb{U}$ and F, where $\mathbb{U}$ is the complex numbers with the $\mathbb{Q}$-module structure and the distinguished subgroup $2\pi i\mathbb{Z}$, and F is the complex numbers as the field. Then $\exp : \mathbb{U} \to \mathbb{G}_m(\mathrm{F})$ is the map between the two sorts. But in fact along with exp we automatically get in abelian cases the family

$$\exp_n : \mathbb{U} \to \mathbb{G}_m(\mathrm{F}), \quad \exp_n : x \mapsto \exp(\frac{x}{n})$$

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which of course agrees with the system of finite covers

$$\mathrm{pr}_{k,n} : \mathbb{G}_m(\mathrm{F}) \to \mathbb{G}_m(\mathrm{F});\ y \mapsto y^{\frac{k}{n}},\ \text{when } n|k.$$

This we describe axiomatically by simple $L_{\omega_1,\omega}$ sentences. In terms of model-theoretic classification such a structure is a *fusion between a locally modular structure* $\mathbb{U}$ *and an algebraically closed field* F. The case $\exp_\Lambda : \mathbb{C} \to \mathrm{E}_\Lambda$ similarly represents a fusion between a locally modular structure $\mathbb{U}_\Lambda$, and an algebraically closed field F, where $\mathbb{U}_\Lambda$ is a $\mathbb{Q}$-module with distinguished $\mathbb{Z}$-module Λ with an alternating form on it (accounting for the Weil pairing).

1.2 It is not clear a priori why such $L_{\omega_1,\omega}$-theories should be categorical in uncountable cardinals and the fact that they are must be of some significance.

The geometric value of the project is perhaps in the fact that the formulation of the categorical theory of the universal cover of a variety $\mathbb{X}$ (essentially the description of $\mathbb{U}$) is a formulation of a complete formal **invariant** of $\mathbb{X}$. By its nature such an $L_{\omega_1,\omega}$-invariant is of "algebraic type" and the fact that it is equivalent to a notion given in topological/analytic terms indicates a possibility of connection to certain key conjectures of algebraic geometry such as the Hodge conjecture.

Indeed, the most interesting outcome of the earlier works was establishing an equivalence between categoricity of the cover of E_Λ and the conjuction of the two arithmetic facts:

(i) the complete classification of the Galois action on the torsion of E_Λ (the open image theorem by J.-P.Serre) and

(ii) the Kummer theory for E_Λ (M.Bashmakov and K.Ribet)

For abelian varieties the success of the program depends on the extension of analogues of (i) and (ii) to abelian varieties, and (ii) is known due to K.Ribet and M.Larsen. However, an analogue of Serre's theorem for abelian varieties is an open problem and therefore the best categoricity result here is under the assumption that the language names points of *the kernel* of the exponential map. This autmomatically removes the problem of determining the Galois action on the torsion points at the cost of weakening the formal invariant of $\mathbb{X}$/k, the theory of the cover of $\mathbb{X}$/k, to the formal invariant of $\mathbb{X}/K$ where K is the field obtained from k by adjoining all torsion points of $\mathbb{X}$.

The case of hyperbolic curves $\mathbb{X}$ was first considered by A.Harris [3] and C.Daw and A.Harris [13] in the context of modular curves $\mathbb{Y}(\Gamma)$ in the formalism (the choice of the language) which has names for each element of

$\mathrm{GL}_2^+(\mathbb{Q})$ acting on the upper half plane $\mathbb{H}$ by Möbius transformations. The proof of categoricity in this setting required essentially Serre's open image theorem for products of non-CM elliptic curves. Interestingly, in the analysis of $(\mathbb{H}, j_N, \mathbb{Y}(N))$ Serre's theorem plays rather the role of (ii), while (i) is not needed since naming elements of $\mathrm{GL}_2(\mathbb{Q})$ fixes the special points (CM-points). This has a cost: one works out the formal invariant of $\mathbb{Y}(\Gamma)$ over the extension of the natural field of definition by special points.

1.3 Our setting. Our current interest is the case of the universal cover of hyperbolic curves, e.g. the complex curve $\mathbb{X} = \mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$. However, before approaching this case we set ourselves a simpler task of the *cover of the modular curve* $\mathbb{Y}(N)$ *universal in the class of modular curves*, which means that our structure incorporates the analytic covering maps

$$j_n : \mathbb{H} \to \mathbb{Y}(n), \text{ for all } n \text{ such that } N|n$$

agreeing with the algebraic finite covers

$$\mathrm{pr}_{n,m} : \mathbb{Y}(n) \to \mathbb{Y}(m), \text{ for all } n, m \text{ such that } N|m,\ m|n.$$

In fact it is enough to classify the case $N = 1$, $\mathbb{Y}(1) = \mathbb{Y} = \mathbf{A}^1$ the affine line (the fact that for some n the covers $j_n : \mathbb{H} \to \mathbb{Y}(n)$ are ramified does not matter in our setting).

Note that $\mathbb{Y}(2) = \mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$ when we consider $\mathbb{Y}(2)$ as an algebraic curve (without the level structure).

The important difference with the case of the proper universal cover is that, instead of the profinite completion $\hat{\Gamma}(N)$ of the respective fundamental group, in the modular setting one has the group $\tilde{\Gamma}(N)$, the completion in the topology based on congruence subgroups, which for $N = 1$ gives us

$$\tilde{\Gamma}(1) = \tilde{\Gamma} = \mathrm{SL}(2, \hat{\mathbb{Z}}),$$

where

$$\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z},$$

the projective limit of residue rings.

1.4 The key problem, similar to earlier cases, is in classifying the saturated version of $\mathbb{H}$ in the structure $(\mathbb{H}, j_n, \mathbb{Y}(n))_{n\in\mathbb{N}}$ which is essentially reducible to understanding the structure on the projective limit

$$\tilde{\mathbb{H}} := \varprojlim \Gamma(n)\backslash\mathbb{H} \cong \varprojlim \mathbb{Y}(n).$$

This includes giving a detailed enough description of the action of the automorphism group $\mathrm{Aut}\,\mathbb{C}$ on $\tilde{\mathbb{H}}$ and, in particular, its action on $\tilde{\mathbb{H}}(\mathrm{CM})$, the special points of the structure (equivalently, the action of $\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}) : \mathbb{Q})$ on $\tilde{\mathbb{H}}(\mathrm{CM})$), the analogue of (i) of 1.2.

It turns out that this problem is closely connected to the theory of *canonical models of Shimura varieties* resulting from Shimura's Conjecture which was developed by G.Shimura, P.Deligne and others. This involves an advanced theory of complex multiplication and Artin's reciprocity map. The results allow one to identify the action of $\mathrm{Aut}\,\mathbb{C}$ on a single CM-point (that is the Galois orbit of the point), see [8], 12.8. We need a stronger result: our paper [10] goes further and identifies the action of $\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}) : \mathbb{Q})$ on $\tilde{\mathbb{H}}(\mathrm{CM})$, at the same time describing all the relations between CM-points. One of the interesting model-theoretic forms of the main result of [10] is that *the field* $\mathbb{Q}(\mathrm{CM})$ *as a structure is* $L_{\omega_1,\omega}$*-bi-interpretable with a certain structure formulated purely in terms of the ring* $\mathbb{A}_f$ *of finite adeles over* $\mathbb{Q}$.

In [10] we defined a certain locally modular structure of trivial type on the set $\tilde{\mathbb{H}}$, which we term $\tilde{\mathbb{H}}^{\mathrm{Pure}}$. This is formulated in terms of the action of a large subgroup $\tilde{\mathrm{G}}$ of $\mathrm{GL}_2(\mathbb{A}_f)$ on $\tilde{\mathbb{H}}$. The main result of [10] determines automorphisms of $\tilde{\mathbb{H}}^{\mathrm{Pure}}$ and then describes $\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}) : \mathbb{Q})$ in terms of $\mathrm{Aut}\,\tilde{\mathbb{H}}^{\mathrm{Pure}}(\mathrm{CM})$. These two groups are "almost" equal: there is an obvious embedding of $\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}) : \mathbb{Q})$ into $\mathrm{Aut}\,\tilde{\mathbb{H}}^{\mathrm{Pure}}(\mathrm{CM})$, and when restricting the two groups to their action on a finite number of G-orbits the first is a finite index subgroup of the second.

$\tilde{\mathbb{H}}^{\mathrm{Pure}}$ has a "standard" version $\mathbb{H}^{\mathrm{Pure}}$ which is based on the actual upper half-plane $\mathbb{H}$ and is given in terms of the action of $\mathrm{GL}_2^+(\mathbb{Q})$ and complex multiplication.

1.5 With the results of [10] in hand we can apply the model-theoretic techniques on categoricity developed in earlier works (see e.g. the survey in [1]) and in particular the proof in [13] which we follow quite closely (and so use the Serre open image theorem).

The axioms Σ of the resulting $L_{\omega_1,\omega}$-theory consist, as in abelian examples, of three parts:

A The axioms describing $\mathbb{U}^{\mathrm{Pure}}$ (the abstract version of $\mathbb{H}^{\mathrm{Pure}}$), a locally modular structure of trivial type with the action of the group G isomorphic to $\mathrm{GL}_2^+(\mathbb{Q})$, and its subgroups corresponding to $\Gamma(n)$.

B The axioms describing an algebraically closed field F of characteristic 0 and curves $\mathbb{Y}(n) \subset \mathbf{P}^3(\mathrm{F})$.

C The axioms describing $j_n : \mathbb{U}^{\mathrm{Pure}} \to \mathbb{Y}(n)$ obtained via the translation of relations on $\mathbb{U}^{\mathrm{Pure}}$ to special relations on the $\mathbb{Y}(n)$.

Main Theorem. *The above system of axioms Σ is satisfied by the standard complex structure $\mathbb{H}$ and every model of the axioms is a quasi-minimal geometry structure.*

For each model $\mathbb{U}^{\mathrm{Full}}$ of Σ there is a simple $L_{\omega_1,\omega}$-sentence Θ which holds on $\mathbb{U}^{\mathrm{Full}}$ and such that $\Sigma \,\&\, \Theta$ has, up to isomorphism, a unique model in any uncountable cardinal.

1.6 The first author whishes to express his gratitude to J.Derakhshan for his interest in the work and his help with some of the mathematical issues in the paper.

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2 Groups acting on $\mathbb{H}$.

2.1 Groups G and Γ and their generators.

Generators of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ are represented by matrices

$$\mathbf{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}:$$

$$\mathbf{s}^2 = -\mathbf{I}, \quad (\mathbf{st})^3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 = -\mathbf{I}.$$

Generators of $\mathrm{G} := \mathrm{GL}_2^+(\mathbb{Q})$:

$$\mathbf{s}, \mathbf{t} \text{ and } \mathbf{d}_q := \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, q \in \mathbb{Q}_+$$

satisfying:

$$\mathbf{d}_q \mathbf{d}_r = \mathbf{d}_{qr},$$

$$\mathbf{s}\mathbf{d}_q = q\mathbf{d}_q^{-1}\mathbf{s},$$

$$\mathbf{d}_n\mathbf{t} = \mathbf{t}^n\mathbf{d}_n, \text{ for } n \in \mathbb{N}.$$

We will use

$$\mathbf{d}'_q := \mathbf{s}\mathbf{d}_q\mathbf{s}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, q \in \mathbb{Q}_+$$

$$\mathbf{t}_- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

which satisfies

$$\mathbf{s}\mathbf{t}\mathbf{s}^{-1} = \mathbf{t}_-; \quad \mathbf{d}_n\mathbf{t}_-^n = \mathbf{t}_-\mathbf{d}_n, \text{ for } n \in \mathbb{N},$$

and subgroups

$$\Delta(\mathbb{Q}_+) = \{\mathbf{d}_q : \ q \in \mathbb{Q}_+\} \text{ and } \Delta'(\mathbb{Q}_+) = \{\mathbf{d}'_q : \ q \in \mathbb{Q}_+\}. \tag{1}$$

Similar notation also make sense for the multiplicative group of a commutative ring R,

$$\Delta(R^\times) = \{\mathbf{d}_q : \ q \in R^\times\} \text{ and } \Delta'(R^\times) = \{\mathbf{d}'_q : \ q \in R^\times\}.$$

2.2 Remark. Note that all automorphisms of $\mathrm{GL}_2^+(\mathbb{Q})$ and of $\mathrm{PGL}_2(\mathbb{Q})$ are inner and the only non-identity automorphism which fixes Δ' element-wise and preserves the subgroup Γ is the involution

$$g \mapsto \check{g} := \mathbf{d}_{-1} \cdot g \cdot \mathbf{d}_{-1}.$$

Note also that since the action of the centre of $\mathrm{GL}_2^+(\mathbb{Q})$ acts trivially on $\mathbb{H}$ we can work equally with $\mathrm{PGL}_2(\mathbb{Q})$, the generators and defining relations of which are the same if we ignore the scalar multipliers in the group relations.

2.3 Special points on $\mathbb{H}$. Let

$$\mathrm{E} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{G} : (d-a)^2 < -4bc\} = \{e \in \mathrm{G} : \mathrm{tr}^2\, e < 4 \det e\},$$

the set of *elliptic* transformations.

These are exactly the elements for which there is a unique fixed point $\tau_e \in \mathbb{H}$ which can be found by solving the equation

$$cx^2 + (d-a)x - b = 0, \quad \tau_e := x, \quad \Im x > 0. \tag{2}$$

Note that elements of the centre

$$\mathrm{Z} = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathrm{G}\}$$

act as identity on $\mathbb{H}$.

2.4 Remark. (i) E is invariant under automorphisms of G since $g \cdot e \cdot g^{-1}$ fixes $g\tau_e$ and thus belongs to E.

(ii) The subgroup $\mathrm{St}_\tau \subset \mathrm{G}$ fixing point $\tau = \tau_e$ is definable by the condition that $\mathrm{St}_\tau = C(e)$, the centraliser of e in the group. This is a general fact for Mobius transformations over any field of characteristic 0.

The following is a technical remark which we are going to use later.

2.5 Remark. Let $I = \bigcup_l \mathrm{E}^l$ and $\{\Phi_h : h \in I\}$ a family of quantifier-free $L_{\omega_1,\omega}$-formulas.

There is an existential $L_{\omega_1,\omega}$-formula Θ in the language with names $\mathbf{d}_q$ for respective elements of G stating that for any $\mathsf{h} \in \mathrm{E}^l$ there exists $\bar{t}_h = \langle t_1, \dots, t_l \rangle$, a tuple of fixed points of h such that $\Phi_i(\bar{t}_h)$.

Indeed, let $\{w_e(\mathbf{s}, \mathbf{t}, \mathbf{d}_\mathbb{Q}) : e \in \mathrm{E}\}$ be the family of group words in generators $\mathbf{s}, \mathbf{t}, \mathbf{d}_q$ $q \in \mathbb{Q}_+$, listing all the elements of E. Using the family of words we can produce the family $\{\bar{w}_h(\mathbf{s}, \mathbf{t}, \mathbf{d}_\mathbb{Q}) : \mathsf{h} \in I\}$ of tuples of words corresponding to tuples $\mathsf{h} \in I$. Let $\mathrm{Fix}(\bar{t}, \bar{h})$ is the formula saying that $\bar{t}$ is a tuple from $\mathbb{H}$ fixed by the tuple $\bar{h}$ from E.

Set

$$\Theta := \begin{array}{ll} \exists s, t \in \mathrm{G}\ \{s^2 = -1\ \& & (st)^3 = -1\ \&\ \bigwedge_{q \in \mathbb{Q}_+} \mathbf{s}\mathbf{d}_q = q\mathbf{d}_q^{-1}\mathbf{s}\ \&\ \bigwedge_{n \in \mathbb{N}} \mathbf{d}_n \mathbf{t} = \mathbf{t}^n \mathbf{d}_n\ \& \\ & \&\ \bigwedge_{\mathsf{h} \in I} \exists \bar{t}_h \mathrm{Fix}(\bar{t}_h, \bar{w}_h(s, t, \mathbf{d}_\mathbb{Q}))\ \&\ \Phi_h(\bar{t}_h)\} \end{array}$$

Since the choice of any $s, t \in \mathrm{G}$ satisfying the group relations in the first line of Θ is conjugated to $\mathbf{s}, \mathbf{t}$ by an automorphism of G (see 2.2) and E is invariant under the automorphisms, the values of words $\bar{w}_h(s, t, \mathbf{d}_\mathbb{Q}))$ run through all elements of E. □

3 Projective limit

3.1 We consider the structure $\tilde{\mathbb{H}}$, the projective limit of structures

$$\tilde{\mathbb{H}} = \varprojlim \Gamma(N) \backslash \mathbb{H}, \tag{3}$$

along the projective system of

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad a \equiv d \equiv 1 \mod N,\ c \equiv b \equiv 0 \mod N \right\},$$

normal subgroups of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

The structures $\Gamma(N)\backslash\mathbb{H}$ are identified with the classical complex modular curves $\mathbb{Y}(N)$ (without the full level N structure). These are non-singular algebraic curves and therefore can be realised as quasi-projective algebraic curves in $\mathbf{P}^3(\mathbb{C})$. Moreover, the realisation can be obtained over $\mathbb{Q}$, see e.g. section 4 of [11] or the paper [12], section 2. Another argument for this fact is given in [10], 3.2(8).

The classical modular functions

$$j_N : \mathbb{H} \to \mathbb{Y}(N); \quad \tau \mapsto \Gamma(N)\cdot\tau$$

are holomorphic and the curves $\mathbb{Y}(N)$ are finite coverings of $\mathbb{Y}(N/d)$, for $d|N$, via the projection maps

$$\mathrm{pr}_{N,N/d} : j_N(\tau) \mapsto j_{N/d}(\tau), \quad \mathbb{Y}(N) \to \mathbb{Y}(N/d). \tag{4}$$

3.2 The definition (3) implies the existence of maps

$$j_N : \tilde{\mathbb{H}} \to \Gamma(N)\backslash\mathbb{H} = \mathbb{Y}(N)$$

which we are going to use below.

By definition, any $\tau \in \tilde{\mathbb{H}}$ is uniquely determined by the sequence

$$j_n(\tau) \in \mathbb{Y}(n),\ n \in \mathbb{N}$$

and the sequence has the property

$$\mathrm{pr}_{n,n/d}(j_n(\tau)) = j_{n/d}(\tau),\ \text{for each } d|n.$$

3.3 Lemma. *Any $\tau \in \mathbb{H}$ is uniquely determined by the sequence*

$$j_n(\tau) \in \mathbb{Y}(n):\ n \in \mathbb{N}.$$

This gives the canonical embedding

$$\mathbb{H} \hookrightarrow \tilde{\mathbb{H}}.$$

Proof. The first statement follows from the fact that $\bigcap_N \Gamma(N) = \{1\}$. The second statement is the consequence of the fact that the sequence satisfies

$$\mathrm{pr}_{n,n/d}(j_n(\tau)) = j_{n/d}(\tau),\ \text{for each } d|n.$$

□

3.4 Remark. The system of covers (4) is not étale. However, by removing finitely many points on $\mathbb{Y}(1)$ and all the points on $\mathbb{Y}(n)$ over these one gets smooth curves $\mathbb{Y}^-(n)$ and a projective system of covers

$$\mathrm{pr}_{n,n/d} : \mathbb{Y}^-(n) \to \mathbb{Y}^-(n/d)$$

which is étale.

Since the construction of $\tilde{\Gamma}$ depends on generic fibres we have the same $\tilde{\Gamma}$ for the construction corresponding to the system of étale covers.

3.5 The projective limit as the structure $\tilde{\mathbb{H}}^{\mathrm{Full}}$.

The analysis and study of the projective limit of modular curves $\mathbb{Y}(n)$ with the *n-level structure defined over* $\mathbb{Q}(\zeta_n)$ is readily reducible to the study of *canonical models of Shimura varieties*, see [8], and more specifically canonical models of modular curves, [9]. The case of curves $\mathbb{Y}(n)$ over $\mathbb{Q}$ as above in the setting appropriate for our purposes is studied in [10]. The structure $\tilde{\mathbb{H}}^{\mathrm{Full}}$ is described therein as the quotient $\Delta(\hat{\mathbb{Z}}^\times)\backslash\mathbf{S}^{\mathrm{Full}}$ of a more fundamental structure $\mathbf{S}^{\mathrm{Full}}$ associated with the Shimura datum $(\mathbb{H} \cup -\mathbb{H}, \mathrm{GL}_2)$.

The main conclusions, see [10], 3.22, are as follows:

(a) There is a group $\tilde{\mathrm{G}}$ acting on $\tilde{\mathbb{H}}$.

$$\tilde{\mathrm{G}} \cong \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f) \subset \mathrm{GL}_2(\mathbb{A}_f),$$

where $\mathbb{A}_f$ is the ring of finite adeles.

One can speak about an action of $\mathbf{g} \in \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f)$ on $\tilde{\mathbb{H}}$ once an isomorphism $\varphi : \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f) \to \tilde{\mathrm{G}}$ is provided. Call φ a **naming isomorphism**. The naming isomorphisms form the family

$$\{\varphi_\mu : \ \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f) \to \tilde{\mathrm{G}} \mid \mu \in \hat{\mathbb{Z}}^\times\}; \ \ \text{set } \mathbf{g}^\mu := \varphi_\mu(\mathbf{g}),$$

which satisfies

$$\mathbf{g}^{\mu\cdot\lambda} = (\mathbf{d}_\lambda^{-1} \cdot \mathbf{g} \cdot \mathbf{d}_\lambda)^\mu$$

where the conjugation by $\mathbf{d}_\lambda$ is in the ambient group $\mathrm{GL}_2(\mathbb{A}_f)$.

(b) Any $\mathbf{g} \in \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f)$ determines the 0-definable subset of $\tilde{\mathrm{G}}$:

$$\mathbf{g}^\Delta = \{\mathbf{g}^\mu : \mu \in \hat{\mathbb{Z}}^\times\}.$$

Each $\mathbf{g}^\mu$ gives rise to the sequence of algebraic curves defined over $\bar{\mathbb{Q}}$,

$$C^\mu_{\mathbf{g},N} \subset \mathbb{Y}(N) \times \mathbb{Y}(N); \ \ C^\mu_{\mathbf{g},N} := \{\langle j_N(u), j_N(\mathbf{g}^\mu \cdot u)\rangle : u \in \tilde{\mathbb{H}}\}$$

(here $C^{\mu}_{\mathbf{g},N}$ corresponds to $C^{\mu}_{\mathbf{g},\approx K}$ of [10]). Note that any special curve $C \subset \mathbb{Y}(N) \times \mathbb{Y}(N)$ has the form $C = C^{\mu}_{\mathbf{g},N}$ for some $\mathbf{g}$.

The construction of the projective limit applied to the sequence $\{C^{\mu}_{\mathbf{g},N} : N \in \mathbb{N}\}$ of curves results in the *limit curve*

$$C^{\mu}_{\mathbf{g}} \subset \tilde{\mathbb{H}} \times \tilde{\mathbb{H}}; \quad C^{\mu}_{\mathbf{g}} := \{\langle u, \mathbf{g}^{\mu} u\rangle : u \in \tilde{\mathbb{H}}\}.$$

(c) For a fixed $\mathbf{g} \in \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f)$ we obtain the finite family of curves on $\mathbb{Y}(N) \times \mathbb{Y}(N)$

$$\{C^{\mu}_{\mathbf{g},N},\ \mu \in \hat{\mathbb{Z}}^{\times}\}$$

where

$$\mu - \lambda \in N\hat{\mathbb{Z}} \Longrightarrow C^{\mu}_{\mathbf{g},N} = C^{\lambda}_{\mathbf{g},N}.$$

These curves are irreducible components of the algebraic curve $C_{\mathbf{g},N}$ defined over $\mathbb{Q}$:

$$C_{\mathbf{g},N} = \bigcup_{\mu \in \hat{\mathbb{Z}}^{\times}} C^{\mu}_{\mathbf{g},N} \subset \mathbb{Y}(N) \times \mathbb{Y}(N).$$

In the limit one obtains the the infinite-component limit curve on $\tilde{\mathbb{H}} \times \tilde{\mathbb{H}}$:

$$C_{\mathbf{g}} = \bigcup_{\mu \in \hat{\mathbb{Z}}^{\times}} C^{\mu}_{\mathbf{g}}.$$

$C_{\mathbf{g}}$ is defined over $\mathbb{Q}$ too.

(d) The irreducible components $C^{\mu}_{\mathbf{g},N}$ of $C_{\mathbf{g},N}$ are Galois conjugated over $\mathbb{Q}$.

Clearly $C_{\mathbf{g}} = C_{\mathbf{g}'}$ for $\mathbf{g}' = \mathbf{g}^{\mu}$. The image of $C_{\mathbf{g}}$ under $j_N \times j_N$ is $C_{\mathbf{g},N}$.

(e) For any $\mathbf{g} \in \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f)$, the definable 4-ary relation on $\tilde{\mathbb{H}}$

$$\mathrm{Comp}_{\mathbf{g}}(s_1, s_2, t_1, t_2) :\equiv \exists h \in \tilde{\mathrm{G}}\ s_2 = h \cdot s_1\ \&\ t_2 = h \cdot t_1\ \&\ C_{\mathbf{g}}(s_1, s_2)$$

determines the condition that $\langle s_1, s_2\rangle$ and $\langle t_1, t_2\rangle$ belong to the same irreducible component of a $C_{\mathbf{g}}$.

For each N the relation $\mathrm{Comp}_{\mathbf{g},N}$ on $\mathbb{Y}(N)$ is the image of $\mathrm{Comp}_{\mathbf{g}}$ under j_N, the relation which determines the decomposition of the algebraic curve $C_{\mathbf{g},N}$ into irreducible components. This relation is invariant under $\mathrm{Gal}_{\mathbb{Q}}$ and so definable over $\mathbb{Q}$.

We also have on $\tilde{\mathbb{H}}$ the definable equivalence

$$j_N(s_1) = j_N(s_2),$$

which can be equivalently given by:

$$\exists \gamma \in \tilde{\Gamma}(n)\ s_2 = \gamma \cdot s_1.$$

(f) The points s in $\tilde{\mathbb{H}}$ which are fixed by a $g \in \tilde{\mathrm{G}} \smallsetminus \varphi_\mu(\mathrm{Z})$ (the centre of $\mathrm{GL}_2^+(\mathbb{Q})$ will be called special, or CM-points of $\tilde{\mathbb{H}}$. $j_N(s)$ is a CM-point in $\mathbb{Y}(N)$ and, in particular, is algebraic.

The paper [10] describes the binary relations on $\tilde{\mathbb{H}}$ written as

$$\langle t_1, t_2 \rangle \in \mathrm{tp}(s_1, s_2)$$

defined for each pair s_1, s_2 of CM-points. The reation is valid if and only if there is an automorphism σ of the projective system (4) such that $\sigma : \langle s_1, s_2 \rangle \mapsto \langle t_1, t_2 \rangle$. The relations are invariant under automorphisms of the projective system (i.e. defined over $\mathbb{Q}$) for each choice of s_1, s_2.

The image of the relation under j_N is the relation on $\mathbb{Y}(N)$

$$\langle y_1, y_2 \rangle \in \mathrm{tp}_N(x_1, x_2)$$

which holds if and only if $\langle y_1, y_2 \rangle$ is Galois conjugated to $\langle x_1, x_2 \rangle$ over $\mathbb{Q}$.

3.6 Definition. $\tilde{\mathbb{H}}^{\mathrm{Pure}}$ is the structure with the universe $\tilde{\mathbb{H}}$ and relations $C_{\mathbf{g}}(s_1, s_2)$, $\mathrm{Comp}_{\mathbf{g}}(s_1, s_2, t_1, t_2)$ ($\mathbf{g} \in \Delta(\mathbb{Q}_+) \cdot \mathrm{SL}_2(\mathbb{A}_f)$), $j_N(s_1) = j_N(s_2)$ and $\langle t_1, t_2 \rangle \in \mathrm{tp}(s_1, s_2)$.

$\tilde{\mathbb{H}}^{\mathrm{Full}}$ is the multisorted structure with sorts $\tilde{\mathbb{H}}$ and $\mathbb{Y}(N)$, $N \in \mathbb{N}$, and relations:

- on $\tilde{\mathbb{H}}$ the relations of $\tilde{\mathbb{H}}^{\mathrm{Pure}}$;
- on the $\mathbb{Y}(N)$ the Zariski closed relations defined over $\mathbb{Q}$;
- the maps $j_N : \tilde{\mathbb{H}} \to \mathbb{Y}(N)$ and $\mathrm{pr}_{N,N/d} : \mathbb{Y}(N) \to \mathbb{Y}(N/d)$.

$\tilde{\mathbb{H}}^{\mathrm{Pure}}(\mathrm{CM})$ and $\tilde{\mathbb{H}}^{\mathrm{Full}}(\mathrm{CM})$ are substructures of the structures with universes restricted to their special points.

Remarks. (i) Since components of the $C_{\mathbf{g}}(s_1, s_2)$ are graphs of actions of elements of the group $\tilde{\mathrm{G}}$, we often look at $\tilde{\mathbb{H}}$ as a $\tilde{\mathrm{G}}$-set.

(ii) A corollary of definitions is that the relations $C_{\mathbf{g}}$, $\mathrm{Comp}_{\mathbf{g}}$, and $\langle t_1, t_2 \rangle \in \mathrm{tp}(s_1, s_2)$ are projective limits of the relations $C_{\mathbf{g},N}$, $\mathrm{Comp}_{\mathbf{g},N}$, and $\langle t_1, t_2 \rangle \in \mathrm{tp}_N(s_1, s_2)$ on the $\mathbb{Y}(N)$. That is the relations on the sort $\tilde{\mathbb{H}}$ are positive-type-definable in terms of pull-backs of the respective relations on the $\mathbb{Y}(N)$ along with the pull-back of equality.

We thus can, up to $L_{\omega_1,\omega}$-bi-interpretability, identify $\tilde{\mathbb{H}}^{\mathrm{Pure}}$ as the structure given by the pull-backs of $C_{\mathbf{g},N}$, $\mathrm{Comp}_{\mathbf{g},N}$, $\langle t_1, t_2\rangle \in \mathrm{tp}_N(s_1, s_2)$ and equality. Call it **the pull-backs version of** $\tilde{\mathbb{H}}^{\mathrm{Pure}}$. In the same sense we speak about the pull-backs version of $\mathbb{H}^{\mathrm{Pure}}$, a substructure of the pull-backs version of $\tilde{\mathbb{H}}^{\mathrm{Pure}}$.

3.7 Lemma. *The pull-backs versions of $\tilde{\mathbb{H}}^{\mathrm{Pure}}$ and of $\mathbb{H}^{\mathrm{Pure}}$ are locally modular of trivial type.*

The structures satisfy

$$\mathbb{H}^{\mathrm{Pure}} \prec \tilde{\mathbb{H}}^{\mathrm{Pure}}.$$

Proof. The structure $\mathbb{H}^{\mathrm{Pure}}$ satisfies the assumptions of Theorems 4.11 and 4.14 of [1]. It follows that its theory has quantifier elimination and is of trivial type. $\tilde{\mathbb{H}}^{\mathrm{Pure}}$ is then its elementary extension in the obvious way. □

3.8 Remarks. It is easy to see that the centre Z of $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\tilde{\mathbb{H}}$ trivially.

Note that in the definitions above the curves $\mathbb{Y}(N)$ are curves over $\mathbb{C}$ (defined over $\mathbb{Q}$) and so the points of $\tilde{\mathbb{H}}$ are limits of $\mathbb{C}$-points.

However the definitions and results are valid in the context of curves $\mathbb{Y}(N)$ over F, an abtract algebraically closed field of characteristic 0. In this case $\mathbb{Y}_{\mathrm{F}}(N)$ are curves over F.

3.9 Define $\tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Pure}}$ and $\tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Full}}$ to be the respective structures obtained as the projective limit of the $\mathbb{Y}_{\mathrm{F}}(N)$, for an arbitrary algebraically closed field of characteristic 0.

3.10 Remark. Note that

$$\tilde{\mathbb{H}}^{\mathrm{Pure}}(\mathrm{CM}) = \tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Pure}}(\mathrm{CM})$$

since both sides are the structures obtained by taking the projective limit of the respective substructures $\mathbb{Y}(n)^{\mathrm{Pure}}(\mathrm{CM})$ on the curves. The same is true for the full structures:

$$\tilde{\mathbb{H}}^{\mathrm{Full}}(\mathrm{CM}) = \tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Full}}(\mathrm{CM}).$$

3.11 Define
$$\tilde{\mathrm{E}} = \{e \in \tilde{\mathrm{G}} \setminus \mathrm{Z}\ \exists u \in \tilde{\mathbb{U}}\ e \cdot u = u\}.$$

Set

$$\tilde{\mathrm{G}}_* = (\tilde{\mathrm{G}}, \tilde{\Gamma}, \tilde{\mathrm{E}}, \{\mathbf{d}_q,\ \mathbf{d}'_q : q \in \mathbb{Q}_+)$$

the structure on the group $\tilde{\mathrm{G}}$ with distinguished subgroup $\tilde{\Gamma}$, distinguished subset $\tilde{\mathrm{E}}$ and distinguished elements $\mathbf{d}_q$ and $\mathbf{d}'_q$.

Analogously,
$$\mathrm{G}_* = (\mathrm{G}, \Gamma, \mathrm{E}, \{\mathbf{d}_q,\ \mathbf{d}'_q : q \in \mathbb{Q}_+\}).$$

Clearly,
$$\mathrm{G}_* \subset \tilde{\mathrm{G}}_*$$

as structures.

3.12 Remark. There is an embeding

$$\tilde{\mathrm{G}} \subset \mathrm{GL}_2(\mathbb{A}_f)$$

which is an identity on the diagonal elements $\mathbf{d}_q,\ \mathbf{d}'_q : q \in \mathbb{Q}_+$. It is easy to see that such an embedding is determined uniquely, up to the conjugation by $\mathbf{d}_\mu$, $\mu \in \hat{\mathbb{Z}}^\times$.

In particular, we may assume that elements g of $\tilde{\mathrm{G}}$ are also elements of $\mathrm{GL}_2(\mathbb{A}_f)$ and thus the conjugation by an element $\mathbf{d}_\lambda \in \Delta$,

$$g \mapsto \mathbf{d}_\lambda \cdot g \cdot \mathbf{d}_\lambda^{-1}$$

is well defined.

3.13 Lemma. *Consider the natural embedding* $\mathrm{G}_* \subset \tilde{\mathrm{G}}_*$ *and let*

$$\psi : \mathrm{G}_* \twoheadrightarrow \mathrm{G}'_* \subset \tilde{\mathrm{G}}_*$$

be a partial isomorphism of $\tilde{\mathrm{G}}_*$.

ψ *can be extended to an automorphism* $\tilde{\psi} : \tilde{\mathrm{G}}_* \to \tilde{\mathrm{G}}_*$. *Moreover, there is* $\lambda \in \hat{\mathbb{Z}}^\times$ *such that*
$$\tilde{\psi} : g \mapsto \mathbf{d}_\lambda \cdot g \cdot \mathbf{d}_\lambda^{-1},\ \textit{for all } g \in \tilde{\mathrm{G}}.$$

Proof. Let $\mathbf{s}' = \psi(\mathbf{s}) \in \mathrm{G}'$. Then

$$\mathbf{s}' \in \tilde{\mathrm{G}}, \quad \mathbf{s}' \cdot \mathbf{d}_{-1} = \mathbf{d}'_{-1}\mathbf{s}' \text{ and } \quad \mathbf{s}'^2 = -\mathbf{I} = \mathbf{d}_{-1} \cdot \mathbf{d}'_{-1}.$$

It is easy to see that the three equations imply that, for some $\lambda \in \hat{\mathbb{Z}}^\times$,

$$\mathbf{s}' = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} = \mathbf{d}_\lambda \cdot \mathbf{s} \cdot \mathbf{d}_\lambda^{-1}.$$

Let $\mathbf{t}' = \psi(\mathbf{t}) \in \mathrm{G}'$. Then for each $n \in \mathbb{N}_{>0}$,

$$\mathbf{t}' \in \tilde{\Gamma}, \quad \mathbf{d}_n \cdot \mathbf{t}' \cdot \mathbf{d}_n^{-1} = \mathbf{t}'^n \text{ and } (\mathbf{s}'\mathbf{t}')^3 = \mathbf{I}.$$

It follows

$$\mathbf{t}' = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \mathbf{d}_\lambda \cdot \mathbf{t} \cdot \mathbf{d}_\lambda^{-1}.$$

Thus, G' is the group generated by $\mathbf{s}'$ and $\mathbf{t}'$ and

$$\psi : g \mapsto \mathbf{d}_\lambda \cdot g \cdot \mathbf{d}_\lambda^{-1} \text{ for all } g \in \mathrm{G}.$$

Take

$$\tilde{\psi} : g \mapsto \mathbf{d}_\lambda \cdot g \cdot \mathbf{d}_\lambda^{-1} \text{ for all } g \in \tilde{\mathrm{G}}.$$

It is clear that $\tilde{\psi}$ preserves $\tilde{\Gamma}$, $\mathbf{d}_q$ and $\mathbf{d}'_q$, $q \in \mathbb{Q}_+$.

Finally note that $g \in \tilde{\mathrm{E}}$ if and only if $\tilde{\psi}(g) \in \tilde{\mathrm{E}}$. This follows from the description of fixed points in 3.22D(b) of [10].

□

The following is essentially a corollary of main results of [10].

3.14 Lemma. *Any* $\psi \in \mathrm{Aut}\,\tilde{\mathrm{G}}_*$ *can be extended to* $\tilde{\psi} \in \mathrm{Aut}\,\tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Pure}}(\mathrm{CM})$.

Proof As noted above $\tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Pure}}(\mathrm{CM}) = \tilde{\mathbb{H}}^{\mathrm{Pure}}(\mathrm{CM})$ so we may argue in the setting of $\tilde{\mathbb{H}}$.

By 3.13 ψ has the form $\mathbf{g}^\mu \mapsto \mathbf{g}^{\mu\cdot\lambda} = \mathbf{d}_\lambda \cdot \mathbf{g}^\mu \cdot \mathbf{d}_\lambda^{-1}$, for some $\lambda \in \hat{\mathbb{Z}}^\times$. And by 3.5(d) ψ is induced by a $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$. In its turn σ acts on the CM-points of $\tilde{\mathbb{U}}_F$ and induces a $\tilde{\psi} \in \mathrm{Aut}\,\tilde{\mathbb{U}}_{\mathrm{F}}^{\mathrm{Pure}}(\mathrm{CM})$, as required. □

4 Axiomatisation of $\mathbb{H}_{\mathrm{F}}^{\mathrm{Full}}$ and $\mathbb{U}_{\mathrm{F}}^{\mathrm{Full}}$.

4.1 Axioms.

The language $\mathcal{L}(j_{n\in\mathbb{N}})$ is 3-sorted, with sorts $\mathbb{U}, \mathrm{G}$ and F. The structure on F is that of a field given in a standard ring language, the structure on G and $\mathbb{U}$ is that of a group acting on $\mathbb{U}$ with distinguished subsets $\mathrm{E}, \Gamma, \{\mathbf{d}_q, \mathbf{d}'_q, q \in \mathbb{Q}^\times\}$. Note that $\{\mathbf{d}_q, q \in \mathbb{Q}^\times\} = \Delta(\mathbb{Q})$ with all its elements named and the same for $\{\mathbf{d}'_q, q \in \mathbb{Q}^\times\} = \Delta'(\mathbb{Q})$. Note that

$$\mathrm{GL}_2^+(\mathbb{Q}) = \Delta(\mathbb{Q}_{>0}) \cdot \mathrm{SL}_2(\mathbb{Z}) = \Delta'(\mathbb{Q}_{>0}) \cdot \mathrm{SL}_2(\mathbb{Z})$$

and so group G is isomorphic to $\mathrm{GL}_2^+(\mathbb{Q})$ with distinguished elements of Δ, Δ' and subgroup $\Gamma \cong \mathrm{SL}_2(\mathbb{Z})$ will have the same structure.

Note that $\mathrm{GL}_2^+(\mathbb{Q})$ is invariant under the involutive transformation

$$g \mapsto \breve{g} := \mathbf{d}_{-1} \cdot g \cdot \mathbf{d}_{-1}$$

where $\mathbf{d}_{-1} \in \mathrm{GL}_2(\mathbb{Q}) \subset \mathrm{GL}_2(\mathbb{A}_f)$.

The maps j_n have $\mathbb{U}$ as their domain and have quasi-projective curves $\mathbb{Y}(n)$ as their range. Σ consists of the following five groups of axioms :

Group axioms:

$$(\mathrm{G}, \Gamma, \mathrm{E}, \{\mathbf{d}_q, \mathbf{d}'_q : q \in \mathbb{Q}_{>0}\}) \cong (\mathrm{GL}_2^+(\mathbb{Q}), \mathrm{SL}_2(\mathbb{Z}), \mathrm{E}(\mathbb{Q}), \{\mathbf{d}_q, \mathbf{d}'_q : q \in \mathbb{Q}_{>0}\}). \tag{5}$$

Note that $\Gamma(n)$ is definable from the data, namely the subgroup $\Gamma_0(n)$ of matrices of the form $\mathbf{g} = \mathbf{t}^{nm}$ are definable by condition

$$\exists \gamma \in \Gamma \ \mathbf{g} = \mathbf{d}_n \cdot \gamma \cdot \mathbf{d}_n^{-1},$$

and $\Gamma(n)$ can be defined as the normal closure of $\Gamma_0(n)$.

Let Z = Centre(G).

Action axiom: G acts on $\mathbb{U}$;

$$\begin{aligned} &\forall g \in \mathrm{G} \smallsetminus (\mathrm{E} \cup \mathrm{Z}) \ \forall u \in \mathbb{U} \ g \cdot u \neq u, \\ &\forall g \in \mathrm{Z} \ \forall u \in \mathbb{U} \ g \cdot u = u, \\ &\forall e \in \mathrm{E} \ \exists! u_e \in \mathbb{U} \ e \cdot u_e = u_e. \end{aligned} \tag{6}$$

Fibre formula:

$$\forall u, v \in \mathbb{U} \ j_n(u) = j_n(v) \leftrightarrow \exists \gamma \in \Gamma(n) \ v = \gamma \cdot u \tag{7}$$

$\mathbf{ACF}_0$ axioms and sorts $\mathbb{Y}(n)$:

$$\mathrm{F} \vDash \mathrm{ACF}_0 \tag{8}$$

and

$$\mathbb{Y}(n) \subset \mathbf{P}^3(\mathrm{F}); \quad \mathrm{pr}_{n.m} : \mathbb{Y}(n) \to \mathbb{Y}(m), \text{ for } m|n$$

are given by specific equations over $\mathbb{Q}$.

Functional equations:

$$j_n : \mathbb{U} \twoheadrightarrow \mathbb{Y}(n); \quad \mathrm{pr}_{n,m} \circ j_n = j_m \text{ for each } m|n; \tag{9}$$

$$\begin{array}{l} \forall g \in \mathrm{G}\ \exists \mathbf{g} \in \mathrm{GL}_2^+(\mathbb{Q}), \exists \mu \in \hat{\mathbb{Z}}^\times : \ \bigwedge_{n\in\mathbb{N}} C^\mu_{\mathbf{g},n} = j_n(\mathrm{graph}\, g) \\ \forall q \in \mathbb{Q}_+\ \forall \mu \in \hat{\mathbb{Z}}^\times : \ \bigwedge_{n\in\mathbb{N}} C^\mu_{\mathbf{d}_q,n} = j_n(\mathrm{graph}\, \mathbf{d}_q) \end{array} \tag{10}$$

$\forall \mathbf{g} \in \mathrm{GL}_2^+(\mathbb{Q}),\ \forall \mu \in \hat{\mathbb{Z}}^\times,\ \forall u, v \in \mathbb{U}$:

$$\bigwedge_{n\in\mathbb{N}} \langle j_n(u), j_n(v)\rangle \in C^\mu_{\mathbf{g},n} \Leftrightarrow \exists g \in \mathrm{G}\ v = g\cdot u\ \&\ \bigwedge_{n\in\mathbb{N}} C^\mu_{\mathbf{g},n} = j_n(\mathrm{graph}\, g) \tag{11}$$

4.2 Proposition. *$\mathbb{H}^{\mathrm{Full}}_{\mathbb{C}}$ is a model of Σ.*

Proof. Axioms (5)-(10) just list general properties of $\mathbb{H}$ and j_n from section 3 (axioms (9)-(10) are established in 3.5).

Axiom (11) is proved for $\mathbb{H}$ in [10] in the Claim 4.8(5). Indeed, the left-hand side of (11) states that $\langle u, v\rangle$ is a point on a graph of an element $\mathbf{g}^\mu \in \tilde{\mathrm{G}}$ whose graph in $\tilde{\mathbb{H}}$ is $C^\mu_{\mathbf{g}}$. By the Claim $\mathbf{g}^\mu \in \mathrm{GL}_2^+(\mathbb{Q})$.

□

4.3 Lemma. *For any model $\mathbb{U}^{\mathrm{Full}}_{\mathrm{F}}$ of Σ there is an embedding*

$$\mathbf{i} : \mathbb{U}^{\mathrm{Full}}_{\mathrm{F}} \hookrightarrow \tilde{\mathbb{U}}^{\mathrm{Full}}_{\mathrm{F}} \tag{12}$$

which is the identity on F *and induces an embedding* $\mathbf{i} : \mathrm{G}/Z \hookrightarrow \tilde{\mathrm{G}}/\{\pm 1\}$ *such that the* $\mathbf{d}_q$*-named elements of* G *map to the* $\mathbf{d}_q$*-named elements of* $\tilde{\mathrm{G}}$, $\mathbf{i}(\Gamma) \subset \tilde{\Gamma}$ *and* $\mathbf{i}(\mathrm{E}) \subset \tilde{\mathrm{E}}$.

Proof. $\tilde{\mathbb{U}}$ in $\tilde{\mathbb{U}}^{\mathrm{Full}}_{\mathrm{F}}$ is given as the universal cover of $\mathbb{Y}(\mathrm{F})$, with respect to the class of modular curves. By axiom (9) $\mathbb{U}$ covers every principal modular curve $\mathbb{Y}(n)$ by $j_n : \mathbb{U} \to \mathbb{Y}(n)$ which agrees with the system $\mathrm{pr}_{n,m}$ by the same rule as $\tilde{j}_n$ does. Hence, for any $u \in \mathbb{U}$ there is a unique $\tilde{u} \in \tilde{\mathbb{U}}$ such that $j_n(u) = \tilde{j}_n(\tilde{u})$ for all n. This determines a unique map

$$\mathbf{i} : u \mapsto \tilde{u};\ \mathbb{U} \to \tilde{\mathbb{U}}.$$

Axioms (5) and (7) ensure that $\mathbf{i}$ is an embedding. (8) and (9) imply that the structure on the $\mathbb{Y}(n)$ in $\mathbb{U}_{\mathrm{F}}$ and in $\tilde{\mathbb{U}}_{\mathrm{F}}$ are identical and that the maps j_n and $\tilde{j}_n$ satisfy the same equations. The first line of (10) ensures that i sends the graph of $g \in \mathrm{G}$ to the relations $C^{\mu}_{\mathbf{g},n}$ for all n, equivalently graph g corresponds to the graph of a $\mathbf{g}^{\mu} \in \tilde{\mathrm{G}}$. Thus $\mathbf{i} : g/Z \mapsto \mathbf{g}^{\mu}/Z$ is an embedding of G/Z into $\tilde{\mathrm{G}}/(Z \cap \tilde{\mathrm{G}}) = \tilde{\mathrm{G}}/\{\pm 1\}$ such that $\mathbf{i}(g * u) = \mathbf{g}^{\mu} * \mathbf{i}(u) = \mathbf{i}(g) * \mathbf{i}(u)$. The second line of (10) tells us that $\mathbf{i}(\mathbf{d}_q) = \mathbf{d}_q$ (note that $\mathbf{d}^{\mu}_q = \mathbf{d}_q$, for all μ).

We get $\mathbf{i}(\Gamma(n)) \subset \tilde{\Gamma}(n)$ by (7), and $\mathbf{i}(\mathrm{E}) \subset \tilde{\mathrm{E}}$ by (5).

□

4.4 Lemma. *Suppose* $\mathbf{g} \in \mathrm{GL}_2^+(\mathbb{Q})$, $g, g' \in \mathrm{G}$ *and for some* $\mu, \mu' \in \hat{\mathbb{Z}}^{\times}$, *in structure* $\tilde{\mathbb{U}}$:

$$\bigwedge_{n\in\mathbb{N}} C^{\mu}_{\mathbf{g},n} = j_n(\mathrm{graph}\, g) \ \textit{and} \ \bigwedge_{n\in\mathbb{N}} C^{\mu'}_{\mathbf{g},n} = j_n(\mathrm{graph}\, g') \tag{13}$$

Then $g' = g$ *or* $g' = \hat{g}$, *where* $\hat{g} := \mathbf{d}_{-1} g \mathbf{d}_{-1}$.

Moreover, there is $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$ *such that* $g' = g^{\sigma}$.

Proof. The equalities imply that $C^{\mu}_{\mathbf{g}} = \mathrm{graph}\, g$ and $C^{\mu'}_{\mathbf{g}} = \mathrm{graph}\, g'$, that is in the imbedding $\mathrm{G} \subset \tilde{\mathrm{G}}$

$$g = \mathbf{g}^{\mu} \text{ and } g' = \mathbf{g}^{\mu'}$$

that is $g' = g^{\lambda}$, for $\lambda \in \hat{\mathbb{Z}}^{\times}$ defining an automorphism $\psi_{\lambda} : \tilde{\mathrm{G}} \to \tilde{\mathrm{G}},\ g \mapsto g^{\lambda}$.

We may assume $\mathrm{G} = \mathrm{GL}_2^+(\mathbb{Q}) \subset \tilde{\mathrm{G}}$. If $g \neq g'$ then $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not a diagonal matrix, say $b \neq 0$, and $g' = g^{\lambda} = \begin{pmatrix} a & \lambda b \\ \lambda^{-1} c & d \end{pmatrix}$. Since b and λb are distinct non-zero rational numbers, we have necessarily $\lambda = -1$.

Finally note that ψ_{λ} is an automorphism of $\tilde{\mathrm{G}}$ which is induced by a Galois automorphism acting on $\tilde{\mathbb{U}}^{\mathrm{Full}}_{\mathrm{F}}$. □

5 An analytic Zariski structure

5.1 Let k ⊆ F be a subfield and $S \subseteq \mathbb{U}^m$ in a model $\mathbb{U}_{\mathrm{F}}$ of Σ. We say S is a **closed** (analytic) set over k if there is a family $\{W_n \subseteq \mathbb{Y}(n)^m : n \in \mathbb{N}\}$ of Zariski closed subsets defined over k such that

$$S = \bigcap_{n\in\mathbb{N}} j_n^{-1}(W_n).$$

S over k is said to be **irreducible** over k′ (k ⊆ k′ ⊆ F) if for any countable family $\{S_i \subseteq \mathbb{U}^m : i \in I\}$ of closed sets over k′

$$S = \bigcup_{i\in I} S_i \Rightarrow S = S_{i_0}, \text{ for some } i_0 \in I.$$

For $\bar{u} \in \mathbb{U}^m$ we call locus($\bar{u}$/k) the smallest closed $S \subseteq \mathbb{U}^m$ over k which contains $\bar{u}$. This is, clearly, irreducible over k.

We say that $\bar{u} = \langle u_1, \dots, u_m \rangle \in \mathbb{U}^m$ is G-generic if there is no $g \in \mathrm{G}$ such that $u_j = g \cdot u_i$ for $1 \le i, j \le m$.

In 5.2-5.8 we prove, as the matter of fact, that the universe $\mathbb{U}$ equipped with predicates for analytic sets over $\mathbb{Q}$ is a one-dimensional *analytic Zariski structure* as defined in [18]. Such a structure, as proved in [19] (see also [20]), is a quasi-minimal geometry structure which, according to [14], can be axiomatised categorically in uncountable cardinals by an $L_{\omega_1,\omega}(Q)$-sentence. The is summarised in the final Theorem 5.10.

5.2 Lemma. *Let $\bar{u} \in \mathbb{U}^k$ be* G*-generic. Then there is an $n \in \mathbb{N}$ and a Zariski closed $Z_{\bar{u},\mathbb{Q}} \subseteq \mathbb{Y}(n)^k$ over $\mathbb{Q}$ satisfying*

$$\mathrm{locus}(\bar{u}/\mathbb{Q}) = j_n^{-1}(Z_{\bar{u}}, \mathbb{Q}).$$

Let $\mathrm{F}_0 \subset \mathrm{F}$ *be an algebraically closed subfield and assume that at least one coordinate of $j(\bar{u})$ is not in* F_0*. Then there is an $n \in \mathbb{N}$ and a Zariski closed $Z_{\bar{u},\mathrm{F}_0} \subseteq \mathbb{Y}(n)^k$ over* F_0 *satisfying*

$$\mathrm{locus}(\bar{u}/\mathrm{F}_0) = j_n^{-1}(Z_{\bar{u},\mathrm{F}_0}).$$

Proof. We use [13], section 5.1. Let $\mathrm{k}_0 = \mathbb{Q}^{\mathrm{ab}}(\mathrm{CM}(1))$, the extension of $\mathbb{Q}^{\mathrm{ab}}$ by the co-ordinates of special points in $\mathbb{Y}(1)$. Let $\bar{a} = j(\bar{u})$.

$\mathrm{Gal}(\mathrm{F}/\mathrm{k}_0(\bar{a}))$ acts on $\tilde{\mathbb{U}}$ so that there is a number $M \in \mathbb{N}$ and a subgroup $\Omega \subseteq \tilde{\Gamma}^k$ of index M,

$$\tilde{\Gamma}^k = \bigcup_{1\le i\le M} \Omega \cdot \bar{\gamma}_i, \quad j^{-1}(\bar{a}) = \dot{\bigcup_{1\le i\le M}} \Omega \cdot \bar{u}_i, \quad \text{where } \bar{u}_i := \bar{\gamma}_i \cdot \bar{u}$$

$$\Omega \cdot \bar{u}_i = \mathrm{Gal}(\mathrm{F}/\mathrm{k}_0) \cdot \bar{u}_i, \text{ for } i = 1, \dots, M.$$

Assume $\bar{u} = \bar{u}_1$.

Each of the M Galois orbits correspond to the algebraic type of the sequence $\{j_n(\bar{u}_i) : n \in \mathbb{N}\}$ given by systems of equations

$$\bar{P}_{i,n}(j_n(\bar{u}_i), \bar{a}) = 0 \tag{14}$$

where $\bar{P}_{i,n}(\bar{x}, \bar{y})$ is over k_0 and defines the locus of $\langle j_n(\bar{u}_i), \bar{a}\rangle$ over k_0. Since by construction $\bar{a} = \mathrm{pr}_{n,1}(j_n(\bar{u}_i))$, for the regular map $\mathrm{pr}_{n,1} : \mathbb{Y}(n) \to \mathbb{Y}(1)$ over $\mathbb{Q}$, we can replace (14) by the equivalent

$$\bar{P}^*_{i,n}(j_n(\bar{u}_i)) = 0, \tag{15}$$

where $\bar{P}^*_{i,n}(\bar{x}) = \bar{P}_{i,n}(\bar{x}, \mathrm{pr}_{n,1}(\bar{x}))$ defines the locus $Z_{i,n}$ of $j_n(\bar{u}_i)$ over k_0.

The systems of Zariski closed subsets $Z_{i,n} \subseteq \mathbb{Y}(n)^m$ have the property that

$$\mathrm{pr}_{n,m}(Z_{i,n}) = Z_{i,m}, \text{ for } m|n.$$

It follows that for some n_0, for any $n, m \ge n_0$

$$\mathrm{pr}^{-1}_{n,m}(Z_{i,m}) = Z_{i,n}, \text{ for } m|n,$$

and thus

$$\mathrm{locus}(\bar{u}/\mathrm{k}_0) = j^{-1}_{n_0}(Z_{1,n_0}),$$

which proves the Lemma for $\mathrm{k} = \mathrm{k}_0$.

Note that since $\tilde{\mathbb{U}}_{\mathrm{F}}$ is invariant under the action of $\mathrm{Gal}(\mathrm{F}/\mathbb{Q})$, for any $\sigma \in \mathrm{Gal}(\mathrm{F}/\mathbb{Q})$,

$$\mathrm{pr}^{-1}_{n,m}(Z^\sigma_{i,m}) = Z^\sigma_{i,n}, \text{ for } m|n,$$

Now let

$$Z^*_{1,n_0} = \bigcup_{\sigma\in\mathrm{Gal}(\mathrm{F}/\mathbb{Q})} Z^\sigma_{1,n_0}.$$

This is defined over $\mathbb{Q}$ and

$$\mathrm{locus}(\bar{u}/\mathbb{Q}) = j^{-1}_{n_0}(Z^*_{1,n_0}).$$

Now consider $\mathrm{F}_0 \subset \mathrm{F}$ algebraically closed and the assumption $\mathrm{tr.deg}_{\mathrm{F}_0}(\bar{a}) > 0$. For this case we refer to the argument [13], section 5.1(b) which show how to deduce the proof of the algebraically closed case from the case of k_0. □

Remark. In case $j(\bar{u}) \in \mathrm{F}_0^k$,

$$\mathrm{locus}(\bar{u}/\mathrm{F}_0) = \{\bar{u}\}.$$

5.3 Remark. Note that in the first part of the proof we could set $\mathrm{k}_0 = \mathbb{Q}^{\mathrm{ab}}(\mathrm{CM})$, where CM is the co-ordinates of all the CM-points on all $\mathbb{Y}(n)$, Indeed, the key property of such a k_0 is the one proved in Lemma 5.2 of [13]: $\mathrm{k}_0(\bar{a})$ is an abelian extension of $\mathbb{Q}^{\mathrm{ab}}(\bar{a})$. This property remains true as we exchange CM(1) by CM by the same argument of class field theory.

Moreover, the first statement of Lemma 5.2 can be extended to the more general:

$$\mathrm{locus}(\bar{u}/\mathrm{k}) = j_n^{-1}(Z_{\bar{u},\mathrm{k}}),$$

where $\mathrm{k} \subseteq \mathrm{k}_0$, any subfield and $Z_{\bar{u},\mathrm{k}}$ is defined over k. The proof of this is the same as the proof for $\mathrm{k} = \mathbb{Q}$.

We keep the notation F_0 for an algebraically closed subfield below.

5.4 Lemma. *Let* $u_1, \ldots, u_k, v_1, \ldots, v_k \in \mathbb{U}$, $\bar{u} := \langle u_1, \ldots, u_k \rangle$ *is* G-*generic,* $u_i \neq v_i$ $\bar{v} := \langle v_1, \ldots, v_k \rangle$ *and* $\bar{v} = \mathsf{g}(\bar{u})$ *for* $\mathsf{g} \in \mathrm{G}^k$ *(that is* $v_i = g_i \cdot v_i$ *for* $g_1, \ldots, g_k \in \mathrm{G}$ *respectively,* $\mathsf{g} = \langle g_1, \ldots, g_k \rangle$*).*

Then

(i)

$$\mathrm{locus}(\bar{u}\bar{v}/\mathbb{Q}) = (S \times \mathbb{U}^k) \cap (\mathrm{graph}(\mathsf{g}) \cup \mathrm{graph}(\hat{\mathsf{g}}))$$

where $S = \mathrm{locus}(\bar{u}/\mathbb{Q})$. *equivalently*

$$\mathrm{locus}(\bar{u}\bar{v}/\mathbb{Q}) = \{\langle x_1, \ldots, x_k, y_1, \ldots, y_k \rangle \in \mathbb{U}^{2k} :\vDash S(x) \ \& \ (y = \mathsf{g}(x) \ \vee y = \hat{\mathsf{g}}(x))\}$$

(ii)

$$\mathrm{locus}(\bar{u}\bar{v}/\mathrm{F}_0) = (S \times \mathbb{U}^k) \cap \mathrm{graph}(\mathsf{g})$$

where $S = \mathrm{locus}(\bar{u}/\mathrm{F})$.

Proof. It follows from the axiom (11) that there are $\mu_1, \ldots, \mu_k \in \hat{\mathbb{Z}}^\times$ such that

$$\bar{u}\bar{v} \in C_{\mathbf{g}_1}^{\mu_1} \times \ldots \times C_{\mathbf{g}_k}^{\mu_k} \subseteq \mathrm{locus}(\bar{u}\bar{v}/\mathbb{Q}) \subseteq C_{\mathbf{g}_1} \times \ldots \times C_{\mathbf{g}_k} \text{ and } C_{\mathbf{g}_i}^{\mu_i} = \mathrm{graph}(g_i).$$

Taking into account that the $C^{\mu_i}_{\mathbf{g}_i,n} \subseteq \mathbb{Y}(n)^2$ are algebraic curves defined over $\bar{\mathbb{Q}}$, we get (ii).

By 4.4 graph$(\mathbf{g}) \cup$ graph$(\hat{\mathbf{g}})$ is the smallest $\mathrm{Gal}_{\mathbb{Q}}$-invariant subset of $\mathbb{U}^{2k}$ containing $C^{\mu_1}_{\mathbf{g}_1} \times \ldots \times C^{\mu_k}_{\mathbf{g}_k}$. The statement (i) of Lemma follows. □

5.5 Lemma. *Let* $\bar{t} = \langle t_1, \ldots, t_l \rangle \in \mathbb{U}^l$ *be the fixed point of* $\mathsf{h} = \langle h_1, \ldots, h_l \rangle \in \mathrm{G}^l$. *Then*

$$\mathrm{locus}(\bar{t}/\mathrm{F}_0) = \{\bar{t}\} \text{ and } \mathrm{locus}(\bar{t}/\mathbb{Q}) = \{\bar{t}, \bar{t}^*\}$$

where $\bar{t}^*$ *is the unique fixed point of* $\hat{\mathsf{h}} := \langle \hat{h}_1, \ldots, \hat{h}_l \rangle$ *in* $\mathbb{U}^l$.

Proof. The first statement is obvious. The second follows from 5.5, since $\bar{t}^*$ is the only point in $\mathbb{U}^l$ Galois conjugated to $\bar{t}$. □

5.6 Lemma. *Let* $u_1, \ldots, u_k, u'_1 \ldots, u'_m, v_1, \ldots, v_k, v'_1, v'_l \in \mathbb{U}$, $\bar{u} = \langle u_1, \ldots, u_k \rangle$, $\bar{u}' = \langle u'_1 \ldots, u'_m \rangle$, $\bar{u}\bar{u}'$ *is* G*-generic, and* $S = \mathrm{locus}(\bar{u}\bar{u}')$. *Suppose* $u_i \neq v_i$, *for each* $1 \leq i \leq k$, *and* $\bar{v} = \mathsf{g}(\bar{u})$ *for* $\mathsf{g} \in \mathrm{G}^k$. *Suppose also that* $v'_1, \ldots, v'_l$ *are special.*

Then

$$\mathrm{locus}(\bar{u}\bar{u}'\bar{v}\bar{v}'/\mathbb{Q}) = \{\bar{x}\bar{x}'\bar{y}\bar{y}' \in \mathbb{U}^{2k+m+l} : S(\bar{x}\bar{x}') \;\&\; (\bar{y} = \mathsf{g}(\bar{x}) \vee \bar{y} = \hat{\mathsf{g}}(\bar{x})) \;\&\; \bar{y}' \in \{\bar{t}, \bar{t}^*\}\}$$

$$\mathrm{locus}(\bar{u}\bar{u}'\bar{v}\bar{v}'/\mathrm{F}_0) = \{\bar{x}\bar{x}'\bar{y}\bar{y}' \in \mathbb{U}^{2k+m+l} : S(\bar{x}\bar{x}') \;\&\; (\bar{y} = \mathsf{g}(\bar{x}) \;\&\; \bar{y}' = \bar{t}\}$$

for some $\bar{t}, \bar{t}^*$ *as in 5.5.*

Proof. Immediate from 5.2-5.5. □

Call

$$T(\bar{x}\bar{x}'\bar{y}\bar{y}') := S(\bar{x}\bar{x}') \;\&\; (\bar{y} = \mathsf{g}(\bar{x}) \;\vee \bar{y} = \hat{\mathsf{g}}(\bar{x})) \;\&\; \bar{y}' \in \{\bar{t}, \bar{t}^*\} \qquad (16)$$

basic predicate over $\mathbb{Q}$. And

$$T(\bar{x}\bar{x}'\bar{y}\bar{y}') := S(\bar{x}\bar{x}') \;\&\; \bar{y} = \mathsf{g}(\bar{x}) \;\&\; \bar{y}' = \bar{t} \qquad (17)$$

basic predicate over F_0, where S, g, $\hat{\mathsf{g}}$, $\bar{t}$ and $\bar{t}^*$ are as in 5.2 – 5.6.

Clearly, basic predicate define closed analytic subsets.

5.7 Proposition. *Let* k *be* $\mathbb{Q}$ *or an algebraically closed subfield of* F, $T(\bar{x}\bar{x}'\bar{y}\bar{y}')$ *be a basic predicate over* k, z *be one of the* M *variables* x_i, x'_j, y_i *or* y'_p. *Let* $\mathrm{pr}T \subseteq \mathbb{U}^{M-1}$ *be the subset defined by the formula* $\exists z \; T(\bar{x}\bar{x}'\bar{y}\bar{y}')$.

Then there is a basic analytic set R and a closed subset $R' \subset R$, both over k, $\dim R' < \dim R$, *such that*

$$R \setminus R' \subseteq \mathrm{pr}T \subseteq R.$$

Proof. Let first k $:= \mathbb{Q}$. We consider four possible cases.

(a) $z = y'_n$. It is immediate from the form of the predicate in (16) that

$$\exists z\ T(\bar{x}\bar{x}'\bar{y}\bar{y}') \equiv S(\bar{x}\bar{x}')\ \&\ (\bar{y} = \mathsf{g}(\bar{x})\ \vee \bar{y} = \hat{\mathsf{g}}(\bar{x}))\ \&\ \bar{y}'_- \in \{\bar{t}_-, \bar{t}^*_-\}$$

where $\bar{y}'_-, \bar{t}_-$ and $\bar{t}^*_-$ stand for the tuples with omitted n-coordinate.

(b) $z = y_i$. In this case, since g_k and $\hat{g}_k$ are operations on the whole of $\mathbb{U}$,

$$\exists z\ T(\bar{x}\bar{x}'\bar{y}\bar{y}') \equiv S(\bar{x}\bar{x}')\ \&\ (\bar{y}_- = \mathsf{g}_-(\bar{x}_-)\ \vee \bar{y}_- = \hat{\mathsf{g}}_-(\bar{x}_-))\ \&\ \bar{y}' \in \{\bar{t}, \bar{t}^*\}$$

where $\bar{y}_-$, g_- and $\bar{x}_-$ stand for the tuples with omitted i - coordinates.

(c) $z = x_i$, say $i = k$. This is the same case as (b) if we rearrange the variables in T by taking $x_1, \dots, x_{k-1}, y_k$ to be the set of variables standing for the G-generic k-tuple $u_1, \dots, u_{k-1}, v_k$ and replace $\langle g_1, \dots, g_{k-1}, g_k\rangle$ by $\langle g_1, \dots, g_{k-1}, g_k^{-1}\rangle$.

(d) $z = x'_j$. Then

$$\exists z\ T(\bar{x}\bar{x}'\bar{y}\bar{y}') \equiv (\exists x'_j\ S(\bar{x}\bar{x}'))\ \&\ (\bar{y} = \mathsf{g}(\bar{x})\ \vee \bar{y} = \hat{\mathsf{g}}(\bar{x}))\ \&\ \bar{y}' \in \{\bar{t}, \bar{t}^*\}.$$

Let pr$S$ be the subset of $\mathbb{U}^{k+m-1}$ defined by $\exists x'_j\ S(\bar{x}\bar{x}')$ and L the subset of $\mathbb{U}^{2k+l}$ defined by $(\bar{y} = \mathsf{g}(\bar{x})\ \vee \bar{y} = \hat{\mathsf{g}}(\bar{x}))\ \&\ \bar{y}' \in \{\bar{t}, \bar{t}^*\}$.

Since $S = j_n^{-1}(W)$ for some Zariski closed subset $W \subseteq \mathbb{X}(n)^{k+m}$, we have $\mathrm{pr}S = j_n^{-1}(\mathrm{pr}W)$, where on the right of the equation we consider the projection along x'_j and on the left the projection along the coordinate corresponding in the image. By standard facts on Zariski topology $\mathrm{pr}W = V \setminus R$, for V Zariski closed and U a boolean combination of Zariski closed, $\dim U < \dim V$. Let $\bar{U}$ be the Zariski closure of U, so the Zariski open set $V \setminus \bar{U}$ is a subset of pr$W$ and pr$W \subset V$.

Then $\mathrm{pr}S = j_n^{-1}(\mathrm{pr}W) = j_n^{-1}(V) \setminus j_n^{-1}(U)$ and so

$$R \setminus R' \subseteq \mathrm{pr}S \subseteq R, \text{ for } R = j_n^{-1}(V),\ R' = j_n^{-1}(\bar{U}).$$

It follows that,

$$((R \setminus R') \times \mathbb{U}^{k+l}) \cap (\mathbb{U}^m \times L) \subseteq \exists z\ T(\bar{x}\bar{x}'\bar{y}\bar{y}') \subseteq (R \times \mathbb{U}^{k+l}) \cap (\mathbb{U}^{m-1} \times L)$$

which proves the Proposition for $\mathrm{k} = \mathbb{Q}$.

For $\mathrm{k} = \mathrm{F}_0$ algebraically closed, use the same arguments (a)-(d) combined with F_0-versions of 5.2-5.6.

□

Let $T \subseteq \mathbb{U}^M$ be a basic predicate over k, that is a predicate of the form (14) or (15).

$$\dim T : \dim S := \dim Z$$

where $Z \subseteq \mathbb{Y}(n)^{k+m}$ is the Zariski closed subset over k such that, according to 5.2, $S = j_n^{-1}(Z)$.

5.8 Proposition. *Suppose* F *is of infinite transcendence degree over* k*, for* $\mathrm{k} = \mathbb{Q}$ *or* $\mathrm{k} = \mathrm{F}_0$*, an algebraically closed subfield of* F*. Then* $\mathbb{U}_\mathrm{F}$ *in the language of basic predicates over* k *is* ω*-homogeneous over* k*:*

for any $\bar{u}, \bar{u}' \in \mathbb{U}^m$ *and* $v \in \mathbb{U}$ *such that* $\mathrm{locus}(\bar{u}/\mathrm{k}) = \mathrm{locus}(\bar{u}'/\mathrm{k})$ *there is* $v' \in \mathbb{U}$ *such that* $\mathrm{locus}(\bar{u}v/\mathrm{k}) = \mathrm{locus}(\bar{u}'v'/\mathrm{k})$.

Proof. Let $T = \mathrm{locus}(\bar{u}v/\mathrm{k})$ and $R = \mathrm{locus}(\bar{u}/\mathrm{k})$. Then by 5.7 there is a closed R' of smaller dimension such that

$$R \smallsetminus R' \subseteq \mathrm{pr}T \subseteq R.$$

Clearly, $\bar{u}' \in R \smallsetminus R'$ and hence $\bar{u}' \in \mathrm{pr}T$, which means that the fibre

$$T(\bar{u}', \mathbb{U}) := \{w \in \mathbb{U} : \ \bar{u}'w \in T\} \neq \emptyset.$$

Note that $T(\bar{u}', \mathbb{U})$ is a closed subset of $\mathbb{U}$ defined over the field k' generated by coordinates of $j_n(\bar{u}')$, all $n \in \mathbb{N}$.

Consider the only two possible cases, $\dim T = \dim R$ and $\dim T = \dim R{+}1$.

In the first case pick up any $v' \in T(\bar{u}', \mathbb{U})$. Then $\mathrm{locus}(\bar{u}'v'/\mathbb{Q}) = T$ since T is irreducible over $\mathbb{Q}$.

In the second case $T(\bar{u}', \mathbb{U}) = \mathbb{U}$. Pick up any $v' \in \mathbb{U}$ generic over k', that is such that $j_1(v') \notin \bar{\mathrm{k}}'$. Clearly, the transcendence degree of k' over $\mathbb{Q}$ is at most the length of $\bar{u}$, and so such v' exists. Now $\dim \mathrm{locus}(\bar{u}'v'/\mathbb{Q}) = \dim R + 1$ and so again by irreducibility of T we have the equality $\mathrm{locus}(\bar{u}'v'/\mathbb{Q}) = T$. □

5.9 For a subset $W \subset \mathbb{U}$ and a point $u \in \mathbb{U}$ we define

$$u \in \mathrm{cl}(W) \Leftrightarrow \exists \bar{w} \subset_{\text{finite}} W : \dim \mathrm{locus}(\bar{w}u) = \dim \mathrm{locus}(\bar{w}).$$

And define the closure of W

$$\mathrm{cl}(W) = \{u \in \mathbb{U} : \; u \in \mathrm{cl}(W)\}.$$

We consider the covering sort $\mathbb{U}$ in $\mathbb{U}_{\mathrm{F}}$ together with basic relations over $\mathbb{Q}$ as a structure.

Recall (see [14]) that one calls $(\mathbb{U}, \mathrm{cl})$ a **quasiminimal pregeometry structure** if the following holds:

QM1. The pregeometry is determined by the language. That is, if $\mathrm{tp}(v\bar{w}) = \mathrm{tp}(v'\bar{w}')$ then $v \in \mathrm{cl}(\bar{w})$ if and only if $v' \in \mathrm{cl}(\bar{w}')$.

QM2. $\mathbb{U}$ is infinite-dimensional with respect to cl.

QM3. (Countable closure property) If $W \subset \mathbb{U}$ is finite then $\mathrm{cl}(W)$ is countable.

QM4. (Uniqueness of the generic type) Suppose that $W, W' \subseteq \mathbb{U}$ are countable subsets, $\mathrm{cl}(W) = W$, $\mathrm{cl}(W') = W'$ and W, W' enumerated so that $\mathrm{tp}(W) = \mathrm{tp}(W')$.

If $v \in \mathbb{U} \setminus W$ and $v' \in \mathbb{U} \setminus W'$ then $\mathrm{tp}(Wv) = \mathrm{tp}(W'v')$ (with respect to the same enumerations for W and W').

QM5. ($\aleph_0$-homogeneity over closed sets and the empty set) Let $W, W' \subseteq \mathbb{U}$ be countable closed subsets or empty, enumerated such that $\mathrm{tp}(W) = \mathrm{tp}(W')$, and let $\bar{w}, \bar{w}'$ be finite tuples from $\mathbb{U}$ such that $\mathrm{tp}(W\bar{w}) = \mathrm{tp}(W'\bar{w}')$, and let $v \in \mathrm{cl}(W\bar{w})$. Then there is $v' \in \mathbb{U}$ such that $\mathrm{tp}(\bar{w}vW) = \mathrm{tp}(\bar{w}'v'W')$.

5.10 Theorem. *For any model $\mathbb{U}_{\mathrm{F}}$ of Σ, with F algebraically closed of infinite transcendence degree, the structure $(\mathbb{U}, \mathrm{cl})$ is a quasiminimal pregeometry.*

There is an existential $L_{\omega_1,\omega}$-sentence $\Theta_{\mathbb{U}}$ such that $\Sigma \& \Theta_{\mathbb{U}}$ defines a categorical AEC containing $\mathbb{U}$.

Proof. We strat with the proof of the firs statement of the theorem by checking conditions QM1–QM5.

QM1 is by definition. QM2 follows fom the assumption on F.

QM3 is due to the fact, implied by the definition, that $v \in \mathrm{cl}(W)$ if and only if $j(v) \in \mathrm{acl}(j(W))$, taking into account that $j^{-1}(j(v))$ is countable.

To tackle QM4 and QM5 note first that $\mathrm{tp}(\bar{u})$ is determined by $T = \mathrm{locus}(\bar{u}/\mathbb{Q})$. More presicely, the quantifier-free part of the type is given by the basic predicate T together with negations of all the basic predicates R of the same arity such that $\dim R < \dim T$. Now we claim that any type is equivalent to a quntifier-free one. Indeed, by homogeneity proved in 5.8, in a countable elementary substructures $\mathbb{U}^0 \prec \mathbb{U}$ the set defined by $\mathrm{tp}(\bar{u})$ is equal to the set defined by the respective quantifier-free type. Hence the same holds in $\mathbb{U}$.

Now note that the condition $v \notin W = \mathrm{cl}(W)$ in QM4 is equivalent to the condition that $j_n(v)$ is generic in $\mathbb{Y}(n)$ over $j_n(W)$), for all (equivalently, for some) $n \in \mathbb{N}$. Since $\mathbb{Y}(n)$ is absolutely irreducible, the condition determines the complete field-theretic type of $j_n(v)$ over W and hence the complete quantifier-free type of v over W, equivalently, the full type of v over W. QM4 follows.

QM5 is a direct consequence of 5.8. This completes the proof of the first statement.

Now we construct the $L_{\omega_1,\omega}$-sentence $\Theta_{\mathbb{U}}$.

For each tuple $\mathsf{h} = \langle h_1, \dots h_l \rangle \in \mathrm{E}^l$ and the respective tuple $\bar{t}_h$ of fixed element $\langle t_1, \dots, t_l \rangle$ of h, for each n consider the minimal Zariski closed subset $Z_{h,n} \subset \mathbb{Y}(n)^l$ over $\mathbb{Q}$ such that $j_n(\bar{t}_h) \in Z_{h,n}$. These depend on the model $\mathbb{U}_{\mathrm{F}}$ of Σ.

Now set

$$\Phi_h = \bigwedge_{n \in \mathbb{N}} j_n(\bar{t}_h) \in Z_{h,n}.$$

This can be seen as a quantifier-free $L_{\omega_1,\omega}$-formula with variables $\bar{t}_h$.

$\Theta_{\mathbb{U}}$ will be the $L_{\omega_1,\omega}$-formula Θ constructed in 2.5 stating that for any $\mathsf{h} \in \mathrm{E}^l$ there exists $\bar{t}_h = \langle t_1, \dots, t_l \rangle$, a tuple of fixed points of h such that $\Phi_i(\bar{t}_h)$.

The rest of the proof is split into Lemmas and Claims below.

Now let F be an uncountable algebraically closed field of characteristic zero consider $\mathbb{U}_{\mathrm{F}}$ and $\mathbb{U}'_F$ models of Σ & Θ. We aim to prove that $\mathbb{U}_{\mathrm{F}} \cong \mathbb{U}'_F$.

Let G and G′ be the realisations of $\mathrm{GL}_2^+(\mathbb{Q})$ in $\mathbb{U}_{\mathrm{F}}$ and $\mathbb{U}'_{\mathrm{F}}$ respectively. By construction, the statement in the formula Θ implies that some group-isomorphism $\mathsf{i}_{\mathrm{G}} : \mathrm{G} \to \mathrm{G}'$ can be uniquely extended to the map $\mathsf{i}_{\mathrm{CM}} : \mathbb{U}(\mathrm{CM}) \to \mathbb{U}'(\mathrm{CM})$ which acts on the fixed points $u_g \mapsto u_{g'}$, if $g' = \mathsf{i}_{\mathrm{G}}(g)$ and takes $j_n(u_g)$ to $j_n(u_{g'})$ so that polynomial equations in the co-ordinates of $j_n(u_g)$ over $\mathbb{Q}$ are preserved. In other words we have a Galois automorphism

$$\mathsf{i}_{\mathrm{k}} : \mathbb{Q}(\mathrm{CM}) \to \mathbb{Q}(\mathrm{CM}), \text{ where } \mathrm{k} := \mathbb{Q}(\mathrm{CM}),$$

which agrees with i_{G}, that is together the pair $(\mathsf{i}_{\mathrm{G}}, \mathsf{i}_{\mathrm{k}})$ is a partial isomorphism $\mathsf{i}_0 : \mathbb{U}_{\mathrm{F}} \to \mathbb{U}'_{\mathrm{F}}$.

Let $\mathrm{F}_0 \subseteq \mathrm{F}$ be a countable algebraically closed subfield.

We consider submodels $\mathbb{U}_{\mathrm{F}_0} \subseteq \mathbb{U}_{\mathrm{F}}$ and $\mathbb{U}'_{\mathrm{F}_0} \subseteq \mathbb{U}'_{\mathrm{F}}$.

Claim 1. There are subsets $V \subset \mathbb{U}_{\mathrm{F}_0}$ and $V' \subset \mathbb{U}'_{\mathrm{F}_0}$ and a partial isomorphism $\mathsf{i}_V : V \to V'$ extending i_0 such that V is a maximal G-generic subset of $\mathbb{U}_{\mathrm{F}_0}$ and V' is maximal G-generic subset of $\mathbb{U}'_{\mathrm{F}_0}$.

Proof. Since $\mathbb{U}_{\mathrm{F}_0}$ and $\mathbb{U}'_{\mathrm{F}_0}$ are countable we can enumerate the sets and use the back-and-forth procedure in constructing $V = \{v_m : m \in \mathbb{N}\}$, $V' = \{v'_m : m \in \mathbb{N}\}$, and i_V.

Suppose $\bar{v} = \langle v_1, \dots, v_m\rangle$ and $\bar{v}' = \langle v'_1, \dots, v'_m\rangle$ be G-generic and satisfy

$$\mathrm{locus}(\bar{v}'/\mathrm{k}) = \mathsf{i}_{\mathrm{k}}(\mathrm{locus}(\bar{v}/\mathrm{k})).$$

which means that $j_n(\bar{v}) \in Z_n$, for some variety $Z_n \subset \mathbb{Y}(n)^m$ over k, if and only if $j_n(\bar{v}') \in Z'_n$ for the variety $Z'_n = \mathsf{i}_{\mathrm{k}}(Z_n)$, for all n.

For the back-and-forth construction, it suffices to prove:

Let w be the first element in $\mathbb{U}_{\mathrm{F}_0}$ such that $\bar{v}w$ is G-generic. There exists $w' \in \mathbb{U}'_{\mathrm{F}_0}$ such that

$$\mathrm{locus}(\bar{v}'w'/\mathrm{k}) = \mathsf{i}_{\mathrm{k}}(\mathrm{locus}(\bar{v}w/\mathrm{k})).$$

In order to establish w' consider $\{W_n \subset \mathbb{Y}(n)^{m+1} : n \in \mathbb{N}\}$ be the family of varieties over k which determine the locus of $\bar{v}w$. By 5.2 together with 5.3 there is n_0 such that the locus is actually determined by any one of the W_n, for $n \geq n_0$. By algebraic geometry the projection of $\mathrm{pr}W_n$ on the first m coordinates contains a Zariski open subset of Z_n, hence contains $j_n(\bar{v})$. Since Z_n is Galois conjugated to Z'_n, there exists respective $W'_n = \mathsf{i}_{\mathrm{k}}(W_n)$ together with the whole family $\{W'_n \subset \mathbb{Y}(n)^{m+1} : n \in \mathbb{N}\}$ such that $j_n^{-1}(W'_n) = j_{n_0}^{-1}(W'_{n_0})$, for $n \geq n_0$, and $\mathrm{pr}W'_n$ contains an open subset of Z'_n. Thus $j_n(\bar{v}') = \bar{z}' \in \mathrm{pr}W'_n$ that is there exists $t \in \mathbb{Y}(n)$, such that $\bar{z}t \in W'_n$, generic over k. Pick up $w' \in j_n^{-1}(t)$, and thus $j_n(\bar{v}'w') \in W'_n$. So w' is as required. Claim 1 proved.

Claim 2. There is a unique extension i of i_V to an isomorphism

$$\mathsf{i} : \mathbb{U}_{\mathrm{F}_0} \to \mathbb{U}'_{\mathrm{F}_0}.$$

Proof. By definition

$$\mathbb{U}_{\mathrm{F}_0} = \mathbb{U}(\mathrm{CM}) \,\dot\cup\, \mathrm{G}\cdot V \text{ and } \mathbb{U}'_{\mathrm{F}_0} = \mathbb{U}'(\mathrm{CM}) \,\dot\cup\, \mathrm{G}'\cdot V',$$

where

$$\mathrm{G}\cdot V = \{\mathrm{G}\cdot v : v \in V\}, \quad \mathrm{G}'\cdot V' = \{\mathrm{G}'\cdot v' : v' \in V'\}$$

the disjoint unions of Hecke orbits of non-CM points. Any i must coincide with i_{CM} on $\mathbb{U}(\mathrm{CM})$, and on the remaining points define

$$\mathsf{i} : g \cdot v \mapsto \mathsf{i}_{\mathrm{G}}(g) \cdot \mathsf{i}_V(v),$$

$$\mathsf{i} : j_n(g \cdot v) \mapsto j_n(\mathsf{i}_{\mathrm{G}}(g) \cdot \mathsf{i}_V(v)).$$

This is as required.

Finally we are ready to prove $\mathbb{U}_{\mathrm{F}} \cong \mathbb{U}'_{\mathrm{F}}$.

Let $\mathrm{F}_\omega \subset \mathrm{F}$ be an algebraically closed subfield of an infinite countable transcendence degree. We follow [14] and consider the class $\mathcal{K}(M)$ (see Theorem 2.3 therein and the definition before it) for $M \cong \mathbb{U}_{\mathrm{F}_\omega} \cong \mathbb{U}'_{\mathrm{F}_\omega}$. The class of countable substructures of $\mathbb{U}_{\mathrm{F}}$ coincides, up to isomorphism, to the class of countable substructures of $\mathbb{U}'_{\mathrm{F}}$, by Claim 2. Thus $\mathbb{U}_{\mathrm{F}}$ and $\mathbb{U}'_{\mathrm{F}}$ belong to $\mathcal{K}(\mathbb{U}_{\mathrm{F}_\omega})$, by definition.

Since $\mathbb{U}_{\mathrm{F}_\omega}$ is a quasiminimal pregeometry structure, $\mathcal{K}(\mathbb{U}_{\mathrm{F}_\omega})$ is categorical in uncountable cardinalities. The statement follows. $\square$

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