The geometry of model theory and Diophantine geometry
London-Paris Number Theory Seminar

B. Zilber

University of Oxford

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Uncountable structures with categorical theories = **logically perfect structures**.

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- G-module, for a group G, insert one orbit in each point of a trivial structure, orbits in nonspecial point must be free.

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   Dual notion: the **dimension of an algebraic variety** V over F

$$\dim V = \max\{\, \mathrm{tr.deg}_F(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in V\}.$$

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NO, in general (E.Hrushovski, 1989)

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This $(\tilde{\mathbf{M}}, \mathbf{f})$ is "near-perfect".

Are Hrushovski structures mathematical pathologies?

Observation: If **M** is a field and we want $\mathbf{f} = \mathrm{ex}$ to be a group homomorphism

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

Then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{lin.dim}\,(X) \geq 0.$$

The Hrushovski inequality, in the case of the complex numbers and $\mathrm{ex} = \exp$, is equivalent to

$$\mathrm{tr.deg}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent (**the Schanuel conjecture**).

Pseudo-exponentiation

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Amalgamation process produces an *algebraically-exponentially closed* **field with pseudo-exponentiation**, $\mathrm{F_{ex}}(\lambda)$.

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Every system of algebraic-exponential equations which does not explicitly contradict SCH must have a solution.

CC: **Countable closure property:**
Analytic subsets of F^{n} of dimension 0 are countable.

Uniqueness theorem

Theorem (2001) *Given an uncountable cardinal* λ, *there is a unique, up to isomorphism, algebraically closed field with pseudo-exponentiation* F_{ex} *of cardinality* λ *satisfying*

$$\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$$

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Remark. The conjecture would be proven if one shows that F_{ex} allows a locally compact topology with continuous ex.

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Projects. j-invariant and $\mathfrak{P}(\tau, z)$ as a function of two variables.

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  These generalise to covers of semi-abelian varieties.

  Every form of Schanuel's conjecture can be interpreted as a statement on atypical (anomalous) intersection

Nonstandard numbers

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This form of "Schanuel" arises when one attempts to introduce a coarse "locally compact" topology on F_{ex}.

Theorem (with M.Bays, 2002-2006)
TFAE:

1. (CIT) Given $W \subseteq \mathbb{C}^n$, an irreducible algebraic variety over $\mathbb{Q}$, there is finite collection $\tau(W)$ of torsion cosets in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$ and an atypical irreducible component $A \subseteq W \cap T$ there is $\mathbf{T} \in \tau(W)$ such that $A \subseteq W \cap \mathbf{T}$.

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2. for all $X \subset_{\text{fin}} {}^*\mathbb{C}: \quad \delta(X/\mathbb{C}) \geq 0, \quad$ for

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1. (CIT) Given $W \subseteq \mathbb{C}^n$, an irreducible algebraic variety over $\mathbb{Q}$, there is finite collection $\tau(W)$ of torsion cosets in $\mathbb{C}^n$ such that for any torus $T \subseteq \mathbb{C}^n$ and an atypical irreducible component $A \subseteq W \cap T$ there is $\mathbf{T} \in \tau(W)$ such that $A \subseteq W \cap \mathbf{T}$.
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Implies the Mordell-Lang conjecture.

The Weierstrass function of two variables

The Weierstrass function $\mathfrak{P}(\tau, x)$ as a function of two variables
For every $\tau \in \mathcal{H}$ define the field k_τ as $\mathbb{Q}$ or $\mathbb{Q}(i_\tau)$, if the corresponding elliptic curve has complex multiplication i_τ.

The corresponding 'Schanuel conjecture' must take into account the "trivial" geometry on $\mathcal{H}$ (with the action of $\mathrm{GL}_2^+(\mathbb{Q})$) and the linear geometry along each elliptic curve. Thus it takes the form:
given $\tau_1, \dots, \tau_m \in \mathcal{H}$ and $x_1, \dots, x_n \in \mathbb{C}$,

$$\mathrm{tr.deg}(\{\tau_i\},\ \{x_j\},\ \{j(\tau_i)\}, \{\mathfrak{P}(\tau_i, x_j)\}) \geq$$

$$|\left(\{\tau_i\} - \mathrm{acl}_{\mathrm{trivial}}(\emptyset)\right)/\mathrm{GL}_2^+(\mathbb{Q})| + \sum_{\{\tau_i\}/\mathrm{GL}_2^+(\mathbb{Q})} \mathrm{lin.dim}_{k_{\tau_i}}\{x_j\}$$

The right-hand side is $\mathrm{mod.dim}(\{\tau_i\},\ \{x_j\})$, the "modular dimension" on $\mathcal{H}^m \times \mathbb{C}^n$. By definition $\mathrm{acl}_{\mathrm{trivial}}(\emptyset)$, the set of "algebraic" elements of the "trivial" pregeometry on $\mathcal{H}$, are the fixpoints under elements of $\mathrm{GL}_2^+(\mathbb{Q})$, i.e. **special points.**
(See also Cristiana Bertolin, J. Number Theory, 2002)

Weaker form of "Schanuel" for $\mathfrak{P}$

Let K be a subfield of $\mathbb{C}$, $K_{\tau_i} = k_{\tau_i} \otimes K$. Then given $\tau_1, \dots, \tau_m \in \mathcal{H}$ and $x_1, \dots, x_n \in \mathbb{C}$,

$$|\left(\{\tau_i\} - \mathrm{acl}^{\mathrm{K}}_{\mathrm{trivial}}(\emptyset)\right)/\mathrm{GL}_2^+(K)| + \sum_{\{\tau_i\}/\mathrm{GL}_2^+(K)} \mathrm{lin.dim}_{K_{\tau_i}}\{x_j\}$$

$$+\mathrm{tr.deg}(K) + \mathrm{tr.deg}(\{j(\tau_i)\}, \{\mathfrak{P}(\tau_i, x_j)\}) \geq$$

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This corresponds to the fusion of a "trivial" $\mathrm{GL}_2^+(K)$-pregeometry on $\mathcal{H}$, linear pregeometry on elliptic curves and the pregeomtery of the field $\mathbb{C}$.

Analogue of CIT

An analogue of CIT will be equivalent to the above "Schanuel" conjecture with $K = {}^*\mathbb{Q}$ and ${}^*\mathbb{C}$ in place of $\mathbb{C}$.

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This is routinely translated into the following (**cf. Pink Conj.**):
Let S be a mixed Shimura variety, the image of $\mathcal{H}^n \times \mathbb{C}^n$ under $(j, \mathfrak{P})$, the product of moduli spaces for elliptic curves.
Given an algebraic subvariety $W \subset S$ over $\mathbb{C}$, there is a finite family σ_W of mixed Shimura subvarieties of S such that any atypical irreducible component of an intersection $W \cap T$, for T a Shimura subvariety is contained in some $\mathbf{T}$ for $\mathbf{T} \in \sigma_W$.

Here and elsewhere *atypical* means :
$\dim W \cap T > \dim W - \mathrm{codim}\, T$. A mixed Shimura subvariety T of S is the image of a submanifold of $\mathcal{H}^n \times \mathbb{C}^n$ defined by equations of the form $x_i = gx_j$, $x_i = a$ and $m_1 z_1 + \ldots + m_n z_n = b$ for variables $x_1, \ldots x_n$ and constants a ranging in $\mathcal{H}$ and variables $z_1, \ldots z_n$ and constants b ranging in $\mathbb{C}$, and some $g \in \mathrm{GL}^+(\mathbb{Q})$, $m_1, \ldots, m_n \in \mathbb{Z}$.